# HOMOTOPICAL TOPOS THEORY

Garth Warner Department of Mathematics University of Washington

## IN THE MOUNTAINS

There is WINTER. Then there is the melting time. Then there is summer. Then there is the waiting time. Then there is WINTER.

----- --

#### ABSTRACT

The purpose of this book is two fold.

(1) To give a systematic introduction to topos theory from a purely categorical point of view, thus ignoring all logical and algebraic issues.

(2) To give an account of the homotopy theory of the simplicial objects in a Grothendieck topos.

\* \* \* \* \* \*

EDITORIAL COMMENT I have always found the traditional homotopical treatments to be somewhat contrived and ad hoc. There is, however, a way out: Use Cisinski's "localizer theory". For then the classical results are mere instances of the output of this powerful machine which has the effect of sweeping all before it.

### REFERENCES

- 1. Borceux, Francis, Handbook of Categorical Algebra, Cambridge University Press, 1994.
- 2. Johnstone, Peter T., Topos Theory, Academic Press, 1977.
- 3. MacLane, Saunders and Moerdijk, Ieke, Sheaves in Geometry and Logic, Springer-Verlag, 1992.
- 4. Warner, Garth, Categorical Homotopy Theory, Available at: www.math.washington.edu/~warner/

## CONTENTS

- §1. PARTIAL ORDERS
- §2. SUBOBJECTS
- **§3.** DECOMPOSITIONS
- §4. SLICES
- §5. CARTESIAN CLOSED CATEGORIES
- §6. SUBOBJECT CLASSIFIERS
- §7. SIEVES
- **§8. HEYTING ALGEBRAS**
- §9. LOCALES
- §10. SITES
- §11. SHEAVES
- §12. LOCAL ISOMORPHISMS
- §13. SORITES
- §14. TOPOS THEORY: FORMALITIES
- §15. TOPOS THEORY: SLICES AND SUBOBJECTS
- §16. TOPOLOGIES
- §17. GEOMETRIC MORPHISMS
- §18. GROTHENDIECK TOPOSES
- §19. POINTS
- § 20. CISINSKI THEORY
- §21. SIMPLICIAL MACHINERY
- §22. LIFTING
- §23. LOCALIZERS OF DESCENT
- §24. LOCAL FIBRATIONS AND LOCAL WEAK EQUIVALENCES
- § 25. COMPARISON PRINCIPLES

\* \* \* \* \*

## INTERNAL AFFAIRS

#### **§1.** PARTIAL ORDERS

Let X be a class -- then a binary relation  $\leq$  on X is said to be a preorder if

- $\leq$  is reflexive:  $\forall x \in X, x \leq x;$
- $\leq$  is transitive:  $\forall x, y, z \in X: x \leq y \& y \leq z \Rightarrow x \leq z$ .

A preorder is a partial order if in addition

$$\forall \mathbf{x}, \mathbf{x}^{\mathsf{i}} \in \mathbf{X}, \qquad \stackrel{\frown}{=} \mathbf{x} \leq \mathbf{x}^{\mathsf{i}} \qquad \Rightarrow \mathbf{x} = \mathbf{x}^{\mathsf{i}}.$$

Every preorder  $(X, \leq)$  gives rise to a category  $C(X, \leq)$ : The objects of  $C(X, \leq)$  are the elements of X and

$$Mor(x,y) = \begin{bmatrix} - \{(x,y)\} & \text{if } x \leq y \\ & \text{id} = (x,x), \\ & \emptyset & \text{otherwise}, \end{bmatrix}$$

and

$$(y, z) \circ (x, y) = (x, z)$$
.

1.1 LEMMA Let  $(X, \leq)$  be a preorder -- then every arrow in  $\underline{C}(X, \leq)$  is both a monomorphism and an epimorphism.

1.2 LEMMA Let  $(X, \leq)$  be a partial order — then the only isomorphisms in  $C(X, \leq)$  are the identities.

1.3 DEFINITION A poset is a set X equipped with a partial order.

If  $(X, \leq)$ ,  $(Y, \leq)$  are posets, then a functor  $f: \underline{C}(X, \leq) \rightarrow \underline{C}(Y, \leq)$  is simply a function  $f: X \rightarrow Y$  which is monotonic, i.e.,

$$x \leq x'$$
 in  $X \Rightarrow f(x) \leq f(x')$  in Y.

1.4 LEMMA Let  $(X, \leq)$ ,  $(Y, \leq)$  be posets and let

$$f:\underline{C}(X,\leq) \rightarrow \underline{C}(Y,\leq)$$
$$g:\underline{C}(Y,\leq) \rightarrow \underline{C}(X,\leq)$$

be functors — then f is a left adjoint for g if for all  $x \in X$  and  $y \in Y$ ,

$$f(x) \le y \le x \le g(y)$$
.

1.5 DEFINITION Suppose that  $(X, \leq)$  is a poset --- then  $(X, \leq)$  is a <u>lattice</u> if  $C(X, \leq)$  has binary products and binary coproducts, written

[Note: Accordingly,

$$\begin{vmatrix} \mathbf{x} & \mathbf{y} \leq \mathbf{x} \\ & \mathbf{k} \\ & \mathbf{x} & \mathbf{y} \leq \mathbf{y} \end{vmatrix} \begin{vmatrix} \mathbf{z} \leq \mathbf{x} \\ & \mathbf{z} \leq \mathbf{y} \end{vmatrix}$$

and

$$\begin{vmatrix} x \leq x \lor y \\ & & \\ & & & & \\ & & & \\ & & & & \\ & & & \\ & & & \\ & & & \\ & & & & \\ & & & \\ &$$

1.6 DEFINITION Suppose that  $(X, \leq)$  is a lattice -- then  $(X, \leq)$  is said to be <u>bounded</u> if  $C(X, \leq)$  admits a final object, denoted by 1, and an initial object, denoted by 0.

[Note: So, 
$$\forall x \in X$$
,  $0 \le x \le 1$  and  $\begin{bmatrix} x \land 1 = x \\ 0 \lor x = x \end{bmatrix}$ 

1.7 LEMMA Let  $(X, \leq)$  be a preorder -- then a commutative diagram



in  $C(X, \leq)$  is a pullback square iff w is a product of x and y or is a pushout square iff z is a coproduct of x and y.

1.8 RAPPEL Let  $\underline{C}$  be a category — then  $\underline{C}$  is finitely complete iff  $\underline{C}$  has pullbacks and a final object and  $\underline{C}$  is finitely cocomplete iff  $\underline{C}$  has pushouts and an initial object.

1.9 SCHOLIUM If  $(X, \leq)$  is a bounded lattice, then  $\underline{C}(X, \leq)$  is finitely complete and finitely cocomplete.

1.10 REMARK Suppose that  $(X, \leq)$  is a bounded lattice -- then  $\underline{C}(X, \leq)$  has products iff it has coproducts. Therefore  $C(X, \leq)$  is complete iff it is cocomplete.

Let  $(X, \leq)$  be a bounded lattice.

•  $(X, \leq)$  is distributive if  $\forall x, y, z \in X$ :

 $\begin{bmatrix} x \land (y \lor z) = (x \land y) \lor (x \land z) \\ x \lor (y \land z) = (x \lor y) \land (x \lor z). \end{bmatrix}$ 

•  $(X, \leq)$  is complemented if  $\forall x \in X, \exists - x \in X$ :

 $x \wedge - x = 0$  and  $x \vee - x = 1$ .

[Note: In a distributive lattice, a complement - x of x, if it exists, is unique.]

1.11 DEFINITION A <u>boolean algebra</u> is a bounded lattice  $(X, \leq)$  which is both distributive and complemented.

N.B. In a boolean algebra  $(X, \leq)$ ,  $\forall x \in X$ , -, -, x = x. [For

$$- - x + - x = 0$$

and complements are unique.]

1.12 LEMMA Let  $(X, \leq)$  be a boolean algebra — then  $\forall x, y \in X$ ,

$$- - - (x \lor y) = - x \lor - y$$
  
- (x \land y) = - x \lor - y.

[Note: These relations are called the laws of de Morgan.]

1.13 EXAMPLE If S is a set, then its power set PS is a boolean algebra.

#### § 2. SUBOBJECTS

Given a category <u>C</u> and an object X in <u>C</u>, let M(X) be the class of all pairs (Y, f), where  $f: Y \rightarrow X$  is a monomorphism -- then M(X) is the object class of a full subcategory <u>M</u>(X) of <u>C</u>/X.

Given (Y,f), (Z,g) in M(X), write (Y,f)  $\leq_X$  (Z,g) if there exists a morphism h:Y  $\rightarrow$  Z such that f = g  $\circ$  h, i.e., if there exists

$$h \in Mor_{C/X} (Y \longrightarrow X, Z \longrightarrow X).$$

[Note: h is necessarily unique and is itself a monomorphism.]

2.1 LEMMA The binary relation  $\leq_X$  is a preorder on M(X).

N.B. So, in the notation of §1,

$$\underline{M}(\mathbf{X}) = \underline{C}(\mathbf{M}(\mathbf{X}), \leq_{\mathbf{X}}).$$

2.2 DEFINITION Two elements (Y,f) and (Z,g) of M(X) are deemed equivalent, written (Y,f)  $\sim_X$  (Z,g), if there exists an isomorphism  $\phi: Y \to Z$  such that  $f = g \circ \phi$ .

2.3 LEMMA The binary relation  $\sim_{X}$  is an equivalence relation on M(X).

2.4 DEFINITION A subobject of X is an equivalence class of monomorphisms under  $\sim_{\rm X}.$ 

2.5 REMARK In practice, people tend to blur the distinction between a monomorphism  $f:Y \rightarrow X$  and its associated subobject, a potentially confusing abuse of the language.

Let  $\operatorname{Sub}_{\underline{C}} X$  stand for  $M(X)/{\sim}_{X'}$  let [] denote an equivalence class, and let [f]  $\leq_{\underline{X}}$  [g] have the obvious connotation — then the preorder on  $\operatorname{Sub}_{\underline{C}} X$  is a partial order. In fact,

$$\begin{array}{c} (Y,f) \leq_X (Z,g) \\ (Z,g) \leq_X (Y,f) \end{array}$$

imply that  $(Y,f) \sim_X (Z,g)$  or still, [f] = [g].

$$\operatorname{Sub}_{\underline{C}(X,\leq)} 1 \iff X.$$

2.7 EXAMPLE Let X be a topological space and take for  $\underline{C}$  the category  $\underline{Sh}(X)$  (the sheaves of sets on X) -- then

$$\operatorname{Sub}_{\underline{Sh}}(X) \xrightarrow{h}_{X} \longleftrightarrow \tau_{X}.$$

[Note:  $\tau_X$  is the topology on X and the correspondence  $\leftarrow$  assigns to  $U \in \tau_X$ ,

the sheaf  $h_U$ , where  $h_U V = \begin{bmatrix} -1 & \text{if } V \subset U \\ 0 & \text{if } V \neq U \end{bmatrix}$ 

2.8 DEFINITION A representative class of monomorphisms in M(X) is a subclass of M(X) which is a system of representatives for  $\sim_X$ .

2.9 EXAMPLE Suppose that <u>C</u> has an initial object  $\emptyset_{\underline{C}}$ . Let  $f: Y \to \emptyset_{\underline{C}}$  be an

element of  $M(p_{\underline{C}})$  — then f is an isomorphism, hence  $f_{\underline{C}} id_{p_{\underline{C}}}$ . Therefore

$$\operatorname{Sub}_{\underline{C}} \overset{\emptyset}{\underline{C}} = [\operatorname{id}_{\overset{\emptyset}{\underline{C}}}].$$

2.10 RAPPEL A category  $\underline{C}$  is said to be <u>wellpowered</u> provided that each of its objects has a representative class of monomorphisms that can be indexed by a set.

2.11 EXAMPLE Take  $\underline{C} = \underline{SET}$  and fix X -- then a subobject of X is an equivalence class of injective maps.

• Every subobject of X contains exactly one inclusion of a subset of X into X and that subset is the image of every element in the subobject.

• The subsets of X together with their inclusion maps form a representative set of monomorphisms in M(X).

[Note: Therefore SET is wellpowered.]

2.12 EXAMPLE TOP is wellpowered.

[Let  $(X, \tau_X)$  be a topological space -- then a representative set of monomorphisms in  $M(X, \tau_X)$  are the pairs  $((Y, \tau_Y), i_Y)$ , where Y is a subset of X,  $\tau_Y$  is a topology on Y finer than  $\tau_X | Y$ , and  $i_Y: Y \to X$  is the (continuous) inclusion.]

2.13 CRITERION If C is a small category and if <u>D</u> is a finitely complete, wellpowered category, then the functor category [C,D] is wellpowered.

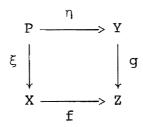
2.14 EXAMPLE If  $\underline{C}$  is a small category, then the presheaf category

 $\hat{\underline{C}} = [\underline{C}^{OP}, \underline{SET}]$ 

$$\underline{\text{SISET}} = [\underline{\triangle}^{\text{OP}}, \underline{\text{SET}}]$$

is wellpowered.

2.15 RAPPEL Consider a pullback square

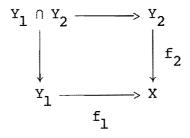


in a category C. Assume: f is a monomorphism — then  $\eta$  is a monomorphism.

2.16 DEFINITION Let C be a category with pullbacks. Given an object X in

 $\underline{C}, \text{ suppose that} \begin{vmatrix} f_1:Y_1 \neq X \\ f_2:Y_2 \neq X \end{vmatrix} \in M(X) - \text{ then their intersection is the pair}$ 

 $(Y_1 \cap Y_2, \triangle_{1,2}) \in M(X)$ , where  $Y_1 \cap Y_2$  is defined by the pullback square



and

$$\Delta_{1,2}:Y_1 \cap Y_2 \to X$$

is the corner arrow.

2.18 DEFINITION Let  $\underline{C}$  be a category. Given an object X in  $\underline{C}$ , suppose that  $\{(\underline{Y}_i, f_i) : i \in I\}$  is a set-indexed collection of elements of M(X) -- then an element  $(\underline{Y}, f) \in M(X)$  is called an <u>intersection</u> of the  $(\underline{Y}_i, f_i)$  provided that

$$\forall$$
 i, (Y,f)  $\leq_X$  (Y<sub>i</sub>,f<sub>i</sub>)

and for any object U  $\longrightarrow$  X in C/X such that

$$\forall i, \exists g_i \in Mor_{\underline{C}/X} (U \longrightarrow X, Y_i \longrightarrow X),$$

there exists a

$$g \in Mor_{C/X}$$
 (U  $\longrightarrow$  X, Y  $\longrightarrow$  X).

[Note: If  $I = \{1,2\}$ , then matters reduce to that of 2.16 (universal property of pullbacks).]

<u>N.B.</u> Intersections are unique up to isomorphism and the intersection of the empty collection of monomorphisms with codomain X is  $id_X: X \rightarrow X$ .

2.19 DEFINITION A category <u>C</u> is said to have <u>(finite) intersections</u> if for each  $X \in Ob \ C$  and any (finite) set-indexed collection of elements of M(X), there exists an intersection.

2.20 LEMMA If  $\underline{C}$  is a finitely complete category, then  $\underline{C}$  has finite intersections, and if  $\underline{C}$  is a complete category, then  $\underline{C}$  has intersections.

[Note: An intersection ("finite or infinite") is a multiple pullback and a multiple pullback is a limit.]

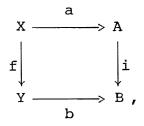
2.21 SCHOLIUM If <u>C</u> is wellpowered and (finitely) complete, then  $\forall X \in Ob \underline{C}$ , the category <u>C</u>(Sub<sub>C</sub> X,  $\leq_X$ ) associated with the poset (Sub<sub>C</sub> X,  $\leq_X$ ) has (finite) products.

#### **§3.** DECOMPOSITIONS

Let <u>C</u> be a category,  $f:X \rightarrow Y$  an epimorphism -- then there are various restrictions that can be imposed on f.

(1) f is a coequalizer, i.e.,  $\exists \ Z \in Ob \ \underline{C} \ and \ u,v \in Mor(Z,X)$  such that  $f = coeq(u,v) \ .$ 

(2) f has the left lifting property w.r.t. monomorphisms, i.e., every commutative diagram



where  $i:A \rightarrow B$  is a monomorphism, admits a filler  $w:Y \rightarrow A$  (thus  $w \circ f = a, i \circ w = b$ , and w is necessarily unique).

[Note: Epimorphisms with this property are closed under composition.]

(3) f is extremal, i.e., in any factorization  $f = h \circ g$ , if h is a monomorphism, then h is an isomorphism.

In general,

$$(1) \implies (2) \implies (3)$$

and none of the implications can be reversed.

3.1 LEMMA Suppose that <u>C</u> is finitely complete — then an epimorphism  $f:X \rightarrow Y$  satisfies (2) iff it satisfies (3).

3.2 EXAMPLE In CAT, there are extremal epimorphisms that are not coequalizers.

3.3 DEFINITION A finitely complete category <u>C</u> fulfills the <u>standard conditions</u> if <u>C</u> has coequalizers and the epimorphisms that are coequalizers are pullback stable.

3.4 EXAMPLE In <u>SET</u>, every epimorphism is a coequalizer and surjective functions are pullback stable. Therefore SET fulfills the standard conditions.

3.5 EXAMPLE In <u>TOP</u>, an epimorphism is extremal iff it is a quotient map, thus "(1) = (3)". Still, <u>TOP</u> does not fulfill the standard conditions since quotient maps are not pullback stable.

3.6 REMARK If <u>C</u> fulfills the standard conditions and if <u>I</u> is small, then the functor category [I,C] fulfills the standard conditions.

3.7 LEMMA Suppose that <u>C</u> fulfills the standard conditions -- then an epimorphism  $f:X \rightarrow Y$  satisfies (1) iff it satisfies (2).

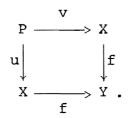
3.8 DEFINITION Let  $f:X \to Y$  be an arrow in a category  $\underline{C}$  --- then a <u>decomposition</u> of f is a pair of arrows  $X \xrightarrow{k} M \xrightarrow{m} Y$  such that  $f = m \circ k$ , where k is an epimorphism and m is a monomorphism. The decomposition (k,m) of f is said to be <u>minimal</u> (and M is said to be the <u>image</u> of f, denoted im f) if for any other factorization  $X \xrightarrow{\ell} N \xrightarrow{n} Y$  of f with n a monomorphism, there is an h:M + N such that  $h \circ k = \ell$  and  $n \circ h = m$  (=>  $(M,m) \leq_{Y} (N,n)$ ).

3.9 LEMMA Suppose that <u>C</u> fulfills the standard conditions -- then every morphism  $f:X \rightarrow Y$  in <u>C</u> admits a decomposition  $f = m \circ k$ , where k is an epimorphism

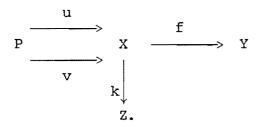
2.

satisfying "(1) = (2)" and m is a monomorphism.

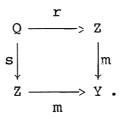
PROOF Form the pullback square



Then u and v are epimorphisms. Pass now to coeq(u,v):



Since  $f \circ u = f \circ v$ , there is a unique  $m: Z \rightarrow Y$  such that  $f = m \circ k$  and the claim is that m is a monomorphism. To see this, form the pullback square



Then

so there is a unique morphism  $q: P \rightarrow Q$  such that

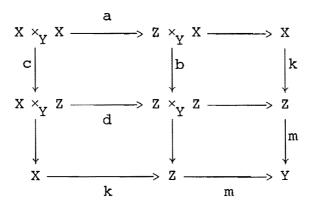
$$\mathbf{r} \circ \mathbf{q} = \mathbf{k} \circ \mathbf{u}, \mathbf{s} \circ \mathbf{q} = \mathbf{k} \circ \mathbf{v}.$$

But q is an epimorphism (cf. infra) and  $k \circ u = k \circ v$ , hence r = s which implies that m is a monomorphism.

[Note: From the definitions

$$P = X \times_{Y} X$$
$$Q = Z \times_{Y} Z$$

and there is a commutative diagram

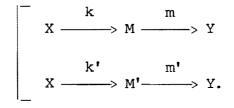


of pullback squares. Since <u>C</u> fulfills the standard conditions and k is a coequalizer, the arrows a,b,c,d are coequalizers as well. Therefore  $q = b \circ a = d \circ c$  is an epimorphism.

3.10 THEOREM Suppose that <u>C</u> fulfills the standard conditions -- then every morphism  $f:X \rightarrow Y$  in <u>C</u> admits a minimal decomposition  $f = m \circ k$  unique up to isomorphism.

<u>N.B.</u> The decomposition of f secured by 3.9 turns out to be minimal but there are two points of detail that will have to be addressed before this can be established.

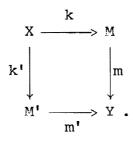
• Suppose given two decompositions of f per 3.9, hence m  $\circ$  k = m'  $\circ$  k', where



Then we claim that there exists an isomorphism  $\phi: M \to M'$  such that

$$\phi \circ k = k' \text{ and } m = m' \circ \phi.$$

Thus consider the commutative diagram



Then by the left lifting property w.r.t. monomorphisms.

$$\exists u: M \to M' \quad \text{st} \begin{vmatrix} & & u \circ k = k' \\ & & \\ & m' \circ u = m \end{vmatrix}$$

anđ

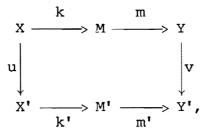
$$\exists u':M' \rightarrow M \quad \text{st} \begin{vmatrix} - & u' \circ k' = k \\ & m \circ u' = m'. \end{vmatrix}$$

Accordingly,

It remains only to take  $\phi = u$ .

[Note: This is what is meant by "unique up to isomorphism" in 3.10.]

• Suppose given a commutative diagram



where  $\begin{bmatrix} f = m \circ k \\ & are decompositions per 3.9 - then there exists a unique \\ f' = m' \circ k' \end{bmatrix}$ 

w:M  $\rightarrow$  M' such that  $\begin{bmatrix} w \circ k = k' \circ u \\ & & \\ m' \circ w = v \circ m \end{bmatrix}$  The uniqueness of w is, of course, clear.

As for the existence of w, use 3.9 again and write

$$\begin{bmatrix} k' \circ u = n \circ l \\ v \circ m = n' \circ l' \end{bmatrix}$$

say

$$X \xrightarrow{\ell' \circ k} N \xrightarrow{m' \circ n} Y'$$
$$X \xrightarrow{\ell' \circ k} N' \xrightarrow{n'} Y'.$$

Since

$$m' \circ k' \circ u = v \circ m \circ k$$

and since

$$(m' \circ n) \circ \ell = m' \circ k' \circ u$$
$$n' \circ (\ell' \circ k) = v \circ m \circ k ,$$

it follows from what has been said above that there exists an isomorphism  $\varphi:N \to N'$  such that

$$\begin{array}{c} & \phi \circ \ell = \ell' \circ k \\ & m' \circ n = n' \circ \phi. \end{array}$$

Now put

$$w = n \circ \phi^{-1} \circ \ell'.$$

Thén

$$w \circ k = n \circ \phi^{-1} \circ \ell' \circ k = n \circ \ell = k' \circ u$$
$$m' \circ w = m' \circ n \circ \phi^{-1} \circ \ell' = n' \circ \ell' = v \circ m,$$

as desired.

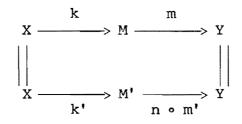
[Note:

$$(u,v) \in Mor_{\underline{C}}(\rightarrow) (f,f')$$

and

$$(\mathbf{u},\mathbf{w}) \in \operatorname{Mor}_{\underline{C}}(\mathbf{A}) (\mathbf{k},\mathbf{k'})$$
$$(\mathbf{w},\mathbf{v}) \in \operatorname{Mor}_{\underline{C}}(\mathbf{A}) (\mathbf{m},\mathbf{m'}).]$$

<u>Proof of 3.10</u> Write  $f = m \circ k$  per 3.9 -- then this decomposition is minimal. For suppose as in 3.8 that  $f = n \circ l$  and using 3.9 once more, write  $l = m' \circ k'$ . Thanks to the preceding discussion, the commutative diagram



gives rise to a unique  $w: M \rightarrow M'$  such that

$$w \circ k = k'$$
 and  $n \circ m' \circ w = m$ .

Put  $h = m' \circ w$  -- then  $h: M \rightarrow N$  and

$$\begin{array}{|c|c|c|c|c|} \hline & h \circ k = m' \circ w = m' \circ k = m' \circ k' = \ell \\ \hline & n \circ h = n \circ m' \circ w = m. \end{array}$$

[Note: Such an h is unique. For  $\begin{bmatrix} n \circ h = m \\ & => h = h', n \text{ being a} \\ & n \circ h' = m \end{bmatrix}$ 

monomorphism.]

3.11 DEFINITION Let <u>C</u> be a category. Given an object X in <u>C</u>, suppose that  $\{(Y_i, f_i) : i \in I\}$  is a set-indexed collection of elements of M(X) — then an element  $(Y, f) \in M(X)$  is called a <u>union</u> of the  $(Y_i, f_i)$  provided that

$$\forall$$
 i,  $(Y_i, f_i) \leq (Y, f)$ 

and for any element  $U \longrightarrow X$  of M(X) such that

$$\forall i, \exists g_i \in Mor_{\underline{C}/X} (Y_i \xrightarrow{f_i} X, U \xrightarrow{u} X),$$

there exists a

\_\_\_\_\_

$$g \in Mor_{\underline{C}/X} (Y \longrightarrow X, U \longrightarrow X).$$

[Note: The definition of union is not the exact analog of the definition of intersection (cf. 2.18).]

3.12 DEFINITION A category <u>C</u> is said to have <u>(finite) unions</u> if for each  $X \in Ob \ \underline{C}$  and any (finite) set-indexed collection of elements of M(X), there exists a union.

3.13 LEMMA Suppose that <u>C</u> fulfills the standard conditions and has finite coproducts — then <u>C</u> has finite unions.

PROOF Fix  $X \in Ob \subseteq$  and let  $\{(Y_i, f_i) : i \in I\}$  be a finite collection of objects of M(X)  $(I \neq \emptyset)$ . Denote by

$$\begin{bmatrix} in_{i}: Y_{i} \longrightarrow \coprod_{i \in I} Y_{i} \\ f: \coprod_{i \in I} Y_{i} \longrightarrow X \end{bmatrix}$$

the canonical arrows. Write  $f = m \circ k$  per 3.10, thus

$$\coprod_{\mathbf{i}\in\mathbf{I}} \overset{\mathbf{k}}{\xrightarrow{}} \mathbf{M} \xrightarrow{\mathbf{m}} \mathbf{X}.$$

Then (M,m) is a union of the  $(Y_i, f_i)$ . To begin with,  $k \circ in_i: Y_i \to M$  and

$$f_i = f \circ in_i = m \circ k \circ in_i \Longrightarrow (Y_i, f_i) \le (M, m).$$

Assume next that U  $\longrightarrow$  X is an element of M(X) and

$$\forall i, \exists g_i \in Mor_{\underline{C}/X} (Y_i \xrightarrow{t_i} X, U \xrightarrow{u} X),$$

so  $f_i = u \circ g_i$  -- then there exists a unique g:  $\coprod_{i \in I} Y_i \rightarrow U$  such that  $g \circ in_i = g_i$ . But

$$u \circ g \circ in_i = u \circ g_i = f_i = f \circ in_i$$
  
=>  $u \circ g = f$  (definition of coproduct)

Now display the data:

Since the decomposition  $f = m \circ k$  is minimal and since u is a monomorphism, there is an  $h:M \rightarrow U$  for which u  $\circ h = m$ , i.e.,

$$(M,m) \leq (U,u).$$

[Note: The union of the empty collection of monomorphisms with codomain X

is initial in M(X).]

<u>N.B.</u> The same argument works for an arbitrary index set so long as  $\underline{C}$  has coproducts.

3.14 SCHOLIUM If <u>C</u> is wellpowered, fulfills the standard conditions, and has (finite) coproducts, then the category  $\underline{C}(\operatorname{Sub}_{\underline{C}} X, \leq_X)$  associated with the poset ( $\operatorname{Sub}_{\underline{C}} X, \leq_X$ ) has (finite) coproducts.

#### §4. SLICES

Let  $\underline{C}$  be a category.

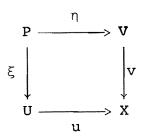
4.1 THEOREM If C is finitely complete, then so are the C/X.

4.2 REMARK It can happen that the C/X are finitely complete, yet <u>C</u> itself is not finitely complete.

[Take  $\underline{C} = \underline{TOP}_{\underline{LH}}$ , the category whose objects are the topological spaces and whose morphisms are the local homeomorphisms — then  $\underline{TOP}_{\underline{LH}}$  has pullbacks but does not have a final object, hence is not finitely complete (cf. 1.8). On the other hand, the  $\underline{TOP}_{\underline{LH}}$ /X are finitely complete.]

4.3 LEMMA If C has pullbacks, then the C/X have binary products.

PROOF Given objects U  $\longrightarrow$  X and V  $\longrightarrow$  X in C/X, form the pullback square



in <u>C</u> -- then the corner arrow  $P \rightarrow X$  is a product of  $U \longrightarrow X$  and  $V \longrightarrow X$  in C/X.

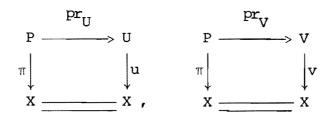
4.4 LEMMA If the C/X have binary products, then <u>C</u> has pullbacks.

PROOF Consider a 2-sink U 
$$\xrightarrow{u}$$
 X  $< \xrightarrow{v}$  V in C, thus  $\begin{bmatrix} -u \\ U \longrightarrow X \\ & \in Ob C/X. \\ V \longrightarrow X \end{bmatrix}$ 

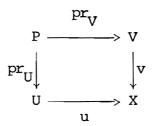
Let

$$P \xrightarrow{\pi} X = (U \xrightarrow{u} X) \times (V \xrightarrow{v} X).$$

Then there are commutative diagrams



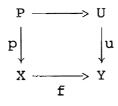
or still, a commutative diagram



which is a pullback square in C.

Let  $X, Y \in Ob \subseteq$  and let  $f: X \to Y$  be a morphism — then f induces a functor  $f_1: \underline{C}/X \to \underline{C}/Y$  via postcomposition.

4.5 LEMMA Suppose that <u>C</u> has pullbacks — then  $\forall$  f, f<sub>1</sub> has a right adjoint f\*. PROOF Given an object U  $\longrightarrow$  Y in C/Y, form the pullback square



and let

$$f^*(U \longrightarrow Y) = P \longrightarrow X.$$

Then this prescription defines a functor  $f^*:C/Y \to C/X$  and  $(f_{!}, f^*)$  is an adjoint pair.

And

$$(g \circ f)_{!} = g_{!} \circ f_{!}$$

but in general

Given  $X \in Ob \ \underline{C}$ , denote by  $i_X$  the inclusion  $\underline{M}(X) \rightarrow \underline{C}/X$ .

4.7 LEMMA Suppose that <u>C</u> fulfills the standard conditions -- then  $i_X$  has a left adjoint

$$\operatorname{im}_{X}: \underline{C}/X \to \underline{M}(X)$$
.

[Given U  $\longrightarrow$  X  $\in$  Ob C/X, write u = m  $\circ$  k per 3.10, so U  $\longrightarrow$  M  $\longrightarrow$  X.

Put

$$\operatorname{im}_{X}(U \longrightarrow X) = M \longrightarrow X.$$

If <u>C</u> has pullbacks and if  $f:X \to Y$  is a morphism, then  $f^*:\underline{C}/Y \to \underline{C}/X$  restricts to a functor  $f^{-1}:\underline{M}(Y) \to \underline{M}(X)$  (cf. 2.15). 4.8 LEMMA Suppose that <u>C</u> fulfills the standard conditions -- then  $f^{-1}$  has a left adjoint

$$\exists_{\mathbf{f}}: \underline{\mathbf{M}}(\mathbf{X}) \rightarrow \underline{\mathbf{M}}(\mathbf{Y}) .$$

[Take for  $\exists_f$  the composite

$$\underbrace{\overset{i_{X}}{\xrightarrow{f_{!}}} \xrightarrow{f_{!}} \underbrace{\underset{Y}{\xrightarrow{im_{Y}}}}_{\underline{Y} \longrightarrow} \underline{C}/\underline{Y} \xrightarrow{im_{Y}} \underline{M}(\underline{Y}).]$$

4.9 REMARK If <u>C</u> fulfills the standard conditions, then so do the <u>C</u>/X.

#### **§5.** CARTESIAN CLOSED CATEGORIES

Let  $\underline{C}$  be a category with finite products.

5.1 DEFINITION <u>C</u> is <u>cartesian closed</u> provided that each of the functors  $-- \times Y: \underline{C} \rightarrow \underline{C}$  has a right adjoint  $\underline{Z} \rightarrow \underline{Z}^{\underline{Y}}$ , so

Mor 
$$(X \times Y, Z) \approx Mor (X, Z^Y)$$
.

N.B. The property of being cartesian closed is invariant under equivalence.

5.2 EXAMPLE <u>SET</u> is cartesian closed but  $\underline{\text{SET}}^{OP}$  is not cartesian closed. The full subcategory of <u>SET</u> whose objects are finite is cartesian closed. On the other hand, the full subcategory of <u>SET</u> whose objects are at most countable is not cartesian closed.

5.3 EXAMPLE TOP is not cartesian closed but does have full, cartesian closed subcategories, e.g., the category of compactly generated Hausdorff spaces.

5.4 EXAMPLE CAT is cartesian closed:

 $Mor(\underline{C} \times \underline{D}, \underline{E}) \approx Mor(\underline{C}, \underline{E}),$ 

where  $\underline{\underline{D}} = [\underline{D}, \underline{E}]$ .

5.5 EXAMPLE Suppose that  $(X_{r\leq})$  is a boolean algebra. Put  $z^{Y} = --_{|} y \vee z$  ---

$$\mathbf{x} \wedge \mathbf{y} \leq \mathbf{z} \iff \mathbf{x} \leq \mathbf{z}^{\mathbf{y}}$$
.

E.g.: Given that  $x \land y \leq z$ , write

$$\mathbf{x} = \mathbf{x} \wedge \mathbf{1} = \mathbf{x} \wedge (--, \mathbf{y} \vee \mathbf{y})$$

$$= (x \land - y) \lor (x \land y)$$

$$\leq (x \land - y) \lor z$$

$$\leq - y \lor z = z^{y}.$$

Therefore

$$Mor(x \land y, z) \approx Mor(x, z^{Y}) \quad (cf. 1.4),$$

hence  $C(X, \leq)$  is cartesian closed.

Let  $\underline{C}$  be a cartesian closed category.

5.6 DEFINITION The object  $Z^Y$  is called an <u>exponential object</u>, the <u>evaluation</u> <u>morphism</u>  $ev_{Y,Z}$  being the arrow

$$z^Y \times Y \rightarrow z$$

with the property that for every f:X  $\times$  Y  $\rightarrow$  Z there is a unique g:X  $\rightarrow$  Z  $^{Y}$  such that

$$f = ev_{Y,Z} \circ (g \times id_Y).$$

One may view the association  $(Y,Z) \ {} \rightarrow \ Z^Y$  as a bifunctor, covariant in Z and contravariant in Y.

• The functor

$$(--)^{\Upsilon}:\underline{C} \rightarrow \underline{C}$$

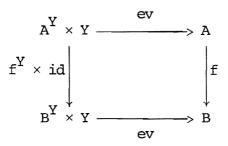
is defined on objects Z by

$$(--)^{Y}z = z^{Y}$$

and on morphisms A  $\xrightarrow{f}$  B by

$$(--)^{Y}(A \xrightarrow{f} B) = A^{Y} \xrightarrow{f^{Y}} B^{Y},$$

where f<sup>Y</sup> is the unique arrow rendering the diagram



commutative.

• The functor

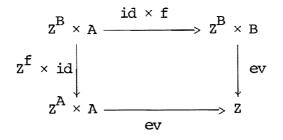
is defined on objects Y by

$$z^{(--)}y = z^{Y}$$

and on morphisms A  $\longrightarrow$  B by

 $z^{(--)} \xrightarrow{f} B = z^{B} \xrightarrow{z^{f}} z^{A}$ ,

where  $\boldsymbol{Z}^{\ensuremath{\boldsymbol{\Sigma}}}$  is the unique arrow rendering the diagram



commutative.

5.7 LEMMA The functor

$$z^{(--)}:\underline{C}^{OP} \rightarrow \underline{C}$$

admits a left adjoint, viz.

$$(z^{(--)})^{OP}:\underline{C} \rightarrow \underline{C}^{OP}.$$

<u>N.B.</u>  $(-)^{Y}$  preserves limits while  $z^{(--)}$  sends colimits to limits. 5.8 LEMMA In a cartesian closed category C,

(1) 
$$X^{Y \times Z} \approx (X^{Y})^{Z}$$
; (3)  $X^{\downarrow \downarrow} \stackrel{Y_{i}}{i} \approx \prod_{i} (X^{Y_{i}})$ ;

(2) 
$$(\prod_{i} x_{i})^{Y} \approx \prod_{i} (x_{i}^{Y});$$
 (4)  $X \times (\prod_{i} Y_{i}) \approx \prod_{i} (X \times Y_{i}).$ 

5.9 LEMMA In a cartesian closed category  $\underline{C}$ , finite products of epimorphisms are epimorphisms.

5.10 RAPPEL A full, isomorphism closed subcategory <u>D</u> of a category <u>C</u> is said to be a <u>reflective</u> subcategory of <u>C</u> if the inclusion  $1:D \rightarrow C$  has a left adjoint R, a <u>reflector</u> for <u>D</u>.

[Note: A reflective subcategory  $\underline{D}$  of a category  $\underline{C}$  is closed under the formation of limits in  $\underline{C}$ .]

Let  $\underline{D}$  be a reflective subcategory of a category  $\underline{C}$ ,  $\underline{R}$  a reflector for  $\underline{D}$  -then one may attach to each  $X \in Ob \underline{C}$  a morphism  $r_X: X \to RX$  in  $\underline{C}$  with the following property: Given any  $Y \in Ob \underline{D}$  and any morphism  $f: X \to Y$  in  $\underline{C}$ , there exists a unique morphism  $g: RX \to Y$  in  $\underline{D}$  such that  $f = g \circ r_X$ .

<u>N.B.</u> Matters can always be arranged in such a way as to ensure that  $R \circ \iota = id_D$ .

5.11 LEMMA Suppose that <u>C</u> is cartesian closed and let <u>D</u> be a reflective subcategory of <u>C</u>. Assume: The reflector  $R: C \rightarrow D$  preserves finite products -- then

D is cartesian closed.

[If  $Y, Z \in Ob \underline{D}$ , then  $Z^{Y}$  is isomorphic to an object in  $\underline{D}$ , hence  $Z^{Y} \in Ob \underline{D}$ .]

Let  $\underline{C}$  be cartesian closed -- then for any final object  $\underline{*}_{\underline{C}}$ , we have

$$(*_{\underline{C}})^{X} \approx *_{\underline{C}} \& X \overset{*_{\underline{C}}}{=} \approx X.$$

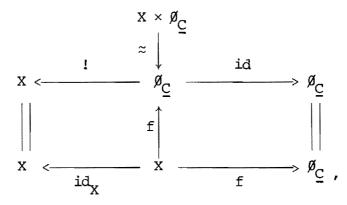
5.12 DEFINITION Let  $\underline{C}$  be a category with an initial object  $\emptyset_{\underline{C}}$  -- then  $\emptyset_{\underline{C}}$  is <u>strict</u> if every morphism f:X  $\rightarrow \emptyset_{\underline{C}}$  with codomain  $\emptyset_{\underline{C}}$  is an isomorphism.

[Note: Any morphism to an initial object is an epimorphism.]

5.13 LEMMA Let  $\underline{C}$  be a category with finite products and an initial object  $\emptyset_{\underline{C}}$  -- then  $\emptyset_{\underline{C}}$  is strict iff  $\forall X \in Ob \underline{C}$ ,

$$\mathbf{X} \times \mathbf{\emptyset}_{\underline{\mathbf{C}}} \approx \mathbf{\emptyset}_{\underline{\mathbf{C}}}$$

PROOF If  $\emptyset_{\underline{C}}$  is strict, then the projection  $X \times \emptyset_{\underline{C}} \to \emptyset_{\underline{C}}$  is an isomorphism. Conversely, let  $f:X \to \emptyset_{\underline{C}}$  be a morphism -- then there is a commutative diagram



from which it follows that f is a split monomorphism (!  $\circ f = id_X$ ). But f is

also an epimorphism. Therefore f is an isomorphism.

5.14 APPLICATION Let C be a cartesian closed category with an initial object  $\beta_{\rm C}$  — then  $\beta_{\rm C}$  is strict.

[The functor — × X preserves colimits, in particular initial objects, so  $\emptyset_{\underline{C}} \times X \approx \emptyset_{\underline{C}}$ . And

$$\emptyset_{\underline{\mathbf{C}}} \times \mathbf{X} \approx \mathbf{X} \times \emptyset_{\underline{\mathbf{C}}}.$$

5.15 EXAMPLE Under the preceding assumptions,

[Given  $A \in Ob C$ ,

$$Mor(\mathbf{A}, \mathbf{X}) \approx Mor(\mathbf{A} \times \mathbf{\emptyset}_{\underline{C}}, \mathbf{X})$$
$$\approx Mor(\mathbf{\emptyset}_{\underline{C}}, \mathbf{X}).$$

But there is a unique arrow  $\emptyset_{\underline{C}} \rightarrow X$ , so there is a unique arrow  $A \rightarrow X$  and this means that X is a final object.]

5.16 LEMMA Let <u>C</u> be a cartesian closed category with an initial object  $\emptyset_{\underline{C}}$  -then  $\forall X \in Ob \underline{C}$ , the canonical arrow  $\emptyset_{\underline{C}} \xrightarrow{!} X$  is a monomorphism, thus is an element of M(X).

PROOF Suppose that  $a, b: A \to \emptyset_{\underline{C}}$  are morphisms such that !  $\circ a = ! \circ b$ . Since A is initial ( $\emptyset_{\underline{C}}$  being strict), a = b, hence  $\emptyset_{\underline{C}} \longrightarrow X$  is a monomorphism.

5.17 EXAMPLE Under the preceding assumptions

$$\mathbf{I}^{X} \in \mathbf{M}(\star_{\underline{C}})$$
.

[The functor ( - )<sup>X</sup> preserves limits, in particular monomorphisms. Therefore

$$(\emptyset_{\underline{C}})^X \xrightarrow{!^X} (\star_{\underline{C}})^X$$

is a monomorphism. But

SO

$$\mathbf{I}^{X} \in \mathtt{M}(\star_{\underline{C}})$$
.]

[Note:  $M(*_{\underline{C}})$  is an <u>exponential ideal</u> in the sense that if  $Z \longrightarrow *_{\underline{C}}$  is a monomorphism, then  $\forall Y \in Ob \underline{C}, Z^{\underline{Y}} \longrightarrow *_{\underline{C}}$  is a monomorphism.]

5.18 RAPPEL An object in a category  $\underline{C}$  is called a <u>zero object</u> if it is both an initial object and a final object.

5.19 LEMMA Suppose that <u>C</u> is cartesian closed — then <u>C</u> has a zero object iff <u>C</u> is equivalent to <u>1</u>.

5.20 EXAMPLE Neither  $\underline{SET}_{\star}$  nor  $\underline{TOP}_{\star}$  is cartesian closed.

5.21 THEOREM Let <u>C</u> be a small category — then  $\hat{\underline{C}}$  is cartesian closed. PROOF Given F,G  $\in$  Ob  $\hat{\underline{C}}$ , define

$$G^{F}:\underline{C}^{OP} \rightarrow \underline{SET}$$

by the rule

$$G^{F}(X) = Nat(h_{X} \times F,G)$$
 (X  $\in$  Ob C).

5.22 EXAMPLE  $\hat{\Delta} = \underline{\text{SISET}}$  is cartesian closed:

$$Nat(X \times Y, Z) \approx Nat(X, Z^Y)$$
,

where

$$Z^{Y}([n]) = \operatorname{Nat}(\Delta[n] \times Y, Z) \quad (\Delta[n] = h_{[n]}).$$

5.23 DEFINITION A category <u>C</u> is <u>locally cartesian closed</u> if  $\forall X \in Ob \underline{C}$ , the category <u>C</u>/X is cartesian closed.

[Note: A locally cartesian closed category with a final object is cartesian closed.]

5.24 EXAMPLE SET is locally cartesian closed. Proof: SET/X is equivalent to  $\underline{\text{SET}}^{X}$ .

5.25 EXAMPLE CAT is cartesian closed but CAT is not locally cartesian closed.

5.26 EXAMPLE  $\underline{\text{TOP}}_{LH}$  is locally cartesian closed but  $\underline{\text{TOP}}_{LH}$  is not cartesian closed.

5.27 THEOREM Let  $\underline{C}$  be a small category — then  $\underline{\hat{C}}$  is locally cartesian closed. PROOF Given  $F \in Ob \ \underline{\hat{C}}$ , write C/F in place of  $\operatorname{gro}_{C} F$  — then the canonical arrow

$$\hat{C}/F \longrightarrow \hat{C}/F$$

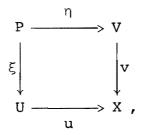
is an equivalence and C/F is cartesian closed (cf. 5.21).

5.28 THEOREM Let  $\underline{C}$  be a category with pullbacks. Assume:  $\forall$  f, f\* has a right adjoint f<sub>\*</sub> --- then  $\underline{C}$  is locally cartesian closed.

PROOF Thanks to 4.3, C/X has binary products. Since C/X also admits a final object (viz.  $id_X: X \rightarrow X$ ), it follows that C/X has finite products. This said, fix

objects  $\begin{vmatrix} - & u:U \rightarrow X \\ & & in C/X and realize u \times v as the corner arrow P \rightarrow X in the$  $v:V \rightarrow X \end{vmatrix}$ 

pullback square



thus

 $\mathbf{u} \times \mathbf{v} = \mathbf{u} \circ \boldsymbol{\xi} = \mathbf{v} \circ \boldsymbol{\eta} = \mathbf{v}_{!} \mathbf{v}^{*} \mathbf{u}.$ 

Then for any  $f: Y \rightarrow X$ , we have

$$Mor(u \times v, f) = Mor(v_{!}v^{*}u, f)$$
$$\approx Mor(v^{*}u, v^{*}f)$$
$$\approx Mor(u, v_{*}v^{*}f).$$

Definition:

 $f^{V} = v_{\star}v^{\star}f.$ 

Suppose that  $\underline{C}$  is finitely complete. Given  $X \in Ob \ \underline{C}$ , denote by

 $X_1:C/X \rightarrow C$ 

the forgetful functor and by

 $X*:C \rightarrow C/X$ 

the functor that sends Y to  $X \times Y \rightarrow X$ .

5.29 CRITERION The functor —  $\times$  X has a right adjoint iff the functor X\* has a right adjoint.

5.30 LEMMA If <u>C</u> is locally cartesian closed, then  $\forall X \in Ob C$ , the category C/X is locally cartesian closed.

PROOF For every object  $A \rightarrow X$  of C/X,

$$\underline{C}/X/A \rightarrow X \approx \underline{C}/A.$$

5.31 LEMMA If <u>C</u> is locally cartesian closed, then  $\forall X \in Ob \underline{C}$ , the category C/X is finitely complete.

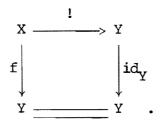
PROOF Since the C/X are cartesian closed, they have products, in particular binary products, hence C has pullbacks (cf. 4.4). So  $\forall X \in Ob C$ , C/X has pullbacks (pullbacks in C/X are computed as in C (cf. 4.1)). But C/X has a final object, thus C/X is finitely complete (cf. 1.8).

5.32 LEMMA If <u>C</u> is locally cartesian closed, then  $\forall$  f, f<sub>1</sub> has a right adjoint f\*.

[Because, as noted above, C has pullbacks.]

5.33 THEOREM If C is locally cartesian closed, then  $\forall$  f, f\* has a right adjoint f<sub>\*</sub>.

[A morphism  $f:X \rightarrow Y$  is an object of C/Y and



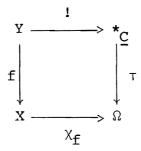
Therefore 5.29 is applicable.]

 $\underline{\text{N.B.}}$  f\* preserves exponential objects.

## **§6.** SUBOBJECT CLASSIFIERS

Let  $\underline{C}$  be a finitely complete category.

6.1 DEFINITION A subobject classifier for <u>C</u> is a pair  $(\Omega, \tau)$ , where  $\tau: *_{\underline{C}} \rightarrow \Omega$ is a monomorphism with the property that for each object X in <u>C</u> and every monomorphism f:Y  $\rightarrow$  X there exists a unique morphism  $\chi_{f}: X \rightarrow \Omega$  such that the diagram



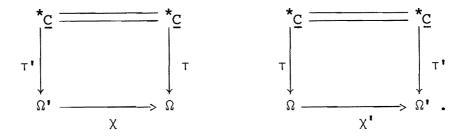
is a pullback square.

[Note: The morphism  $\chi_f: X \to \Omega$  is called the <u>classifying arrow</u> of (Y,f) in X.]

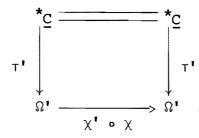
6.2 EXAMPLE  $\operatorname{id}_{\Omega}$  is the classifying arrow of  $(*_{C'}^{\mathsf{T}})$  in  $\Omega$ .

6.3 LEMMA If  $(\Omega, \tau)$  and  $(\Omega', \tau')$  are subobject classifiers, then  $\Omega$  and  $\Omega'$  are isomorphic.

PROOF From the definitions, there are pullback squares

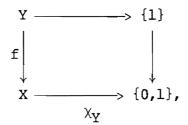


Therefore  $\chi' \circ \chi$  is the classifying arrow of  $(*_{\underline{C}'}^{\top})$  in  $\Omega'$ :



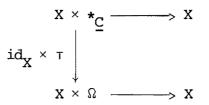
So, by uniqueness,  $\chi' \circ \chi = id$ . And, analogously,  $\chi \circ \chi' = id_{\Omega}$ .

6.4 EXAMPLE Take  $\underline{C} = \underline{SET}$ , let  $*\underline{C} = \{1\}$ ,  $\Omega = \{0,1\}$ , and define  $\top : *\underline{C} \to \Omega$  by sending 1 to 1. Given X, if Y is a subset of X and if  $f:Y \to X$  is the inclusion, then there is a pullback square



where  $\chi_{\gamma}$  is the characteristic function of Y.

6.5 LEMMA Let  $(\Omega, \tau)$  be a subobject classifier -- then  $\forall X \in Ob C$ ,



is a subobject classifier in C/X.

[Note: Recall that C/X is finitely complete (cf. 4.1).]

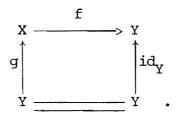
6.6 RAPPEL A category C is balanced if every morphism that is simultaneously

a monomorphism and an epimorphism is an isomorphism.

6.7 EXAMPLE SET is balanced but TOP is not balanced.

6.8 LEMMA Let <u>C</u> be a category and let  $f:X \rightarrow Y$  be a morphism. Assume: f is an equalizer and an epimorphism -- then f is an isomorphism.

PROOF Suppose that f = eq(u,v), hence  $u \circ f = v \circ f$ , so u = v (f being an epimorphism). But the equalizer of u = v is  $id_{Y}$ , hence there is a unique arrow  $g:Y \rightarrow X$  such that  $f \circ g = id_{Y}$ :



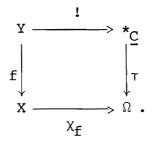
And then

$$f \circ g \circ f = id_{Y} \circ f = f = f \circ id_{X}$$
$$\Longrightarrow$$
$$g \circ f = id_{X}.$$

Therefore f is an isomorphism.

6.9 LEMMA If <u>C</u> admits a subobject classifier  $(\Omega, \tau)$ , then every monomorphism f:Y  $\Rightarrow$  X is an equalizer.

PROOF Consider the pullback square



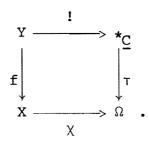
Then  $\tau$  is a split monomorphism, hence the same is true of f. And a split monomorphism is an equalizer.

6.10 SCHOLIUM A category with a subobject classifier is balanced.

Assume: C admits a subobject classifier  $(\Omega, \tau)$ .

6.11 LEMMA Let (Y,f), (Z,g) be elements of M(X) -- then (Y,f)  $\sim_X$  (Z,g) iff  $\chi_f$  =  $\chi_q$  -

6.12 LEMMA Given  $\chi \in Mor(X, \Omega)$ , form the pullback square

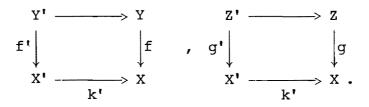


Then  $\chi_f = \chi$ .

6.13 THEOREM The map [f]  $\rightarrow \chi_f$  is a bijection between the class  $\operatorname{Sub}_{\underline{C}} X$  of subobjects of X and the set Mor(X,  $\Omega$ ).

[Note: Therefore  $\text{Sub}_{\underline{C}} X$  "is a set", i.e., has a representative class of monomorphisms which is a set, thus  $\underline{C}$  is wellpowered.]

Consider pullback squares



6.14 LEMMA If (Y,f) 
$$\sim_X$$
 (Z,g), then (Y',f')  $\sim_X$  (Z',g').

Therefore not only is a pullback of a monomorphism a monomorphism but a pullback of a subobject is a subobject.

Denote by  $Sub_{\underline{C}}$  the association  $\underline{C}^{OP}\to \underline{SET}$  that sends X to  $Sub_{\underline{C}}\ X$  and  $k':X'\to X$  to  $Sub_{\underline{C}}\ k'$ , where

$$\operatorname{Sub}_{\underline{C}} k': \operatorname{Sub}_{\underline{C}} X \to \operatorname{Sub}_{\underline{C}} X'$$

is the arrow  $[f] \rightarrow [f']$ .

6.15 LEMMA  $Sub_{C}$  is a functor.

PROOF It is clear that  $Sub_{\underline{C}}$  sends the identity of X to the identity of  $Sub_{\underline{C}} X$ . As for compositions, if

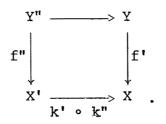
$$\begin{bmatrix} k': X' \to X \\ k'': X'' \to X', \end{bmatrix}$$

then the claim is that

$$\operatorname{Sub}_{\underline{C}} (k' \circ k'') = \operatorname{Sub}_{\underline{C}} k'' \circ \operatorname{Sub}_{\underline{C}} k'.$$

To see this, pass from the pullback squares

to the pullback square



6.16 THEOREM The presheaf  ${\rm Sub}_{\mathbb C}$  is represented by  $\Omega \colon \forall \ X \in {\rm Ob} \ \underline{C}$  ,

$$\operatorname{Sub}_{\underline{C}} X \approx \operatorname{Mor}(X, \Omega).$$

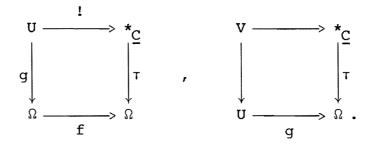
[Note: The natural isomorphism

$$\operatorname{Sub}_{\underline{C}} \neq \operatorname{Mor}(--,\Omega)$$

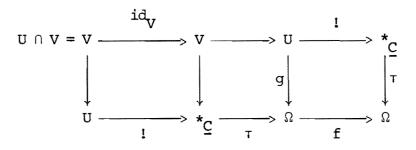
sends a subobject [f] of X to its classifying arrow  $\chi_{\mathbf{f}}.]$ 

6.17 LEMMA Every monomorphism  $f: \Omega \rightarrow \Omega$  is an isomorphism.

PROOF It suffices to show that f  $\circ$  f = id<sub> $\Omega$ </sub>. Form the pullback squares



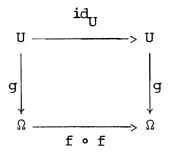
Since f is a monomorphism, the arrow  $U \longrightarrow *_{\underline{C}}$  is a monomorphism and since g is a monomorphism, the arrow  $V \longrightarrow *_{\underline{C}}$  is a monomorphism, thus the squares in the diagram



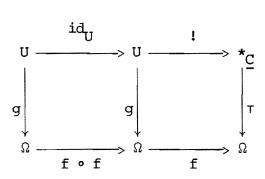
are pullback squares, so by uniqueness, f  $\circ \tau \circ ! = g$ , which implies that

$$f \circ f \circ g = f \circ \tau \circ ! = g = g \circ id_U$$

or still, that the square



commutes. Working through the definitions and bearing in mind that  $f \circ f$  is a monomorphism, it follows that this square is in fact a pullback square. Therefore the outer rectangle



is a pullback square, hence by uniqueness,

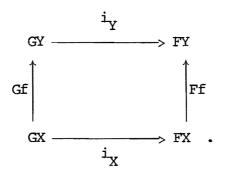
$$f \circ f \circ f = f = f \circ id_{\Omega} \Longrightarrow f \circ f = id_{\Omega}$$

## §7. SIEVES

Let <u>C</u> be a small category.

7.1 DEFINITION Let  $X \in Ob \subseteq --$  then a sieve over X is a subset \$\$ of Ob  $\subseteq X$ such that the composition  $Z \xrightarrow{g} Y \xrightarrow{f} X$  belongs to \$\$ if  $Y \xrightarrow{f} X$  belongs to \$\$.

7.2 DEFINITION A <u>subfunctor</u> of a functor  $F:\underline{C}^{OP} \rightarrow \underline{SET}$  is a functor  $G:\underline{C}^{OP} \rightarrow \underline{SET}$ such that  $\forall X \in Ob \underline{C}$ , GX is a subset of FX and the corresponding inclusions constitute a natural transformation  $G \rightarrow F$ , so  $\forall f:Y \rightarrow X$  there is a commutative diagram



7.3 LEMMA Fix an object X in C -- then there is a one-to-one correspondence between the sieves over X and the subfunctors of  $h_X$ .

PROOF If § is a sieve over X, then the designation

$$GY = \{f: Y \rightarrow X \& f \in \$\}$$

defines a subfunctor of  $h_X$  (given  $Z \xrightarrow{g} Y$ ,  $Gg:GY \rightarrow GZ$  is the map  $f \rightarrow f \circ g$ ). Conversely, if G is a subfunctor of  $h_X$ , then  $GY \subset Mor(Y,X)$  and

is a sieve over X.

7.4 EXAMPLE The <u>maximal sieve</u> over X is  $\$_{\max} = Ob C/X$  and the associated subfunctor of  $h_X$  is  $h_X$  itself. The <u>minimal sieve</u> over X is  $\$_{\min} = \emptyset$  and the associated subfunctor of  $h_X$  is  $\emptyset_{\hat{C}}$  (the initial object of  $\hat{C}$ ).

Consider now the functor category

$$\hat{\underline{C}} = [\underline{C}^{OP}, \underline{SET}].$$

<u>N.B.</u>  $\hat{\underline{C}}$  is wellpowered (cf. 2.14).

7.5 LEMMA The monomorphisms in  $\hat{\underline{C}}$  are levelwise, i.e., an arrow  $\Xi: G \rightarrow F$  in  $\hat{\underline{C}}$  is a monomorphism iff  $\forall X \in Ob \underline{C}$ ,

$$\Xi_X$$
:GX  $\rightarrow$  FX

is a monomorphism in SET.

Suppose that  $E:G \Rightarrow F$  is a monomorphism in  $\hat{\underline{C}}$  — then  $(G,E) \in M(F)$ , so  $\forall X \in Ob \underline{C}$ ,

$$(GX, \Xi_X) \in M(FX)$$

and

$$(GX, \Xi_X) \sim_{FX} (G'X, \Xi_X'),$$

where G'X is a subset of FX and  $E_X^*$  is the inclusion G'X  $\Rightarrow$  FX.

7.6 LEMMA G' is a subfunctor of F.

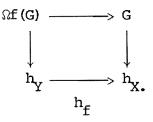
It follows that there is a one-to-one correspondence between the subobjects of F and the subfunctors of F.

Definition of  $\Omega$  There are two ways to proceed.

• Define

$$\Omega:\underline{C}^{OP} \rightarrow \underline{SET}$$

on an object X by letting  $\Omega X$  be the set of all subfunctors of  $h_X$  and on a morphism f:Y  $\rightarrow$  X by letting  $\Omega f: \Omega X \rightarrow \Omega Y$  operate via the pullback square



• Define

$$\Omega:\underline{\mathbf{C}}^{\mathsf{OP}} \to \underline{\mathsf{SET}}$$

on an object X by letting  $\Omega X$  be the set of all sieves over X and on a morphism f:Y  $\rightarrow$  X by letting  $\Omega f:\Omega X \rightarrow \Omega Y$  be the rule  $\mathfrak{F} \rightarrow \mathfrak{F} \cdot f$ , where  $\mathfrak{F} \cdot f = \{g: f \circ g \in \mathfrak{F}\}.$ 

Definition of  $\tau: *_{\hat{C}} \rightarrow \Omega$  In terms of subfunctors,  $\tau_X(*) = h_X$  and in terms of sieves,  $\tau_X(*) = \$_{max}$ .

The claim then is that the pair  $(\Omega, \tau)$  is a subobject classifier for  $\hat{\underline{C}}$  and for this we shall work with sieves, the details in the subfunctor picture being analogous. So let  $\Xi: G \to F$  be a monomorphism, where w.l.o.g., G is a subfunctor of F -- then the classifying arrow  $\chi_{\underline{\Xi}}: F \to \Omega$  of (G, $\Xi$ ) in F at a given  $X \in Ob \underline{C}$ is the map

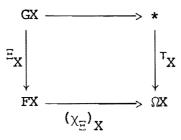
$$(\chi_{\equiv})_{X}$$
:FX  $\rightarrow \Omega X$ 

$$(\chi_{\Xi})_{X}(x) = \{Y \longrightarrow X: (Ff)x \in GY\}.$$

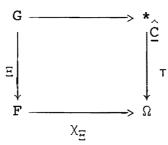
Since

$$(\chi_{\Xi})_{X}(x) = \mathfrak{F}_{max} \iff x \in GX,$$

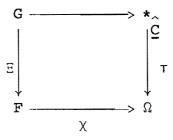
the diagram



is a pullback square in SET, thus the diagram



is a pullback square in  $\hat{\underline{C}}.$  This completes the verification, modulo uniqueness, i.e., if



is a pullback square, then  $\chi$  =  $\chi_{\Xi}$  ... .

7.8 EXAMPLE Let G be a group, considered as a category <u>G</u> -- then the category of right G-sets is the functor category [ $\underline{G}^{OP}$ , <u>SET</u>], thus is cartesian closed (cf. 5.21) and admits a subobject classifier (cf. 7.7).

Let  $\underline{C}$  be a small category -- then

- $\hat{\underline{C}}$  fulfills the standard conditions (cf. 3.4 and 3.6);
- $\hat{\underline{C}}$  admits a subobject classifier (cf. 7.7).

7.9 LEMMA Every epimorphism in  $\hat{\underline{C}}$  is a coequalizer.

PROOF Suppose that  $E:F \rightarrow G$  is an epimorphism. Write  $E = m \circ k$  per 3.9, thus m is a monomorphism and k is a coequalizer. But then m is necessarily an epimorphism and  $\hat{C}$  is balanced (cf. 6.10). Therefore m is an isomorphism, hence E is a coequalizer.

#### **§8.** HEYTING ALGEBRAS

A bounded lattice  $(X, \leq)$  is called a <u>Heyting algebra</u> if  $C(X, \leq)$  is cartesian closed (as a category with finite products).

N.B. If  $x, y, z \in X$ , then

$$x \land y \le z \iff x \le z^{Y}$$
 (cf. 1.4).

So, e.g.,

$$y \leq z \ll z^{y} = 1.$$

In particular:  $\forall x \in X, x^{X} = 1$ . And

$$z^{Y} \wedge y \leq z$$
.

In particular:  $\forall x \in X, x \land 0^X = 0.$ 

8.1 EXAMPLE Every boolean algebra is a Heyting algebra (cf. 5.5).

8.2 LEMMA Let  $(X, \leq)$  be a poset which is linearly ordered ( $\forall x, y \in X$ , either  $x \leq y$  or  $y \leq x$ ) and with least and greatest elements 0 and 1 -- then  $(X, \leq)$  is a bounded lattice and, as such, is a Heyting algebra.

PROOF  $C(X, \leq)$  has binary products:

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} - & \mathbf{x} & \mathbf{if} & \mathbf{x} \leq \mathbf{y} \\ & & \\ - & \mathbf{y} & \mathbf{if} & \mathbf{y} \leq \mathbf{x} \end{vmatrix}$$

and binary coproducts:

$$\mathbf{x} \lor \mathbf{y} = \begin{vmatrix} -\mathbf{y} & \text{if } \mathbf{x} \le \mathbf{y} \\ -\mathbf{x} & \text{if } \mathbf{y} \le \mathbf{x} \end{vmatrix}$$

This said, the prescription

$$y^{\mathbf{X}} = \begin{bmatrix} -1 & \text{if } \mathbf{x} \leq \mathbf{y} \\ y & \text{if } \mathbf{y} \leq \mathbf{x} & \text{if } \mathbf{y} \neq \mathbf{x} \end{bmatrix}$$

defines an exponential object, so  $C(X, \leq)$  is cartesian closed.

8.3 EXAMPLE The closed unit interval  $[0,1] \subset \underline{R}$  in its usual ordering is a Heyting algebra (but not a boolean algebra).

8.4 LEMMA A Heyting algebra is necessarily a distributive lattice.

The difference between a boolean algebra and a Heyting algebra lies in the notion of complement.

8.5 DEFINITION Let  $(X, \le)$  be a Heyting algebra. Given  $x \in X$ , put —  $x = 0^{X}$  — then — x is called the <u>pseudocomplement</u> of x.

N.B. In a boolean algebra  $(X, \leq)$ ,

$$0^{x} = - x \vee 0 = - x$$
 (cf. 5.5).

8.6 LEMMA Let  $(X, \leq)$  be a Heyting algebra -- then  $\forall x \in X$ ,

$$--_{1} x = v \{y: x \land y = 0\}.$$

8.7 EXAMPLE Let S be an infinite set and let X be the subset of the power set PS consisting of all finite subsets of S together with S itself — then  $(X, \underline{c})$  is a distributive lattice but it is not a Heyting algebra.

[If  $x \in X$  and  $x \neq \emptyset$ , then the set of  $y \in S$  such that  $x \cap y = \emptyset$  has no largest member.]

To recapitulate:

boolean algebra => Heyting algebra => distributive lattice

and none of the implications are reversible.

8.8 RULES In a Heyting algebra  $(X, \leq)$ ,

$$(1) - 1 = 1, -1 = 0; \qquad (6) \ x \le y = > -1 - 1 \ x \le -1 - 1 \ y;$$

$$(2) \ x \le y = > -1 \ y \le -1 \ x; \qquad (7) \ x \le -1 - 1 \ x;$$

$$(3) - 1 \ x = -1 - 1 \ x; \qquad (8) - 1 - 1 \ x = -1 - 1 \ x;$$

$$(4) - 1 \ (x \lor y) = -1 \ x \land -1 \ y; \qquad (9) - 1 - 1 \ (x \land y) = -1 \ x \land -1 \ y;$$

$$(5) - 1 \ x \lor y \le y^{x}; \qquad (10) - 1 - 1 \ (y^{x}) = (-1 - 1 \ y)^{-1} \ x.$$

[Note: This list is by no means exhaustive but suffices for our purposes (there is another list to the effect that any Heyting algebra satisfies the axioms of the intuitionistic propositional calculus).]

8.9 LEMMA Let  $(X, \leq)$  be a Heyting algebra — then  $(X, \leq)$  is a boolean algebra iff  $\forall x \in X, x \lor - x = 1$ .

[Note: In any Heyting algebra,  $x \land -- x = 0$ .]

8.10 LEMMA Let  $(X, \le)$  be a Heyting algebra — then  $(X, \le)$  is a boolean algebra iff  $\forall x \in X$ , —  $(X, \le)$  x = x.

8.11 EXAMPLE Given a topological space X, let O(X) be the set of open subsets of X, thus under the operations

$$U \leq V \iff U \subset V,$$

$$U \leq V \iff U \subset V,$$

$$U \leq V \ll U \cap V$$

$$U \leq V \leq U \cup V$$

O(X) is a bounded lattice. Denote by O(X) the category underlying O(X) -- then

O(X) is cartesian closed:

$$v^{U} = \cup W (W \cap U \subset V)$$
.

Therefore O(X) is a Heyting algebra. Here

$$- U = \beta^{U} = int(X - U) = X - cl U$$

$$=>$$

$$- U = int cl U > U.$$

[Note: In general, O(X) is not a boolean algebra (cf. 8.9 and 8.10).]

8.12 DEFINITION Let  $(X, \le)$  be a Heyting algebra — then an  $x \in X$  is boolean if -x = x.

[Note: It is always the case that  $x \leq -$  , - , x.]

8.13 EXAMPLE In 8.11, an open set U is boolean iff it coincides with the interior of its closure.

8.14 NOTATION  $(X_{b'} \leq)$  is the subposet of  $(X, \leq)$  whose elements are the boolean elements of X.

8.15 THEOREM  $(X_{b'} \leq)$  is a boolean algebra.

PROOF First,

$$---_{1} ---_{1} 0 = 0$$
$$---_{1} ---_{1} 1 = 1,$$

so 0 and 1 are boolean. Next, if  $x, y \in X$  are boolean, then

 $-- | -| (x \land y) = -- | -| x \land -- | -- | y = x \land y,$ 

thus  $x \wedge y$  is boolean. On the other hand,  $x \vee y$  is not necessarily boolean. To remedy this, put

$$x \underline{\vee} y = - (x \vee y).$$

Then

$$-- | -| (x \vee y) = -- | -| -| (x \vee y)$$
$$= -- | (x \vee y) = x \vee y.$$

So, with these definitions,  $(X_b, \leq)$  is a bounded lattice (which, in general, is not a sublattice of  $(X, \leq)$ ). There remains the claim that  $(X_b, \leq)$  is distributive and complemented.

• 
$$\forall x, y, z \in X_{b}$$
:  
 $x \land (y \lor z) = x \land ---_{1} ---_{1} (y \lor z)$   
 $= --_{1} ---_{1} x \land ---_{1} ---_{1} (y \lor z)$   
 $= --_{1} ---_{1} (x \land (y \lor z))$   
 $= --_{1} ---_{1} ((x \land y) \lor (x \land z))$   
 $= (x \land y) \lor (x \land z).$ 

Analogously,

$$x \underline{\vee} (y \wedge z) = (x \underline{\vee} y) \wedge (x \underline{\vee} z).$$

•  $\forall x \in x_b$ :

$$\mathbf{x} \wedge - \mathbf{x} = \mathbf{0}$$

and

$$x \ \underline{v} \ -1 \ x = -1 \ -1 \ (x \ v \ -1 \ x)$$
  
= --1 (x \ v \ -1 \ x)

$$= - \frac{1}{1} (- \frac{1}{1} \times \sqrt{-1} - \frac{1}{1} \times \frac{1}{1})$$
$$= - \frac{1}{1} (- \frac{1}{1} \times \sqrt{-1})$$
$$= - \frac{1}{1} 0$$

8.16 THEOREM Let <u>C</u> be a small category — then  $\forall F \in Ob \stackrel{\circ}{\underline{C}}$ , the poset  $Sub_{\stackrel{\circ}{\underline{C}}} F \stackrel{\circ}{\underline{C}}$  is a Heyting algebra.

PROOF Suppose that  $G_1, G_2$  are subfunctors of F — then under the operations

$$(G_1 \land G_2) X = G_1 X \cap G_2 X$$
  
,  $0X = \emptyset$ ,  $1X = FX$   
$$(G_1 \lor G_2) X = G_1 X \cup G_2 X$$

 $\operatorname{Sub}_{\hat{C}} F$  is a bounded lattice. As for the exponential object  $\operatorname{G}_{2}^{G_{1}}$ , take  $(\operatorname{G}_{1}^{G_{2}})X$  to  $\hat{C}$  be the set of  $x \in FX$  which have the property that if  $f:Y \to X$  and if  $(Ff)x \in \operatorname{G}_{1}Y$ , then  $(Ff)x \in \operatorname{G}_{2}Y$ .

[Note: So, if G is a subfunctor of F, then (--, G)X is the set of  $x \in FX$  such that for all  $f:Y \to X$ ,  $(Ff)x \notin GY$ .]

8.17 EXAMPLE Consider the functor category  $[\underline{G}^{OP}, \underline{SET}]$  per 7.8 — then for every right G-set X, the Heyting algebra Sub, X is actually a boolean algebra.  $\underline{G}$ 

# §9. LOCALES

A <u>locale</u> is a Heyting algebra  $(X, \leq)$  for which the category  $\underline{C}(X, \leq)$  is complete and cocomplete (cf. 1.10).

[Note: If  $\underline{C}(X, \leq)$  is complete and cocomplete, then  $\underline{C}(X_b, \leq)$  is complete and cocomplete, hence the boolean algebra  $(X_b, \leq)$  (cf. 8.15) is also a locale.]

9.1 EXAMPLE The closed unit interval  $[0,1] \subset \underline{R}$  in its usual ordering is a locale (cf. 8.3).

9.2 EXAMPLE If X is a topological space, then O(X) is a locale (cf. 8.11). [Here  $\lor U_i = \bigcup U_i$  while  $\land U_i$  is the largest open set contained in all  $i \in I$   $i \in I$   $i \in I$   $i \in I$   $i \in I$  the  $U_i$ .]

9.3 EXAMPLE If <u>C</u> is a small category and if  $F \in Ob \stackrel{\circ}{C}$ , then Sub<sub> $\hat{C}$ </sub> F is a locale (cf. 8.16).

9.4 LEMMA Suppose that  $(X, \leq)$  is a locale — then for any index set I,

$$x \land (v y_i) = v (x \land y_i).$$
  
 $i \in I \quad i \in I$ 

[Recall that left adjoints preserve colimits.]

[Note: If  $(X, \leq)$  is a bounded lattice for which the category  $\underline{C}(X, \leq)$  is complete and cocomplete (cf. 1.10) and with the property that "arbitrary joins distribute over finite meets", i.e., the conclusion of 9.4, then  $(X, \leq)$  is a Heyting algebra or still, is a locale. Proof: Put

$$z^{Y} = v\{x:x \land y \leq z\}.$$

Generically, locales are denoted by L,M, ... and are to be regarded as categories.

9.5 LEMMA Let L be a locale. Given  $x \in L$ , put

$$\uparrow \mathbf{x} = \{ \mathbf{y} \in L : \mathbf{x} \le \mathbf{y} \}$$
$$\downarrow \mathbf{x} = \{ \mathbf{y} \in L : \mathbf{y} \le \mathbf{x} \}.$$

Then the subposets  $\begin{vmatrix} - & \uparrow x \\ & & \text{are locales.} \\ & \downarrow x \end{vmatrix}$ 

9.6 DEFINITION Let L, M be locales --- then a <u>localic arrow</u> f:  $L \rightarrow M$  is a pair of functors

$$f_*:L \rightarrow M$$

$$f^*:M \rightarrow L$$

such that  $f^*$  is a left adjoint for  $f_*$  and  $f^*$  preserves finite products.

9.7 REMARK There is a one-to-one correspondence between the localic arrows  $f:L \rightarrow M$  and the functors  $f^*:M \rightarrow L$  such that

(1)  $f^*(v \ y_i) = v \ f^*(y_i),$   $i \in I$   $i \in I$ (2)  $f^*(y \land y') = f^*(y) \land f^*(y'),$ (3)  $f^*(1) = 1,$ 

for all indexing sets I and elements  $y_{i'}y_{i'}y_{i'}y_{i'}$  of M.

[If f\* satisfies these conditions, then by quoting the appropriate "adjoint functor theorem" one infers the existence of  $f_*$  ( $f_*$  is uniquely determined by f

(in a poset, the only isomorphisms are the identities (cf. 1.2))). Specifically:

$$f_{*}(x) = v\{y \in M: f^{*}(y) \le x\}$$
 (cf. 1.4).]

9.8 EXAMPLE Let X,Y be topological spaces and let  $f:X \rightarrow Y$  be a continuous function -- then f induces a localic arrow  $f:O(X) \rightarrow O(Y)$ .

[Take  $f^* = f^{-1}$ , hence

$$f_{\star}(U) = \cup \{ V \in O(Y) : f^{-1}(V) \subset U \}$$

or still,

$$f_{\star}(U) = Y - \overline{f(X-U)}.$$

9.9 NOTATION  $\underline{LOC}$  is the category whose objects are the locales and whose morphisms are the localic arrows.

9.10 THEOREM LOC is complete and cocomplete.

N.B. An initial object for LOC is {\*} and a final object for LOC is {0,1}.
[E.g.: Given L, a localic arrow f:L → {0,1} must have the property that
f\*(0) = 0, f\*(1) = 1 implying thereby the uniqueness of f as well as its existence
(cf. 9.7).]

9.11 DEFINITION A point of a locale L is a localic arrow  $p:\{0,1\} \rightarrow L$ .

9.12 DEFINITION An element x of a locale L is prime if  $\forall$  a,b  $\in$  L,

 $a \land b \le x \Rightarrow a \le x \text{ or } b \le x.$ 

9.13 LEMMA Let L be a locale -- then there is a bijection between the points of L and the prime elements of L.

PROOF Given a point p of L, put

$$x = v\{a \in L: p^*(a) = 0\}.$$

Then  $p^*(x) = 0$ , hence  $x \neq 1$  ( $p^*(1) = 1$ ). And x is prime:

$$a \wedge b \le x \Longrightarrow p^*(a \wedge b) = 0$$
  
 $\implies p^*(a) \wedge p^*(b) = 0$   
 $\implies p^*(a) = 0 \text{ or } p^*(b) = 0$   
 $\implies a \le x \text{ or } b \le x.$ 

Conversely, if  $x \in L$  is prime, define  $p^*: L \rightarrow \{0, 1\}$  by

$$p^{*}(a) = \begin{vmatrix} - & 0 & \text{if } a \leq x \\ & - & 1 & \text{if } a \neq x. \end{vmatrix}$$

Then p\* satisfies (1), (2), (3) of 9.7, so p\* is the left adjoint constituent of a localic arrow  $p:\{0,1\} \rightarrow L$ .

• Start with a point p, form the prime element x as above, and consider the point q associated with x. Given  $a \in L$ ,

$$q^{*}(a) = 0 \iff a \le x \iff p^{*}(a) = 0.$$

Therefore  $q^* = p^*$  or still, q = p.

• Start with a prime element x, pass to the point p corresponding to x, thence to the prime element y corresponding to p. Given  $a \in L$ ,

$$a \le x \iff p^*(a) = 0 \iff a \le y$$
.

Therefore x = y.

9.14 EXAMPLE Let X be a topological space — then each  $x \in X$  determines a

point  $p_x: \{0,1\} \rightarrow O(X)$ , thus

$$p_{X}^{\star}(U) = 0 \iff x \notin U,$$

the prime element per  $\boldsymbol{p}_{\boldsymbol{X}}$  being  $\boldsymbol{X}$  -  $\overline{\{\boldsymbol{X}\}}.$ 

9.15 NOTATION Given a locale L, let

$$pt(L) = Mor(\{0,1\},L),$$

the set of points of L.

[Note: It can happen that  $pt(L) = \emptyset$ . E.g., take the real line <u>R</u> in its usual topology and let

$$L = (O(\underline{R})_{b}, \underline{c}).$$

Then L has no prime element, thus  $pt(L) = \emptyset$  (cf. 9.13).]

9.16 LEMMA Let L be a locale. Given  $x \in L$ , put

$$U_{x} = \{p \in pt(L) : p^{*}(x) = 1\}.$$

Then the collection  $\{U_x : x \in L\}$  is a topology on pt(L).

[Note: We have

$$\begin{bmatrix} U_0 = \emptyset \\ & \bigcup & U_x = U \\ i \in I & i & i \in I \end{bmatrix} \begin{bmatrix} U_1 & U_1 & U_2 & U_3 \\ U_1 = pt(L), \end{bmatrix}$$

<u>N.B.</u> If  $f: L \rightarrow M$  is a localic arrow, then postcomposition  $pt(f): pt(L) \rightarrow pt(M) \quad (p \rightarrow f \circ p)$ 

is continuous.

[In fact,

$$pt(f)^{-1}(U_x) = U_{f^*(x)}$$
.]

Therefore these definitions give rise to a functor

pt:LOC 
$$\rightarrow$$
 TOP.

In the other direction, let

$$loc: TOP \rightarrow LOC$$

be the functor that sends X to O(X) and  $f:X \rightarrow Y$  to its associated localic arrow  $f:O(X) \rightarrow O(Y)$  (cf. 9.8).

9.17 THEOREM The functor pt is a right adjoint for the functor loc.

[Note: The arrows of adjunction

$$\mu \in \operatorname{Nat}(\operatorname{id}_{\underline{\operatorname{TOP}}}, \operatorname{pt} \circ \operatorname{loc})$$
$$v \in \operatorname{Nat}(\operatorname{loc} \circ \operatorname{pt}, \operatorname{id}_{\underline{\operatorname{LOC}}})$$

are

• Given a topological space X,

$$\mu_X: X \longrightarrow pt(O(X))$$

sends  $x \in X$  to  $\textbf{p}_{_{\! X}}$  (cf. 9.14);

• Given a locale L, the left adjoint part of

$$v_L:O(pt(L)) \longrightarrow L$$

is the functor

$$v_i^*:L \longrightarrow O(pt(L))$$

that sends  $x \in L$  to  $U_x$ .]

9.18 RAPPEL Let X be a topological space -- then a nonempty closed subset  $S \subset X$  is <u>irreducible</u> if for all closed subsets  $S_1, S_2$  of X,

$$S \subset S_1 \cup S_2 \Longrightarrow S \subset S_1 \text{ or } S \subset S_2,$$

i.e., if X - S  $\in$  O(X) is prime. E.g.:  $\forall$  x  $\in$  X,  $\overline{\{x\}}$  is an irreducible closed subset of X.

[Note: The only irreducible closed subsets of a Hausdorff space are singletons.]

9.19 DEFINITION A topological space X is <u>sober</u> provided that every irreducible closed subset S of X is the closure of a unique point  $x \in X:S = \overline{\{x\}}$ .

[Note: Consider the map  $x \rightarrow \overline{\{x\}}$  from the points of X to the irreducible closed subsets of X -- then X is  $T_0$  iff this map is injective and X is sober iff this map is bijective.]

9.20 EXAMPLE The spectrum of a commutative ring with unit in its Zariski topology is sober.

9.21 CRITERION A topological space X is sober iff the arrow of adjunction

$$\mu_X: X \rightarrow pt(O(X))$$

is bijective.

9.22 LEMMA Let L be a locale -- then pt(L) is a sober topological space. PROOF It is a question of applying 9.21 when X = pt(L). So let

$$Q: \{0,1\} \rightarrow O(pt(L))$$

be an element of pt(O(pt(L))) -- then there is a unique point  $q \in pt(L)$  such that

7.

 $p_q = Q$  (here

$$p_q^*(U_x) = 0 \iff q \notin U_x$$
 (cf. 9.14)).

To see this, let

$$y = v \{ x \in L: Q^*(U_x) = 0 \}.$$

Then  $Q^*(U_y) = 0$ , hence  $y \neq 1$   $(Q^*(U_1) = Q^*(pt(L)) = 1)$  and it is immediate that y is prime. Let now  $q \in pt(L)$  be the point corresponding to y, thus

$$q^{*}(x) = \begin{bmatrix} 0 & \text{if } x \le y \\ & & (\text{cf. 9.13}). \\ 1 & \text{if } x \ne y \end{bmatrix}$$

Claim:  $p_q = Q$ . Proof:  $\forall x \in L$ ,

$$p_{q}^{\star}(U_{x}) = 0 \iff q \notin U_{x}$$
$$\iff q^{\star}(x) = 0$$
$$\iff x \le y$$
$$\iff Q^{\star}(U_{y}) = 0$$

That q is unique can be established by a similar calculation.

9.23 DEFINITION A locale L is spatial if  $U_x = U_y \Rightarrow x = y$ .

N.B. In other words, L is spatial if

$$v^*:L \rightarrow O(pt(L))$$

is injective (it is surjective by definition).

9.24 EXAMPLE Let X be a topological space -- then the locale O(X) is spatial.

[Given 
$$U \in O(X)$$
,

$$\psi_{O(X)}^{*}(U) = \{ p \in pt(O(X)) : p^{*}(U) = 1 \}.$$

And

$$p_x \in v^*_{O(X)}(U) \iff x \in U.$$

Therefore

$$v^{*}_{O(X)}:O(X) \rightarrow O(pt(O(X)))$$

is injective.]

The reason for introducing "sober topological spaces" and "spatial locales" is the following easy consequence of 9.17.

9.25 THEOREM The category of sober topological spaces is equivalent to the category of spatial locales.

Details:

• A topological space X is sober iff the arrow of adjunction

$$\mu_X: X \rightarrow pt(O(X))$$

is a homeomorphism.

[If X is a topological space, then  $\mu_X$  is continuous (being a morphism in <u>TOP</u>) and if in addition X is sober, then  $\mu_X$  is bijective (cf. 9.21), hence open:

$$\mu_{\rm X}({\rm U}) = {\rm U}_{\rm H} \dots ]$$

• A locale L is spatial iff the arrow of adjunction

$$v_1:O(pt(L)) \rightarrow L$$

is an isomorphism of locales.

[If L is a spatial locale, then  $v_L^*$  is bijective. Moreover,  $v_L^*$  preserves the poset structure (clear) and reflects it:

$$U_{\mathbf{X}} \subset U_{\mathbf{Y}} \Longrightarrow U_{\mathbf{X} \land \mathbf{Y}} = U_{\mathbf{X}} \cap U_{\mathbf{Y}} = U_{\mathbf{X}},$$

so by injectivity,  $x \land y = x$  or still,  $x \le y$ .]

Turning to 9.25, the image of the functor pt is contained in the full subcategory of <u>TOP</u> whose objects are the sober topological spaces (cf. 9.22) and the image of the functor loc is contained in the full subcategory of <u>LOC</u> whose objects are the spatial locales (cf. 9.24). Therefore the adjunction (loc, pt) restricts to an adjunction on these smaller subcategories and by the above observations, the restricted arrows of adjunction are natural isomorphisms.

9.26 SCHOLIUM Let X be a topological space -- then the locale O(X) is isomorphic to the locale of open subsets of a sober topological space.

[For O(X) is spatial (cf. 9.24), hence

$$v_{O(X)} : O(pt(O(X))) \rightarrow O(X)$$

is an isomorphism of locales. But pt(O(X)) is sober (cf. 9.22).]

## §10. SITES

Let C be a small category.

10.1 NOTATION Given a sieve \$ over X and a morphism f:Y  $\rightarrow$  X, put

$$f^*\mathfrak{F} = \{g: \operatorname{cod} g = Y \& f \circ g \in \mathfrak{F}\}.$$

Then f\*\$ is a sieve over Y.

10.2 DEFINITION A Grothendieck topology on <u>C</u> is a function  $\tau$  that assigns to each  $X \in Ob \ \underline{C}$  a set  $\tau_X$  of sieves over X subject to the following assumptions.

(1) The maximal sieve  $\pmb{\$}_{max} \in \tau_X.$ 

(2) If  $\mathfrak{F} \in \tau_{\mathbf{X}}$  and if  $f: \mathbf{Y} \to \mathbf{X}$  is a morphism, then  $f^*\mathfrak{F} \in \tau_{\mathbf{Y}}$ .

(3) If  $\mathfrak{F} \in \tau_X$  and if  $\mathfrak{F}'$  is a sieve over X such that  $f^*\mathfrak{F}' \in \tau_Y$  for all

 $\texttt{f:Y} \twoheadrightarrow \texttt{X} \text{ in } \texttt{$\texttt{$\texttt{s}$, then $\texttt{$\texttt{s}$'}$}} \in \texttt{$\texttt{T}_X$}\text{.}$ 

10.3 DEFINITION A site is a pair  $(\underline{C}, \tau)$ , where  $\underline{C}$  is a small category and  $\tau$  is a Grothendieck topology on  $\underline{C}$ .

10.4 EXAMPLE Let *L* be a locale. Given  $x \in L$ , a sieve over x is a subset 3 of 4x (cf. 9.5) which is hereditary in the sense that

 $\forall s \in S$ ,  $\forall a \in L$ ,  $a \leq s \Rightarrow a \in S$ .

One then says that  $\mathfrak{F}$  <u>covers</u> x if x = v  $\mathfrak{F}$ . Denoting by  $\tau_x$  the set of all such  $\mathfrak{F}$ , the assignment x  $\rightarrow \tau_x$  is a Grothendieck topology  $\tau$  on L.

[It is straightforward to check (1), (2), and (3).

Ad (1) Here  $\$_{max} = 4x$  and it is obvious that

$$\mathbf{f*}\mathbf{\mathcal{F}} = \{\mathbf{s} \le \mathbf{y}; \mathbf{s} \in \mathbf{\mathcal{F}}\} = \{\mathbf{s} \land \mathbf{y}; \mathbf{s} \in \mathbf{\mathcal{F}}\}\$$

and the claim is that  $f^{\star}\mathfrak{F}\in\tau_{_{\mathbf{V}}}.$  In fact,

$$y = x \land y = (v \mathfrak{F}) \land y = v \{s \land y : s \in \mathfrak{F}\} = v f \mathfrak{F}.$$

Ad (3) Given \$', suppose that

$$y = v\{s' \land y:s' \in \mathfrak{Z}'\} \quad (y \in \mathfrak{Z}).$$

Then

 $= \vee (s^{1} \land (\vee s)) = \vee s^{1} \land x = \vee s^{1}.$  $s^{1} \in \mathfrak{F}^{1} \qquad s^{1} \in \mathfrak{F}^{1} \qquad s^{1} \in \mathfrak{F}^{1}$ 

Therefore  $\mathfrak{F}' \in \tau_x$ .]

<u>N.B.</u> Take L = O(X), where X is a topological space -- then a sieve over an open subset U of X is a set of open subsets V  $\subset$  U such that V'  $\subset$  V  $\in$  => V'  $\in$  . And

$$\begin{array}{ccc} \boldsymbol{\mathfrak{F}} \in \boldsymbol{\tau}_{\mathbf{U}} <=> & \boldsymbol{\cup} & \boldsymbol{\mathbf{V}} = \boldsymbol{\mathbf{U}}. \\ & \boldsymbol{\mathbf{V}} \in \boldsymbol{\boldsymbol{\mathfrak{F}}} \end{array}$$

10.5 LEMMA Let  $(\underline{C}, \tau)$  be a site --- then  $\forall X \in Ob \ \underline{C}$ ,

$$\boldsymbol{\mathfrak{z}} \in \boldsymbol{\tau}_{\mathbf{X}} ~\boldsymbol{\mathtt{\&}} ~\boldsymbol{\mathfrak{z}} \subset \boldsymbol{\mathfrak{Z}'} => \boldsymbol{\mathfrak{Z}'} \in \boldsymbol{\tau}_{\mathbf{X}}$$

and

$$\mathfrak{S}, \mathfrak{S}' \in \mathfrak{T}_X \Longrightarrow \mathfrak{S} \cap \mathfrak{S}' \in \mathfrak{T}_X.$$

10.6 REMARK Suppose that we have an assignment  $X \ \ \tau_X$  satisfying (1), (2) of

10.2 and for which

$$\boldsymbol{\boldsymbol{\mathfrak{F}}} \in \boldsymbol{\boldsymbol{\tau}}_X ~\boldsymbol{\boldsymbol{\mathfrak{F}}} \subset \boldsymbol{\boldsymbol{\mathfrak{F}}}' => \boldsymbol{\boldsymbol{\mathfrak{F}}}' \in \boldsymbol{\boldsymbol{\tau}}_X.$$

Then to check (3) of 10.2, it suffices to consider those  $\mathfrak{F}'$  such that  $\mathfrak{F}' \subset \mathfrak{F}$ .

Let  $\underline{C}$  be a small category -- then by  ${}^{T}\underline{C}$  we shall understand the set of Grothendieck topologies on  $\underline{C}$ .

10.7 EXAMPLE Take  $\underline{C} = \underline{1}$  -- then  $\underline{C}$  has two Grothendieck topologies:  $\{\$_{\max}\}\$ and  $\{\$_{\min}, \$_{\max}\}$ .

10.8 DEFINITION

• The minimal Grothendieck topology on <u>C</u> is the assignment  $X \rightarrow \{\$_{\max}\}$ .

• The maximal Grothendieck topology on <u>C</u> is the assignment  $X \rightarrow \{\$\}$ , where \$ runs through all the sieves over X.

Given  $\tau, \tau' \in \tau_C$ , write  $\tau \leq \tau'$  if  $\forall X \in Ob C$ ,  $\tau_X \subset \tau_X'$ .

10.9 LEMMA The poset  $\boldsymbol{\tau}_{C}$  is a bounded lattice.

PROOF If  $\tau, \tau' \in \tau_{\underline{C}}$ , let  $\tau \wedge \tau'$  be their set theoretical intersection and let  $\tau \vee \tau'$  be the smallest Grothendieck topology containing their set theoretical union. As for 0 and 1, take 0 to be the minimal Grothendieck topology and 1 to be the maximal Grothendieck topology.

10.10 THEOREM The bounded lattice  $\tau_{\mbox{C}}$  is a locale.

Let  $\underline{C}$  be a small category with pullbacks.

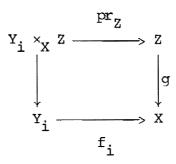
10.11 DEFINITION A coverage on <u>C</u> is a function K that assigns to each  $X \in Ob \ \underline{C}$  a set  $K_X$  of subsets of Ob <u>C</u>/X subject to the following assumptions.

- (1) If  $f:Y \rightarrow X$  is an isomorphism, then  $\{f:Y \rightarrow X\}$  is in  $K_X$ .
- (2) If  $\{f_i:Y_i \rightarrow X \ (i \in I)\}$  is in  $K_X$ , then for any morphism  $g:Z \rightarrow X$ ,

$$\{Y_{i} \times_{X} Z \longrightarrow Z (i \in I)\}$$

is in  $K_{z}$ .

[Note: Here



is a pullback square.] (3) If  $\{f_i:Y_i \rightarrow X \ (i \in I)\}$  is in  $K_X$  and if  $\forall i \in I$ ,  $\{g_{ij}:Z_{ij} \rightarrow Y_i \ (j \in I_i)\}$ is in  $K_{Y_i}$ , then

$$\{f_i \circ g_{ij}: Z_{ij} \rightarrow X (i \in I, j \in I_i)\}$$

is in K<sub>x</sub>.

10.12 EXAMPLE Let *L* be a locale. Given  $x \in L$ , let  $K_x$  be the set of all subsets of  $\forall x$  consisting of those set indexed collections  $\{x_i : i \in I\}$  such that  $\bigvee x_i = x - i \in I$ then the assignment  $x \neq K_x$  is a coverage K on *L*. 10.13 DEFINITION Let K be a coverage on C -- then the Grothendieck topology  $\tau$  on C generated by K is the prescription

$$\mathfrak{F} \in \mathfrak{T}_X \iff \exists \mathbf{R} \in \mathfrak{K}_X : \mathbf{R} \subset \mathfrak{F}.$$

10.14 EXAMPLE Let L be a locale — then the Grothendieck topology on L per 10.4 is generated by the coverage on L per 10.12.

10.15 REMARK Suppose still that <u>C</u> is a small category with pullbacks. Let  $\tau$  be a Grothendieck topology on <u>C</u> -- then there is a coverage K that generates  $\tau$ , viz.

$$R \in K_X \iff R \in \tau_X'$$

where

.....

$$\langle \mathbf{R} \rangle = \{ \mathbf{f} \circ \mathbf{g} : \mathbf{f} \in \mathbf{R}, \text{ dom } \mathbf{f} = \operatorname{cod } \mathbf{g} \}.$$

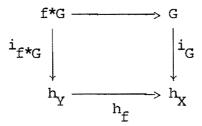
#### §11. SHEAVES

Let C be a small category.

11.1 RAPPEL For any  $X \in Ob \ \underline{C}$ , the sieves over X are in a one-to-one correspondence with the subfunctors of  $h_X$  (cf. 7.3).

Because of this, the notion of Grothendieck topology can be reformulated.

11.2 NOTATION Given a subfunctor G of  $h_X$  and a morphism  $f:Y \to X,$  define  $f^*G$  by the pullback square



in  $\hat{\underline{C}}$  — then f\*G is a subfunctor of  $h_{\gamma}$ .

11.3 DEFINITION A <u>Grothendieck topology</u> on <u>C</u> is a function  $\tau$  that assigns to each  $X \in Ob \ \underline{C}$  a set  $\tau_X$  of subfunctors of  $h_X$  subject to the following assumptions.

(1) The subfunctor  $h_x \in \tau_x$ .

(2) If  $G \in \tau_X$  and if  $f: Y \to X$  is a morphism, then  $f^*G \in \tau_Y$ .

(3) If  $G\in\tau_X$  and if G' is a subfunctor of  $h_X$  such that  $f^*G'\in\tau_Y$  for all  $f\in GY,$  then  $G'\in\tau_X.$ 

[Note: For use below, observe that 10.5 and 10.6 can be stated in terms of

subfunctors instead of sieves.]

Suppose that  $\underline{S}$  is a reflective subcategory of  $\hat{\underline{C}}$ . Denote the reflector by  $\underline{a}$  -- then there is an adjoint pair  $(\underline{a}, \iota)$ ,  $\iota: \underline{S} \rightarrow \hat{\underline{C}}$  the inclusion.

Assume: a preserves finite limits.

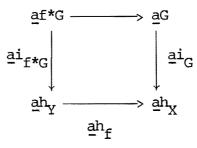
[Note: It is automatic that a preserves colimits.]

11.4 THEOREM Given  $X \in Ob C$ , let  $\tau_X$  be the set of those subfunctors  $G \xrightarrow{1}{G} h_X$ such that  $\underline{ai}_G$  is an isomorphism -- then the assignment  $X \rightarrow \tau_X$  is a Grothendieck topology  $\tau$  on C.

PROOF Since

$$\underline{a}(id_{h_X}) = id_{\underline{a}h_X},$$

it follows that  $h_X \in \tau_X$ , hence (1) is satisfied. As for (2), by assumption <u>a</u> preserves finite limits, so in particular <u>a</u> preserves pullbacks, thus



is a pullback square in S. But  $\underline{ai}_{G}$  is an isomorphism. Therefore  $\underline{ai}_{f^{*}G}$  is an isomorphism, i.e.,  $f^{*}G \in \tau_{v}$ . The verification of (3), however, is more complicated.

• Suppose that  $G \in \tau_{\chi}$  and G is a subfunctor of  $G^{t}\colon$ 

$$\begin{array}{c} - i_{G}:G \rightarrow h_{X} \\ , i:G \rightarrow G'. \end{array}$$

Then

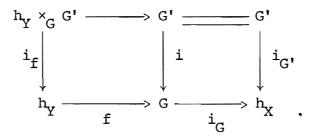
$$i_G = i_G, \circ i \Rightarrow \underline{a}i_G = \underline{a}i_G, \circ \underline{a}i.$$

But  $\underline{ai}_G$  is an isomorphism, hence

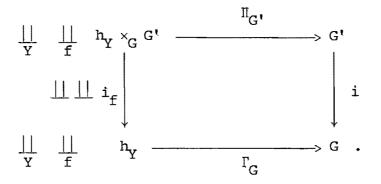
$$id = \underline{ai}_{G}, \circ \underline{ai} \circ (\underline{ai}_{G})^{-1},$$

which implies that  $\underline{ai}_{G}$ , is a split epimorphism. On the other hand,  $\underline{a}$  preserves monomorphisms, hence  $\underline{ai}_{G}$ , is a monomorphism. Therefore  $\underline{ai}_{G}$ , is an isomorphism, i.e.,  $G' \in \tau_X$ .

• It remains to establish (3) under the restriction that G' is a subfunctor of G. Using the Yoneda lemma, identify each  $f \in GY$  with  $f \in Nat(h_Y,G)$  and display the data in the diagram



There is one such diagram for each Y and each  $f \in GY$ , so upon consolidation we have



Now i is an equalizer (all monomorphisms in  $\hat{C}$  are equalizers), thus <u>a</u>i is an equalizer (by the assumption on <u>a</u>). But the assumption on G' is that  $\forall$  Y and  $\forall$  f  $\in$  GY, <u>a</u>i<sub>f</sub> is an isomorphism, thus <u>a</u>i is an epimorphism (see 11.8 below). And this means that <u>a</u>i is an isomorphism (cf. 6.8). Finally,

$$i_{G'} = i_{G} \circ i \Longrightarrow \underline{a}i_{G'} = \underline{a}i_{G} \circ \underline{a}i_{G'}$$

Therefore  $\underline{ai}_{G}$ , is an isomorphism, i.e.,  $G' \in \tau_{\chi}$ .

11.5 RAPPEL Given a category C, a set U of objects in C is said to be a <u>separating set</u> if for every pair X  $\xrightarrow{f}$  Y of distinct morphisms, there exists  $\xrightarrow{g}$ 

a  $U \in U$  and a morphism  $\sigma: U \to X$  such that  $f \circ \sigma \neq g \circ \sigma$ .

11.6 EXAMPLE Suppose that <u>C</u> is small — then the  $h_{\underline{Y}}$  (Y  $\in$  Ob <u>C</u>) are a separating set for  $\hat{\underline{C}}.$ 

11.7 LEMMA Let <u>C</u> be a category with coproducts and let U be a separating set -then  $\forall X \in Ob C$ , the unique morphism

$$\frac{||}{U \in \mathcal{U}} \quad \frac{||}{f \in Mor(U,X)} \quad \text{dom f} \quad \xrightarrow{^{1}X} X$$

such that  $\forall$  f,  $\Gamma_X \circ in_f = f$  is an epimorphism.

11.8 APPLICATION Suppose that <u>C</u> is small. Working with  $\hat{\underline{C}}$ , take X = G in 11.7 -- then

is an epimorphism.

[Note: To finish the argument that <u>a</u>i is an epimorphism, start with the relation

$$\Gamma_{G} \circ \coprod \coprod i_{f} = i \circ \Pi_{G'}.$$

Then

$$\underline{a}\Gamma_{G} \circ \underline{a}(\coprod \coprod \underline{i}_{f}) = \underline{a} \circ \underline{a}\Pi_{G'}$$

Since  $\Gamma_{G}$  is an epimorphism, the same is true of  $\underline{a}\Gamma_{G}$  (left adjoints preserve epimorphisms). And

$$\underline{a}(\parallel \parallel \underline{l} i_{f}) = \parallel \parallel \underline{a} i_{f}$$

is an isomorphism, call it 
$$\Phi$$
, hence

$$\underline{a}\Gamma_{G} = \underline{a}i \circ (\underline{a}\Pi_{G}, \circ \Phi^{-1}).$$

Therefore ai is an epimorphism.]

11.9 NOTATION Denote by  $\underline{S}_{\underline{C}}$  the "set" of reflective subcategories  $\underline{S}$  of  $\hat{\underline{C}}$  with the property that the inclusion  $\iota:\underline{S} \rightarrow \underline{\hat{C}}$  has a left adjoint  $\underline{a}:\underline{\hat{C}} \rightarrow \underline{S}$  that preserves finite limits.

11.10 DEFINITION Fix a Grothendieck topology  $\tau \in \tau_{\underline{C}}$  -- then a presheaf  $F \in Ob \ \hat{\underline{C}}$  is called a <u> $\tau$ -sheaf</u> if  $\forall X \in Ob \ \underline{C}$  and  $\forall G \in \tau_X$ , the precomposition map

$$i_{G}^{*}:Nat(h_{X},F) \rightarrow Nat(G,F)$$

is bijective.

Write  $\underline{Sh}_{T}(\underline{C})$  for the full subcategory of  $\hat{\underline{C}}$  whose objects are the  $\tau$ -sheaves.

11.11 EXAMPLE Take for  $\tau$  the minimal Grothendieck topology on <u>C</u> -- then  $\underline{Sh}_{\tau}(\underline{C}) = \hat{\underline{C}}.$  [Note: In particular,  $\underline{Sh}_{T}(\underline{1}) = \hat{\underline{1}} \approx \underline{SET}$ .]

11.12 EXAMPLE Take for  $\tau$  the maximal Grothendieck topology on <u>C</u> -- then the objects of <u>Sh</u><sub> $\tau$ </sub>(<u>C</u>) are the final objects in <u>C</u>.

[First,  $\forall X \in Ob \ \underline{C}, \ \underline{\emptyset} \to h_X$ . But  $\underline{\emptyset}$  is initial, thus the condition that F  $\underline{\hat{C}} \to h_X$ . But  $\underline{\emptyset}$  is initial, thus the condition that F be a  $\tau$ -sheaf amounts to the existence for each X of a unique morphism  $h_X \to F$ . Meanwhile, by Yoneda, Nat $(h_x, F) \approx FX$ .]

11.13 EXAMPLE Given  $\tau \in \tau_{\textbf{C'}}$  define  $\boldsymbol{0}_{\tau}$  by the rule

$$O_{\tau}(X) = \begin{bmatrix} - & \{0\} \text{ if } \emptyset \in \tau_{X} \\ & \hat{C} \end{bmatrix}$$
$$\emptyset \text{ if } \emptyset \notin \tau_{X}.$$

Then  $0_{\tau}$  is a  $\tau$ -sheaf and, moreover, is an initial object in  $\underline{Sh}_{\tau}(\underline{C})$ .

11.14 THEOREM The inclusion  $\iota_{\tau}: \underline{Sh}_{\tau}(\underline{C}) \rightarrow \underline{\hat{C}}$  admits a left adjoint  $\underline{a}_{\tau}: \underline{\hat{C}} \rightarrow \underline{Sh}_{\tau}(\underline{C})$  that preserves finite limits.

[Note: We can and will assume that  $\underline{a}_{\tau} \circ \iota_{\tau}$  is the identity.]

Various categorical generalities can then be specialized to the situation at hand.

11.15 DEFINITION A morphism  $f:A \rightarrow B$  and an object X in a category <u>C</u> are said to be <u>orthogonal</u> ( $f \perp X$ ) if the precomposition map  $f^*:Mor(B,X) \rightarrow Mor(A,X)$  is bijective. 11.16 RAPPEL Let  $\underline{D}$  be a reflective subcategory of a category  $\underline{C}$ , R a reflector for  $\underline{D}$  (cf. 5.10). Let  $W_{\underline{D}}$  be the class of morphisms in  $\underline{C}$  rendered invertible by R.

- Let  $X \in Ob \subseteq$  -- then  $X \in Ob \supseteq iff \forall f \in W_D$ ,  $f \perp X$ .
- Let  $f \in Mor \ \underline{C}$  -- then  $f \in W_{\underline{D}}$  iff  $\forall X \in Ob \ \underline{D}$ ,  $f \perp X$ .

ll.17 NOTATION Let  $\textit{W}_{\tau}$  be the class of morphisms in  $\hat{\underline{C}}$  rendered invertible by  $\underline{a}_{\tau}.$ 

11.18 EXAMPLE If  $F \in Ob \ \hat{\underline{C}}$ , then F is a  $\tau$ -sheaf iff  $\forall \Xi \in W_{\tau}$ ,  $\Xi \perp F$ .

11.19 EXAMPLE If  $\Xi \in Mor \hat{C}$ , then  $\Xi \in W_{\tau}$  iff for every  $\tau$ -sheaf F,  $\Xi \perp$  F. [Note: If  $X \in Ob \ C$  and if  $G \in \tau_X$ , then for every  $\tau$ -sheaf F,  $i_G \perp$  F, thus  $i_G \in W_{\tau}$ .]

11.20 RAPPEL Let <u>D</u> be a reflective subcategory of a category <u>C</u>, R a reflector for <u>D</u> (cf. 5.10) -- then the localization  $W_D^{-1}\underline{C}$  is equivalent to <u>D</u>.

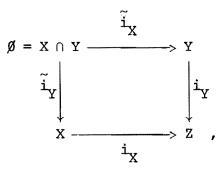
11.21 APPLICATION The localization  $W_{\tau}^{-1}\hat{\underline{C}}$  is equivalent to  $\underline{Sh}_{\tau}(\underline{C})$ .

11.22 RAPPEL Let <u>D</u> be a reflective subcategory of a finitely complete category <u>C</u>, R a reflector for <u>D</u> (cf. 5.10) -- then R preserves finite limits iff  $W_{\underline{D}}$  is pull-back stable.

[Note: When this is the case,  $W_{\underline{D}}$  is saturated (i.e.,  $f \in W_{\underline{D}}$  iff Rf is an isomorphism).]

11.23 APPLICATION Since  $\underline{a}_{\tau}: \hat{\underline{C}} \to \underline{Sh}_{\tau}(\underline{C})$  preserves finite limits, it follows that  $W_{\tau}$  is pullback stable (and saturated).

11.24 EXAMPLE Take  $\underline{C} = \underline{1}$ , so  $\underline{\hat{1}} \approx \underline{\text{SET}}$  -- then  $\#_{\underline{1}} = 2$ . On the other hand, <u>SET</u> has precisely 3 reflective subcategories: <u>SET</u> itself, the full subcategory of final objects, and the full subcategory of final objects plus the empty set  $(\#RX = 1 \text{ if } X \neq \emptyset, R\emptyset = \emptyset)$ . In terms of Grothendieck topologies, the first two are accounted for by 11.11 and 11.12. But the third cannot be a category of sheaves per a Grothendieck topology on  $\underline{C} = \underline{1}$ . To see this, note that the class of morphisms rendered invertible by R consists of all functions  $f:X \neq Y$  with  $X \neq \emptyset$ as well as the function  $\emptyset \neq \emptyset$  (thus the arrows  $\emptyset \neq X$  ( $X \neq \emptyset$ ) are excluded). Suppose now that Z is a nonempty set and X,Y are nonempty subsets of Z with an empty intersection. Consider the pullback square



where  $i_X, i_Y$  are the inclusions — then  $\operatorname{Ri}_Y$  is an isomorphism but  $\operatorname{Ri}_Y$  is not an isomorphism. Therefore the class of morphisms rendered invertible by R is not pullback stable.

11.25 NOTATION Let  $F \in Ob \ \hat{\underline{C}}$  be a presheaf. Given  $X \in Ob \ \underline{C}$ , let  $\tau_X(F)$  be the

8.

set of subfunctors  $i_{G}:G \to h_{X}$  such that for any morphism  $f:Y \to X,$  the precomposition arrow

$$(i_{f^{*}G})^{*}:Nat(h_{Y},F) \rightarrow Nat(f^{*}G,F)$$

is bijective.

11.26 LEMMA The assignment  $X \rightarrow \tau_{\chi}(F)$  is a Grothendieck topology  $\tau(F)$  on <u>C</u>.

N.B.  $\tau(F)$  is the largest Grothendieck topology in which F is a sheaf.

11.27 SCHOLIUM For any class F of presheaves, there exists a largest Grothendieck topology  $\tau(F)$  on C in which the  $F \in F$  are sheaves.

11.28 DEFINITION The <u>canonical</u> Grothendieck topology  $\tau_{can}$  on <u>C</u> is the largest Grothendieck topology on <u>C</u> in which the  $h_X(X \in Ob \underline{C})$  are sheaves.

[Note: Let  $\tau \in \tau_{\underline{C}}$  -- then  $\tau$  is said to be <u>subcanonical</u> if the  $h_{\underline{X}}$  ( $\underline{X} \in Ob \ \underline{C}$ ) are  $\tau$ -sheaves.]

11.29 EXAMPLE Let L be a locale -- then the Grothendieck topology  $\tau$  on L defined in 10.4 is the canonical Grothendieck topology.

[Note: This applies in particular to the locale O(X), where X is a topological space,  $\underline{Sh}_{T}(O(X))$  being the traditional sheaves of sets on X, i.e.,  $\underline{Sh}(X)$ .]

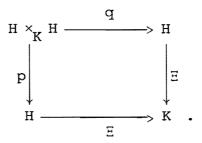
11.30 EXAMPLE Take for X the Sierpinski space (so  $X = \{0,1\}$  with topology  $\{X, \emptyset, \{0\}\}$ ) — then Sh(X) (cf. 11.29) is the arrow category SET( $\Rightarrow$ ).

#### §12. LOCAL ISOMORPHISMS

Let  $\underline{C}$  be a small category.

12.1 RAPPEL  $\hat{\underline{C}}$  fulfills the standard conditions (cf. 3.4 and 3.6) and is balanced (cf. 6.10 and 7.7).

Let  $H,K\in Ob\ \hat{\underline{C}}$  be presheaves and let  $\Xi\in Nat(H,K)$  . Form the pullback square



Then p and q are epimorphisms.

12.2 NOTATION  $\delta_H: H \to H \times_K H$  is the canonical arrow associated with  $id_H$ , thus p  $\circ \delta_H = id_H = q \circ \delta_H$ .

 $\underline{\text{N.B.}}~\delta_{H}$  is a monomorphism.

12.3 LEMMA  $\Xi$  is a monomorphism iff  $\boldsymbol{\delta}_{_{_{_{\!\!\!\!H}}}}$  is an epimorphism.

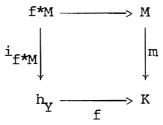
[Note: Consequently, if  $\Xi$  is a monomorphism, then  $\boldsymbol{\delta}_{H}$  is an isomorphism.]

Fix a Grothendieck topology  $\tau \in \tau_C$ .

12.4 DEFINITION Let  $H, K \in Ob \ \hat{\underline{C}}$  be presheaves and let  $\Xi \in Nat(H, K)$ . Factor  $\Xi$  per 3.9:

$$\begin{array}{cccc} k & m \\ H & \longrightarrow & M & \longrightarrow & K. \end{array}$$

Then E is a <u>t-local epimorphism</u> if for any  $f:h_Y \rightarrow K$ , the subfunctor  $f^M$  of  $h_Y$  defined by the pullback square



is in  $\tau_{y}$ .

12.5 LEMMA Every epimorphism in  $\hat{\underline{C}}$  is a  $\tau$ -local epimorphism.

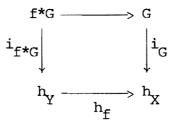
12.6 DEFINITION Let  $H, K \in Ob \stackrel{\circ}{\underline{C}}$  be presheaves and let  $\Xi \in Nat(H, K)$  -- then  $\Xi$  is a  $\underline{\tau}$ -local monomorphism if  $\delta_{\underline{H}}$  is a  $\underline{\tau}$ -local epimorphism (cf. 12.3).

12.7 LEMMA Every monomorphism in  $\hat{\underline{C}}$  is a  $\tau$ -local monomorphism.

12.8 DEFINITION Let  $H, K \in Ob \ \hat{C}$  be presheaves and let  $\Xi \in Nat(H, K)$  -- then  $\Xi$  is a <u>t-local isomorphism</u> if  $\Xi$  is both a t-local epimorphism and a t-local monomorphism.

12.9 EXAMPLE If  $G \in \tau_X'$  then  $i_G: G \to h_X$  is a  $\tau$ -local isomorphism.

[For any  $f: Y \rightarrow X$ , there is a pullback square



in  $\hat{\underline{C}}$  and  $f^{*}G \in \tau_{\underline{Y}}$  (cf. 11.3), thus  $\underline{i}_{\underline{G}}$  is a  $\tau$ -local epimorphism. On the other hand,

 $i_{G}$  is a monomorphism, hence  $i_{G}$  is a  $\tau$ -local monomorphism (cf. 12.7).]

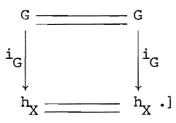
12.10 THEOREM  $\textit{W}_{\tau}$  is the class of  $\tau\text{-local}$  isomorphisms.

12.11 APPLICATION Let  $H\in Ob\ \hat{\underline{C}}$  -- then the canonical arrow  $H\longrightarrow \iota_{\tau}\underline{a}_{\tau}H$ 

is a  $\tau$ -local isomorphism.

12.12 APPLICATION Let  $G \in \tau_X$  -- then  $\underline{a}_{\tau} \underline{i}_G$  is an isomorphism (cf. 11.19).

[Note: Suppose that  $i_G: G \to h_X$  is a subfunctor -- then  $i_G$  is a monomorphism, hence  $i_G$  is a  $\tau$ -local monomorphism (cf. 12.7). Assume in addition that  $i_G$  is a  $\tau$ -local epimorphism. Claim:  $G \in \tau_X$ . Proof: Take  $f = id_X$  and consider



We shall now proceed to establish the "fundamental correspondence".

12.13 THEOREM The arrows

$$\begin{array}{cccc} \underline{s}_{\underline{C}} & \longrightarrow & \tau_{\underline{C}} & (cf. 11.4) \\ \\ & & \tau_{\underline{C}} & \longrightarrow & \underline{s}_{\underline{C}} & (cf. 11.14) \end{array}$$

are mutually inverse.

To dispatch the second of these, consider the composite

$${}^{\tau}\underline{c} \longrightarrow \underline{s}\underline{c} \longrightarrow {}^{\tau}\underline{c}.$$

Take a  $\tau \in \tau_{\underline{C}}$  and pass to  $\underline{Sh}_{\tau}(\underline{C})$  -- then the Grothendieck topology on  $\underline{C}$  determined by  $\underline{Sh}_{\tau}(\underline{C})$  via 11.4 assigns to each  $X \in Ob \underline{C}$  the set of those subfunctors  $i_{\underline{G}}: \underline{G} \neq h_X$ such that  $\underline{a}_{\tau}i_{\underline{G}}$  is an isomorphism or, equivalently, those subfunctors  $i_{\underline{G}}: \underline{G} \neq h_X$ such that  $i_{\underline{G}}$  is a  $\tau$ -local isomorphism (cf. 12.10). But, as has been seen above, the subfunctors of  $h_X$  with this property are precisely the elements of  $\tau_X$ . Therefore the composite

$$\tau_{\underline{c}} \longrightarrow \underline{s}_{\underline{c}} \longrightarrow \tau_{\underline{c}}$$

is the identity map.

It remains to prove that the composite

$$\underline{s}_{\underline{C}} \longrightarrow \underline{\tau}_{\underline{C}} \longrightarrow \underline{s}_{\underline{C}}$$

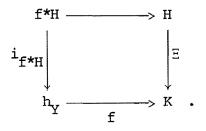
is the identity map. So take an  $\underline{S} \in \underline{S}_{\underline{C}}$ , produce a Grothendieck topology  $\tau$  on  $\underline{C}$ per 11.4, and pass to  $\underline{Sh}_{\tau}(\underline{C})$  -- then  $\underline{S} \subset \underline{Sh}_{\tau}(\underline{C})$ . Thus let  $F \in Ob \underline{S}$ , the claim being that  $F \in Ob \underline{Sh}_{\tau}(\underline{C})$  or still, that F is a  $\tau$ -sheaf, or still, that  $\forall X \in Ob \underline{C}$  and  $\forall G \in \tau_X, i_{\underline{G}} \perp F$ , which is clear since  $i_{\underline{G}} \in W_{\tau}$  (cf. 11.19). To reverse matters and deduce that  $\underline{Sh}_{\tau}(\underline{C}) \subset \underline{S}$ , one has only to show that if  $\Xi: H \rightarrow K$  is a morphism in  $\underline{\hat{C}}$  and if  $\underline{a}\Xi$  is an isomorphism, then  $\underline{a}_{\tau}\Xi$  is an isomorphism. To this end, factor  $\Xi$  per 3.9:

$$\begin{array}{cccc} & \kappa & m \\ H & \longrightarrow & M & \longrightarrow & K. \end{array}$$

Then  $\underline{a} \equiv \underline{a} \\ \underline{$ 

• Assume that  $\underline{a}\Xi$  is an isomorphism, where  $\Xi$  is a monomorphism -- then  $\underline{a}_{\tau}\Xi$  is an isomorphism.

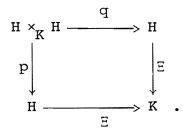
[Bearing in mind that here H = M, consider a pullback square



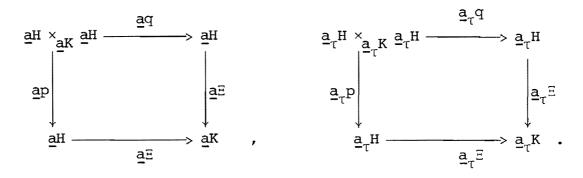
Then the assumption that  $\underline{a}\Xi$  is an isomorphism implies that  $\underline{a}i_{f*H}$  is an isomorphism which in turn implies that  $i_{f*H} \in \tau_Y$ . Therefore  $\Xi$  is a  $\tau$ -local epimorphism or still,  $\Xi$  is a  $\tau$ -local isomorphism, hence  $\Xi \in W_{\tau}$  (cf. 12.10), so  $\underline{a}_{\tau}\Xi$  is an isomorphism.]

• Assume that as is an isomorphism, where  $\Xi$  is a coequalizer -- then  $\underline{a}_{\tau}\Xi$  is an isomorphism.

[Because  $\underline{a}_{\tau} \Xi$  is a coequalizer, to conclude that  $\underline{a}_{\tau} \Xi$  is an isomorphism, it suffices to verify that  $\underline{a}_{\tau} \Xi$  is a monomorphism (cf. 6.8). For this purpose, consider the pullback square



Then  $\boldsymbol{\delta}_{H}$  is a monomorphism and there are pullback squares



But  $\underline{a}\delta_{H} = \delta_{\underline{a}H}$  is an isomorphism (cf. 12.3), thus  $\underline{a}_{\tau}\delta_{H} = \delta_{\underline{a}_{\tau}H}$  is an isomorphism (cf. supra), so  $\underline{a}_{\tau}\Xi$  is a monomorphism (cf. 12.3).]

12.14 THEOREM Let  $H, K \in Ob \ \hat{\underline{C}}$  be presheaves and let  $\Xi \in Nat(H, K)$  -- then  $\underline{a}_{\tau} \Xi : \underline{a}_{\tau} H \Rightarrow \underline{a}_{\tau} K$  is an epimorphism in  $\underline{Sh}_{\tau}(\underline{C})$  iff  $\Xi$  is a  $\tau$ -local epimorphism.

12.15 APPLICATION The epimorphisms in  $\underline{Sh}_{\tau}(\underline{C})$  are pullback stable. [The class of  $\tau$ -local epimorphisms is pullback stable.]

# \$13, SORITES

The category  $\underline{Sh}_{\tau}(\underline{C})$  associated with a site  $(\underline{C}, \tau)$  has a number of properties that will be cataloged below.

13.1 LEMMA  $\underline{Sh}_{T}(\underline{C})$  is complete and cocomplete.

[This is because  $\underline{Sh}_{\tau}(\underline{C})$  is a reflective subcategory of  $\hat{\underline{C}}$  which is both complete and cocomplete. Accordingly, limits in  $\underline{Sh}_{\tau}(\underline{C})$  are computed as in  $\hat{\underline{C}}$  while colimits in  $\underline{Sh}_{\tau}(\underline{C})$  are computed by applying  $\underline{a}_{\tau}$  to the corresponding colimits in  $\hat{\underline{C}}$ .]

13.2 LEMMA  $\underline{Sh}_{\tau}(\underline{C})$  is cartesian closed.

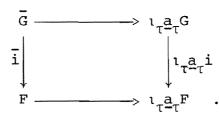
[Since  $\underline{a}_{\tau}: \hat{\underline{C}} \rightarrow \underline{Sh}_{\tau}(\underline{C})$  preserves finite limits, it preserves finite products so one can quote 5.11.]

[Note: Recall that  $\hat{\underline{C}}$  is cartesian closed (cf. 5.21).]

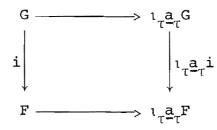
13.3 LEMMA  $\underline{Sh}_{\tau}(\underline{C})$  admits a subobject classifier. [Note: Therefore  $\underline{Sh}_{\tau}(\underline{C})$  is wellpowered (cf. 6.13).]

The proof of this result will be broken up into several steps (tacitly employing the license provided by 7.6).

<u>Step 1</u> Given  $F \in Ob \stackrel{\circ}{\underline{C}}$  and a subfunctor  $i:G \rightarrow F$ , define a subfunctor  $\overline{i:G} \rightarrow F$ by the pullback square



Step 2 There is a commutative diagram



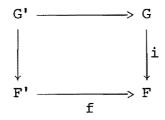
from which an arrow  $\gamma: G \rightarrow \overline{G}$  such that the diagrams



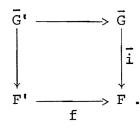
commute.

<u>Step 3</u> Definition: G is <u>closed</u> if  $G = \overline{G}$ . We have (1)  $G \subset \overline{G}$ ; (2)  $G \subset H \Longrightarrow \overline{G} \subset \overline{H}$ ; (3)  $\overline{\overline{G}} = \overline{G}$ . In addition, closed subfunctors are stable under pullbacks.

[Note: To make the last point precise, suppose given an arrow  $f:F' \to F$  in  $\hat{\underline{C}}$ . Define G' by the pullback square



and define  $\overline{G}'$  by the pullback square



Then  $\overline{G'} = \overline{G'}$ , so

$$G = \overline{G} => G' = \overline{G'} = \overline{G'}.$$

 $\underline{\text{Step 4}} \quad \forall \ F \in \text{Ob} \ \hat{\underline{C}},$ 

 $\overline{F} = F$ .

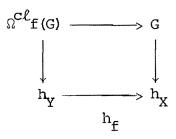
In particular:  $\forall X \in Ob C$ ,

$$\overline{h_X} = h_X$$

Step 5 Let  $(\Omega, \tau)$  be the subobject classifier for  $\hat{\underline{C}}$  (cf. 7.7). Define

$$\Omega^{cl}:\underline{C}^{OP} \rightarrow \underline{SET}$$

on an object X by letting  $\Omega^{cl}X$  be the set of all closed subfunctors of  $h_X$  and on a morphism  $f:Y \to X$  by letting  $\Omega^{cl}f:\Omega^{cl}X \to \Omega^{cl}Y$  operate via the pullback square



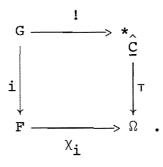
and define

$$\tau^{cl}: * \stackrel{\rightarrow}{\underline{C}} \Omega^{cl}$$

by factoring

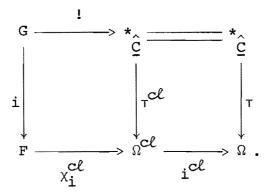
through  $\Omega^{c\ell}$  (which makes sense since  $\overline{h_X} = h_X$ ). With these agreements,  $\Omega^{c\ell}$  is a subfunctor of  $\Omega$ , say  $i^{c\ell}: \Omega^{c\ell} \to \Omega$ .

Step 6 Consider the pullback square

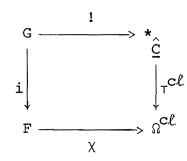


Then the classifying arrow  $\chi_{\rm i}$  factors through  $\Omega^{\rm C\ell}$  iff G is closed.

<u>Step 7</u> If F is a  $\tau$ -sheaf, then it and its  $\tau$ -subsheaves G are closed. This said, consider the commutative diagram



Here  $\chi_i = i^{cl} \circ \chi_i^{cl}$  and both squares are pullbacks. If  $\chi: F \to \Omega^{cl}$  is another morphism and if



is a pullback square, then  $i^{c\ell} \circ \chi$  is a classifying arrow of (G,i) in F, so  $i^{c\ell} \circ \chi = \chi_i = i^{c\ell} \circ \chi_i^{c\ell}$ , hence  $\chi = \chi_i^{c\ell}$ .

 $\frac{\text{Step 8}}{\underline{C}} * \text{ is a } \tau \text{-sheaf (obvious) and } \Omega^{\text{Cl}} \text{ is a } \tau \text{-sheaf (...). Therefore}$ the pair  $(\Omega^{\text{Cl}}, \tau^{\text{Cl}})$  is a subobject classifier for  $\underline{Sh}_{\tau}(\underline{C})$ .

13.4 LEMMA  $\underline{Sh}_{T}(\underline{C})$  is balanced.

[Taking into account 13.3, one has only to cite 6.10.]

13.5 LEMMA Every monomorphism in  $\underline{Sh}_{T}(\underline{C})$  is an equalizer.

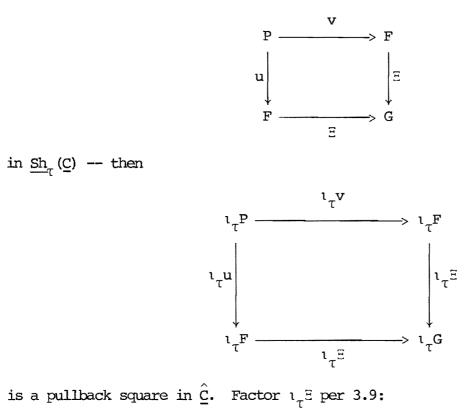
[In view of 13.3, this is a special case of 6.9.]

[Note: It is easy to proceed directly. Thus let  $E:F \neq G$  be a monomorphism in  $\underline{Sh}_{T}(\underline{C})$  -- then  $\iota_{T}E:\iota_{T}F \neq \iota_{T}G$  is a monomorphism in  $\underline{\hat{C}}$ , hence is an equalizer. But  $\underline{a}_{T}$  preserves equalizers (since it preserves finite limits).]

N.B. Monomorphisms in  $\underline{Sh}_{T}(\underline{C})$  are pushout stable.

13.6 LEMMA Every epimorphism in  $\underline{Sh}_{T}(\underline{C})$  is a coequalizer.

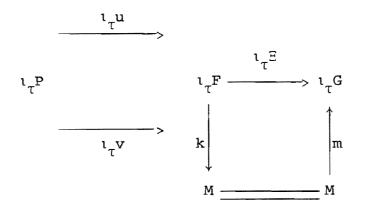
**PROOF** Given an epimorphism  $E:F \rightarrow G$  in  $\underline{Sh}_T(\underline{C})$ , form the pullback square



is a pullback square in  $\hat{\underline{C}}.$  Factor  $\iota_{\tau}\Xi$  per 3.9:

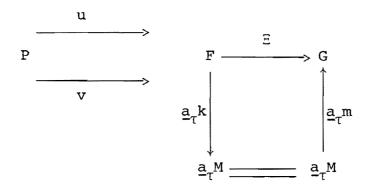
 $\iota_{\tau} F \xrightarrow{k} M \xrightarrow{m} \iota_{\tau} G.$ 

Then by construction there is a coequalizer diagram



in  $\hat{\underline{C}}$ . Now apply  $\underline{a}_{\tau}$  to get a coequalizer diagram

6.



in  $\underline{Sh}_{\tau}(\underline{C})$ . Since

$$\Xi = \underline{a}_{T} \mathbf{m} \circ \underline{a}_{T} \mathbf{k}$$

and since  $\Xi$  is an epimorphism, it follows that  $\underline{a}_{T}$  is an epimorphism. But  $\underline{a}_{T}$  is also a monomorphism. Therefore  $\underline{a}_{T}$  is an isomorphism (cf. 13.4) and  $\Xi$  is a coequalizer, thus being the case of  $\underline{a}_{T}$  k.

13.7 LEMMA  $\underline{Sh}_{T}(\underline{C})$  fulfills the standard conditions.

[Epimorphisms in  $\underline{Sh}_{T}(\underline{C})$  are pullback stable (cf. 12.15) and every epimorphism in  $\underline{Sh}_{T}(\underline{C})$  is a coequalizer (cf. 13.6).]

13.8 LEMMA In  $\underline{Sh}_{T}(\underline{C})$ , filtered colimits commute with finite limits.

13.9 RAPPEL Coproducts in  $\hat{\underline{C}}$  are disjoint.

[In other words, if  $F = \coprod_{i \in I} F_i$  is a coproduct of a set of presheaves  $F_i$ , then  $\forall i \in I, in_i: F_i \neq F$  is a monomorphism and  $\forall i, j \in I$  ( $i \neq j$ ), the pullback  $F_i \times_F F_j$  is the initial object in  $\hat{C}$ .] 13.10 LEMMA Coproducts in  $\underline{Sh}_{\tau}(\underline{C})$  are disjoint.

13.11 RAPPEL Coproducts in  $\hat{\underline{C}}$  are pullback stable.

[In other words, if  $F = \prod_{i \in I} F_i$  is a coproduct of a set of presheaves  $F_i$ , then for every arrow  $F' \neq F$ ,

$$\coprod_{i \in I} F' \times_F F_i \approx F'.$$

13.12 LEMMA Coproducts in  $\underline{\mathrm{Sh}}_{\mathbb{T}}(\underline{C})$  are pullback stable.

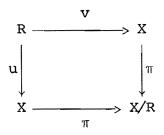
13.13 DEFINITION Let C be a category which fulfills the standard conditions.

Suppose that R  $\xrightarrow{u}$  X is an equivalence relation on an object X in C. Consider  $\xrightarrow{v}$ 

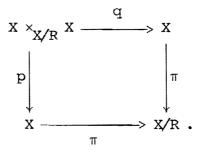
the coequalizer diagram

$$R \xrightarrow{u} X \xrightarrow{\pi} X/R \equiv coeq(u,v).$$

Then there is a commutative diagram



and a pullback square



One then says that R is effective if the canonical arrow

$$R \longrightarrow X \times_{X/R} X$$

is an isomorphism (it is always a monomorphism).

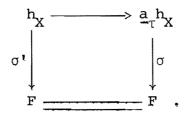
[Note:  $\underline{C}$  has <u>effective equivalence relations</u> if every equivalence relation is effective.]

13.14 LEMMA Equivalence relations in  $\underline{Sh}_{T}(\underline{C})$  are effective.

[The usual methods apply: Equivalence relations in <u>SET</u> are effective, hence equivalence relations in  $\hat{\underline{C}}$  are effective etc.]

13.15 LEMMA The  $\underline{a}_T \underline{h}_X$  (X  $\in$  Ob C) are a separating set for  $\underline{Sh}_T(\underline{C})$ .

PROOF Let  $E, E': F \neq G$  be distinct arrows in  $\underline{Sh}_{T}(\underline{C})$  — then the claim is that  $\exists X \in Ob \underline{C}$  and  $\sigma: \underline{a}_{T} \underline{h}_{X} \neq F$  such that  $E \circ \sigma \neq E' \circ \sigma$ . But  $E \neq E'$  implies that  $E_{X} \neq E_{X}'$  ( $\exists X \in Ob \underline{C}$ ) which implies that  $E_{X} x \neq E_{X}' x$  ( $\exists x \in FX$ ). Owing to the Yoneda lemma,  $FX \approx Nat(\underline{h}_{X}, F)$ , so x corresponds to a  $\sigma' \in Nat(\underline{h}_{X}, F)$ , thus  $E \circ \sigma' \neq E' \circ \sigma'$ . Determine  $\sigma: \underline{a}_{T} \underline{h}_{X} \neq F$  by the diagram



Then  $\Xi \circ \sigma \neq \Xi' \circ \sigma$ .

<u>N.B.</u> All epimorphisms in  $\underline{Sh}_{\tau}(\underline{C})$  are coequalizers (cf. 13.6). So, for every  $\tau$ -sheaf F, the epimorphism  $\Gamma_{F}$  of 11.7 is automatically a coequalizer. Therefore the  $\underline{a}_{\tau}\underline{h}_{X}$  (X  $\in$  Ob <u>C</u>) are a "strong" separating set for  $\underline{Sh}_{\tau}(\underline{C})$ .

[Note: This baroque technicality is used implicitly in 13.16 below.]

A summary of the theory of presentable categories can be found in the Appendix to CHT and will not be repeated here.

[Note: As a point of terminology, let  $\underline{C}$  be a cocomplete category and let  $\kappa$  be a regular cardinal -- then an object  $X \in Ob \underline{C}$  is  $\kappa$ -definite if Mor(X,--) preserves  $\kappa$ -filtered colimits.]

13.16 LEMMA  $\underline{Sh}_{\tau}(\underline{C})$  is presentable.

PROOF Fix a regular cardinal  $\kappa > \#Mor \underline{C}$  — then  $\forall X \in Ob \underline{C}$ ,  $h_X \in Ob \underline{\hat{C}}$  is  $\kappa$ -definite, the contention being that  $\forall X \in Ob \underline{C}$ ,  $\underline{a}_T h_X \in Ob \underline{Sh}_T(\underline{C})$  is  $\kappa$ -definite, which suffices (cf. 13.15). To see this, note first that a  $\kappa$ -filtered colimit of  $\tau$ -sheaves can be computed levelwise, i.e., its  $\kappa$ -filtered colimit per  $\underline{\hat{C}}$  is a  $\tau$ -sheaf. Now fix a  $\kappa$ -filtered category  $\underline{I}$  and let  $\Delta: \underline{I} \to \underline{Sh}_T(\underline{C})$  be a diagram — then

$$\begin{split} \operatorname{Nat}(\underline{a}_{\tau}\mathbf{h}_{X},\operatorname{colim}_{\underline{I}} \Delta_{\underline{i}}) &\approx \operatorname{Nat}(\underline{a}_{\tau}\mathbf{h}_{X},\operatorname{colim}_{\underline{I}} \iota_{\tau}\Delta_{\underline{i}}) \\ &\approx \operatorname{Nat}(\mathbf{h}_{X},\operatorname{colim}_{\underline{I}} \iota_{\tau}\Delta_{\underline{i}}) \\ &\approx \operatorname{colim}_{\underline{I}} \operatorname{Nat}(\mathbf{h}_{X},\iota_{\tau}\Delta_{\underline{i}}) \\ &\approx \operatorname{colim}_{\underline{I}} \operatorname{Nat}(\underline{a}_{\tau}\mathbf{h}_{X},\Delta_{\underline{i}}) . \end{split}$$

13.17 REMARK It is a fact that a presentable category is complete and cocomplete, wellpowered and cowellpowered.

# **§14.** TOPOS THEORY: FORMALITIES

Let  $\underline{E}$  be a category.

14.1 DEFINITION E is a topos if

- E is finitely complete;
- <u>E</u> is cartesian closed;
- $\underline{E}$  has a subobject classifier  $(\Omega, \tau)$ .

[Note: The defining properties of a topos are invariant under equivalence.]

N.B. Every topos is wellpowered.

14.2 EXAMPLE SET is a topos.

[Note: The full subcategory of <u>SET</u> whose objects are finite is a topos. On the other hand, the full subcategory of <u>SET</u> whose objects are at most countable has a subobject classifier but is not cartesian closed, hence is not a topos.]

14.3 EXAMPLE Let <u>C</u> be a small category -- then  $\hat{\underline{C}}$  is a topos (cf. 5.21 and 7.7).

14.4 EXAMPLE Let  $(\underline{C}, \tau)$  be a site -- then  $\underline{Sh}_{\tau}(\underline{C})$  is a topos (cf. 13.2 and 13.3).

14.5 THEOREM Every topos is finitely cocomplete.

14.6 THEOREM Every topos fulfills the standard conditions.

14.7 LEMMA Let  $\underline{E}$  be a topos.

(1) Every monomorphism in  $\xi$  is an equalizer.

(2) Every epimorphism in  $\underline{E}$  is a coequalizer.

(3) Every morphism in  $\underline{E}$  which is both a monomorphism and an epimorphism is an isomorphism.

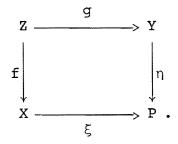
(4) Every morphism in  $\underline{E}$  admits a minimal decomposition unique up to isomorphism.

14.8 EXAMPLE Not all monomorphisms in <u>CAT</u> are equalizers and not all epimorphisms in CAT are coequalizers. Therefore CAT is not a topos.

14.9 LEMMA Every topos has effective equivalence relations.

14.10 EXAMPLE In POS (the category whose objects are the posets and whose morphisms are the order preserving maps), not all equivalence relations are effective.

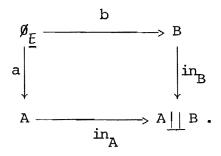
14.11 CRITERION In a topos E, consider a pushout square



Assume: f is a monomorphism -- then  $\eta$  is a monomorphism and the square is a pullback.

14.12 LEMMA In a topos E, finite coproducts are disjoint.

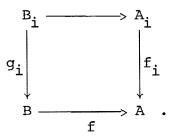
PROOF Let  $A, B \in Ob \not E$  -- then on general grounds, there is a pushout square



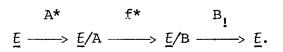
On the other hand, a and b are monomorphisms (cf. 5.16). Therefore in  $_{A}$  and in  $_{B}$  are monomorphisms and the square is a pullback (cf. 14.11).

14.13 LEMMA In a topos E, finite coproducts are pullback stable.

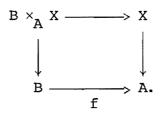
[Note: Finiteness is not needed provided that the coproducts in question exist. Thus suppose that  $\{A_i \xrightarrow{f_i} A: i \in I\}$  is a coproduct diagram in  $\underline{E}$ . Let  $B \xrightarrow{f} A$ and for each  $i \in I$ , define  $B_i$  by the pullback square



Then  $\{B_i \longrightarrow B: i \in I\}$  is a coproduct diagram in  $\underline{E}$ . To see this, use 15.3: Consider the composition



Each of the functors  $A^*$ ,  $f^*$ ,  $B_1$  has a right adjoint, hence preserve colimits, in particular coproducts. On the other hand, given an arrow  $X \rightarrow A$ , define an arrow

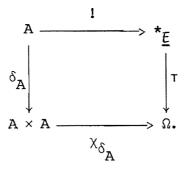


Then

$$B_{I} \circ f^{*} \circ A^{*}(X \rightarrow A) = B \times_{A} X \rightarrow B.$$

Let  $\underline{E}$  be a topos.

14.14 NOTATION Given  $A \in Ob \not E$ , let  $\delta_A : A \rightarrow A \times A$  be the diagonal -- then  $\delta_A$  is a monomorphism, so there is a pullback square



Abbreviate  $\chi_{\delta_A}$  to =<sub>A</sub>.

We have

Mor 
$$(\mathbf{A} \times \mathbf{A}, \Omega) \approx \operatorname{Mor}(\mathbf{A}, \Omega^{\mathbf{A}})$$
.

Therefore

$$=_{A} \in Mor(A \times A, \Omega)$$

corresponds to an element

$$\{\cdot\}_{A} \in Mor(A, \Omega^{A}),$$

the singleton on A.

14.15 LEMMA {.}, is a monomorphism, hence

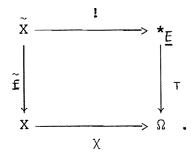
$$(\mathbf{A}, \{\cdot\}_{\mathbf{A}}) \in \mathbf{M}(\Omega^{\mathbf{A}}).$$

14.16 EXAMPLE Take  $\underline{E} = \underline{SET}$  -- then  $\{\cdot\}_A : A \to \Omega^A$  sends  $a \in A$  to the characteristic function of  $\{a\}$  (cf. 6.4). Identifying  $\Omega^A$  with PA (the power set of A), it follows that  $\{\cdot\}_A : A \to \Omega^A$  sends a to  $\{a\}$ .

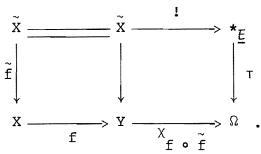
14.17 RAPPEL Given a category C, an object Q in C is said to be <u>injective</u> if for each monomorphism  $f:X \rightarrow Y$  and each morphism  $\phi:X \rightarrow Q$ , there exists a morphism  $g:Y \rightarrow Q$  such that  $g \circ f = \phi$ .

14.18 LEMMA In a topos E, the object  $\Omega$  is injective.

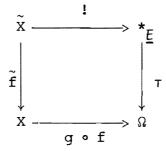
PROOF Let  $f: X \to Y$  be a monomorphism and let  $\chi: X \to \Omega$  be a morphism. Define  $\tilde{(X,f)} \in M(X)$  by the pullback square



Then  $\chi_{\tilde{f}} = \chi$  (cf. 6.12). Consider now the commutative diagram



Put g =  $\chi$  . Since the squares are pullbacks, the commutative diagram f  $\circ$  f

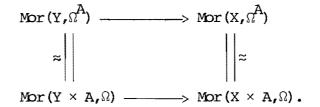


is a pullback square, so  $\chi_{\sim} = g \circ f$ . But f

$$\chi_{\widetilde{f}} = \chi \Longrightarrow g \circ f = \chi.$$

14.19 LEMMA In a topos  $\underline{E}$ , the object  $\Omega^{A}$  ( $A \in Ob \underline{E}$ ) is injective.

PROOF Let  $f:X \to Y$  be a monomorphism and let  $\phi:X \to \Omega^A$  be a morphism -- then there is a commutative diagram



Since  $\Omega$  is injective, the bottom map is surjective, thus the same is true of the top map.

14.20 RAPPEL A category <u>C</u> has <u>enough injectives</u> provided that for any  $X \in Ob \underline{C}$ , there is a monomorphism  $X \rightarrow Q$  with Q injective.

14.21 LEMMA A topos  $\underline{E}$  has enough injectives.

PROOF If  $A \in Ob \underline{E}$ , then  $\alpha^A$  is injective and  $\{\cdot\}_A : A \neq \alpha^A$  is a monomorphism (cf. 14.15).

14.22 LEMMA The injective objects in  $\underline{E}$  are the retracts of the  $\Omega^{\mathbf{A}}$  ( $\mathbf{A} \in \mathbf{Ob} \ \underline{E}$ ).

# \$15. TOPOS THEORY: SLICES AND SUBOBJECTS

Let  $\underline{E}$  be a topos.

15.1 THEOREM For every  $A \in Ob E$ , the category E/A is a topos.

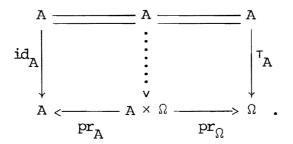
[Since  $\underline{E}$  is finitely complete, the same is true of  $\underline{E}/A$  (cf. 4.1). Let  $T_A$ 

be the composition  $A \longrightarrow *_{\underline{E}} \longrightarrow \Omega$ . Bearing in mind that  $id_A: A \to A$  is a final object in E/A, define

$$(\operatorname{id}_{A}, \operatorname{T}_{A}) : (\operatorname{id}_{A}: A \to A) \to (\operatorname{pr}_{A}: A \times \Omega \to A)$$

by consideration of

------



Then  $\langle id_A, \tau_A \rangle$  is a monomorphism (its domain being a final object in <u>E</u>/A) and the pair

$$(\text{pr}_{A}: A \times \Omega \rightarrow A, < \text{id}_{A}, \tau_{A} >)$$

is a subobject classifier for  $\underline{E}/A$ . The crux is therefore to establish that  $\underline{E}/A$  is cartesian closed.]

In particular: E is locally cartesian closed (cf. 5.23).

15.2 EXAMPLE  $\forall X, \underline{TOP}_{\underline{IH}}/X$  is a topos but  $\underline{TOP}_{\underline{IH}}$  is not a topos (recall that

 $\underline{\text{TOP}}_{LH}$  is not finitely complete (cf. 4.2)).

15.3 THEOREM Suppose that  $f:A \rightarrow B$  is a morphism in  $\underline{E}$  -- then  $f^*:\underline{E}/B \rightarrow \underline{E}/A$ has a left adjoint  $f_{\underline{1}}:\underline{E}/A \rightarrow \underline{E}/B$  and a right adjoint  $f_*:\underline{E}/A \rightarrow \underline{E}/B$ .

[This is a special case of 5.32 and 5.33.]

[Note: f\* preserves exponential objects and subobject classifiers.]

15.4 LEMMA Let  $A \in Ob \not E$  — then the poset  $Sub_{\underline{E}} A$  is a bounded lattice. [Simply apply 2.21 and 3.14. However, for the record, suppose that

$$\begin{array}{c} - & \sigma \\ \mathbf{S} & \longrightarrow \mathbf{A} \\ - & \tau \\ - & \tau \end{array}$$

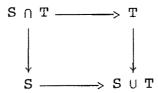
are monomorphisms. Definition:

$$S \land T = S \cap T$$
$$S \lor T = S \cup T.$$

To complete the picture, let

$$\begin{bmatrix} 1 = (id_{A}: A \to A) \\ 0 = (!: \emptyset_{\underline{E}} \to A) \quad (cf. 5.14 \text{ and } 5.16). \end{bmatrix}$$

15.5 REMARK The square



is both a pullback and a pushout.

15.6 THEOREM Let  $A \in Ob \not E$  -- then the bounded lattice  $Sub_{\underline{E}} A$  is a Heyting algebra.

PROOF Given monomorphisms

define T<sup>S</sup> as the equalizer

$$T^{S} \longrightarrow A \xrightarrow{\longrightarrow} \Omega$$

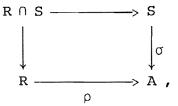
of  $\chi_{\sigma}$  and  $\chi_{\theta}$  (where  $S \cap T \xrightarrow{\theta} A$  is the corner arrow). Let  $R \xrightarrow{\rho} A$  be a monomorphism -- then, from the properties of an equalizer,

$$\mathbb{R} \leq_{\mathbf{A}} \mathbb{T}^{\mathbf{S}} \iff \chi_{\sigma} \circ \rho = \chi_{\theta} \circ \rho.$$

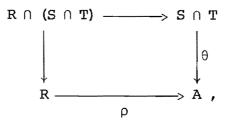
But

$$\chi_{\sigma} \circ \rho = \chi_{\theta} \circ \rho <=> R \cap S \leq_A T.$$

[Note: There is a pullback square



pullback square



the classifying arrow of the monomorphism R  $\cap$  (S  $\cap$  T)  $\rightarrow$  A being  $\chi_{\theta}$   $\circ$   $\rho.]$ 

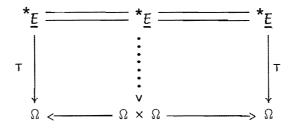
15.7 REMARK If  $(\underline{C}, \tau)$  is a site and if  $\underline{E} = \underline{Sh}_{\tau}(\underline{C})$ , then  $Sub_{\underline{E}} A$  is a locale.

15.8 NOTATION

• Define a monomorphism

$$<\tau, \tau> :*_{\underline{E}} \rightarrow \Omega \times \Omega$$

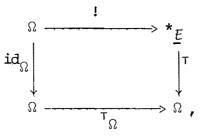
by consideration of the diagram



and denote its classifying arrow by  $\cap$  , thus

• Let  $T_{\Omega}$  be the composition  $\Omega \xrightarrow{!} *_{\underline{E}} \xrightarrow{T} \Omega$  -- then there is a pullback

square

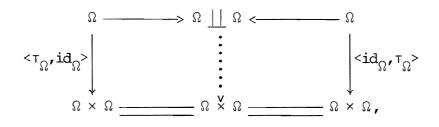


so 
$$\chi_{id_{\Omega}} = \tau_{\Omega}$$
.

• Define a morphism

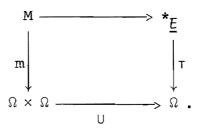
$$< \tau_{\Omega}, id_{\Omega} > \coprod < id_{\Omega} \tau_{\Omega} > :\Omega \coprod \Omega \to \Omega \times \Omega$$

by consideration of the diagram



$$\Omega \coprod \Omega \longrightarrow M \longrightarrow X \times \Omega,$$

and put  $U = \chi_m$ :



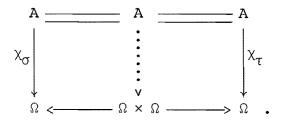
Given monomorphisms

$$\begin{bmatrix} \sigma & \sigma \\ s & \longrightarrow A \\ & \tau & & \\ T & \longrightarrow A, \end{bmatrix}$$

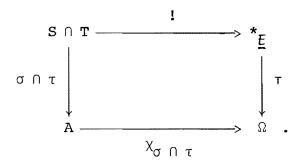
define a morphism

$$\langle \chi_{\alpha}, \chi_{\tau} \rangle : \mathbf{A} \neq \Omega \times \Omega$$

by consideration of the diagram



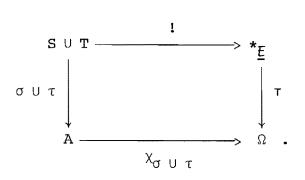
15.9 LEMMA Form the pullback square



Then

$$\chi_{\sigma \cap \tau} = \cap \circ \langle \chi_{\sigma'} \chi_{\tau} \rangle.$$

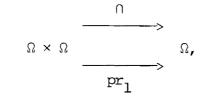
15.10 LEMMA Form the pullback square



Then

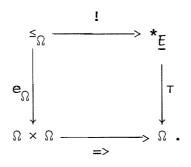
$$\chi_{\sigma \cup \tau} = \cup \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle.$$

15.11 NOTATION Let ( $\boldsymbol{s}_{\Omega}, \boldsymbol{e}_{\Omega})$  be the equalizer of

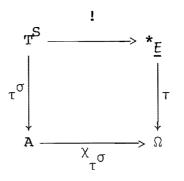


thus

and let =>:  $\Omega$   $\times$   $\Omega$   $\rightarrow$   $\Omega$  be its classifying arrow, thus



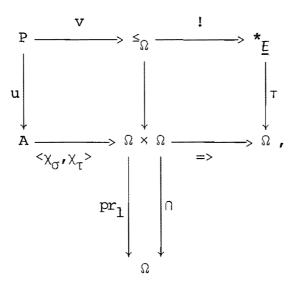
15.12 LEMMA Form the pullback square



Then

$$\chi_{\tau^{\sigma}} = \Rightarrow \circ < \chi_{\sigma'} \chi_{\tau} > \cdot$$

PROOF Consider the diagram



where the squares are pullbacks and

$$pr_1 \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle = \chi_{\sigma}$$
$$\cap \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle = \chi_{\sigma \cap \tau}$$

By construction, the classifying arrow of u is =>  $\circ \langle \chi_{\sigma}, \chi_{\tau} \rangle$  and the claim is that P = T<sup>S</sup> (cf. 15.6) or still, that u is the equalizer of  $\chi_{\sigma}$  and  $\chi_{\sigma\cap\tau}$  or still, that u is the equalizer of pr<sub>1</sub>  $\circ \langle \chi_{\sigma}, \chi_{\tau} \rangle$  and  $\cap \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle$ . But

$$pr_{1} \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle \circ u = pr_{1} \circ e_{\Omega} \circ v$$
$$= \cap \circ e_{\Omega} \circ v$$
$$= \cap \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle \circ u.$$

And if

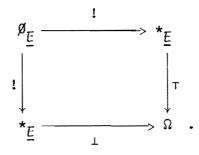
$$pr_1 \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle \circ x = 0 \circ \langle \chi_{\sigma}, \chi_{\tau} \rangle \circ x \quad (x: X \neq P),$$

then

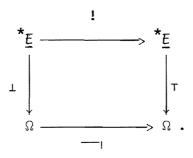
$$\langle \chi_{\sigma}, \chi_{\tau} \rangle \circ \mathbf{x} = \mathbf{e}_{\Omega} \circ \mathbf{y} \quad (\mathbf{y}: \mathbf{X} \neq \leq_{\Omega})$$
  
from which a unique  $\mathbf{z}: \mathbf{X} \neq \mathbf{P}$  such that 
$$\begin{vmatrix} -\mathbf{x} = \mathbf{u} \circ \mathbf{z} \\ \mathbf{y} = \mathbf{v} \circ \mathbf{z}. \end{vmatrix}$$

15.13 NOTATION

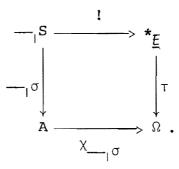
• Denote the classifying arrow of the monomorphism  $\emptyset_{\underline{E}} \xrightarrow{!} *_{\underline{E}}$  by 1. Schematically:



• Denote the classifying arrow of the monomorphism  $*_{\underline{E}} \xrightarrow{\perp} \Omega$  by  $--_{1}$ . Schematically:



15.14 LEMMA Given a monomorphism S  $\xrightarrow{\sigma}$  A, form the pullback square



Then

 $x_{\sigma} = -1 \circ x_{\sigma}$ 

[Note: The monomorphism  $-S \longrightarrow A$  represents the pseudocomplement of [ $\sigma$ ] in the Heyting algebra  $\operatorname{Sub}_{\underline{E}} A$ . E.g.: Take  $A = \Omega$ ,  $S = *_{\underline{E}}, \sigma = \tau$  -- then

$$\chi_{-1} = -1 \circ \chi_{T} = -1 \circ id_{\Omega} = -1 = \chi_{1}$$

Therefore 1 is the pseudocomplement of  $\tau$  in  $Sub_E \Omega$ .]

15.15 DEFINITION A topos  $\underline{E}$  is a <u>boolean</u> topos if for every  $A \in Ob \underline{E}$ , the Heyting algebra  $Sub_{\underline{E}} A$  is a boolean algebra.

15.16 THEOREM A topos  $\underline{E}$  is a boolean topos iff  $\operatorname{Sub}_{\overline{E}} \Omega$  is a boolean algebra.

15.17 REMARK If  $\underline{E}$  is a boolean topos, then for every  $A \in Ob \underline{E}$ , the topos  $\underline{E}/A$  (cf. 15.1) is a boolean topos.

15.18 LEMMA A topos  $\underline{E}$  is a boolean topos iff -  $\circ$  - =  $id_{\Omega}$ .

[To see that the condition is sufficient, consider a monomorphism S  $\xrightarrow{\circ}$  A --

then

$$\chi_{--1} = -1 \circ -1 \circ \chi_{\sigma} = \chi_{\sigma}$$
 (cf. 15.14),

SO

$$- \sigma_{A} \sigma$$
 (cf. 6.11).

Therefore  $Sub_E$  A is a boolean algebra (cf. 8.12 and 8.15).

15.19 LEMMA A topos  $\underline{E}$  is a boolean topos iff the pair

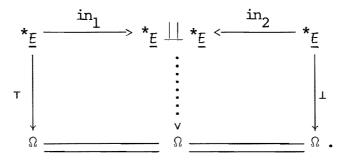
$$(*_{\underline{E}} \perp *_{\underline{E}}, in_1)$$

is a subobject classifier.

[To see that the condition is sufficient, define an isomorphism

$$\mathsf{T} \coprod \mathsf{T} : \mathsf{*}_{\underline{\mathsf{E}}} \coprod \mathsf{*}_{\underline{\mathsf{E}}} \longrightarrow \Omega$$

by consideration of the diagram



15.20 EXAMPLE Let  $\underline{C}$  be a small category -- then the topos  $\hat{\underline{C}}$  is a boolean

topos iff <u>C</u> is a groupoid (in particular, <u>SET</u>  $\approx \hat{\underline{1}}$  is a boolean topos).

[Note: Let G be a group -- then the category of right G-sets is a boolean topos (cf. 7.8).]

15.21 EXAMPLE Let X be a topological space and take  $\underline{Sh}(X)$  per 11.29 — then Sh(X) is a boolean topos iff every open subset of X is closed.

[In fact,  $\underline{Sh}(X)$  is a boolean topos iff  $\forall U \in O(X)$ ,  $U \cup -_{|} U = X$ . But  $-_{|} U =$ int(X - U) (cf. 8.11), thus  $\underline{Sh}(X)$  is a boolean topos iff  $\forall U \in O(X)$ , X - U = int(X - U) or still, iff  $\forall U \in O(X)$ , X - U  $\in O(X)$ .]

[Note: This condition is met if X is discrete, the converse being true if X is in addition  $T_0$ . For if every open set is closed, then every closed set is open, so X: $T_0 \Rightarrow X:T_2$ . But then every subset is a union of closed subsets, hence is a union of open subsets, hence is open.]

15.22 DEFINITION A topos  $\underline{E}$  is said to satisfy the <u>axiom of choice</u> if every epimorphism in  $\underline{E}$  has a section.

15.23 REMARK If  $\underline{E}$  satisfies the axiom of choice, then for every  $A \in Ob \underline{E}$ , the topos  $\underline{E}/A$  (cf. 15.1) satifies the axiom of choice.

15.25 EXAMPLE Let G be a group -- then the category of right G-sets is a boolean topos (cf. 15.20) but it satisfies the axiom of choice iff G is trivial.

[Suppose that G is nontrivial and view G as operating to the right on itself.

Let  $\{*\}$  be the final right G-set -- then G  $\longrightarrow$   $\{*\}$  is an epimorphism but there is no morphism  $\{*\} \rightarrow G$  of right G-sets.]

15.26 EXAMPLE Let [ be a locale and take  $\underline{Sh}(L)$  per 11.29 -- then the following conditions are equivalent.

(1) Sh(L) satisfies the axiom of choice.

(2) Sh(L) is a boolean topos.

(3) L is a boolean algebra.

[Note: Recall that by definition L is a Heyting algebra whose underlying category is complete and cocomplete.

15.27 DEFINITION Let <u>C</u> be a category with a final object  $*_{\underline{C}}$  -- then an object X is said to be <u>subfinal</u> if the arrow X ----->  $*_{\underline{C}}$  is a monomorphism.

15.28 LEMMA Suppose that the topos  $\underline{E}$  satisfies the axiom of choice — then there is a set of subfinal objects of  $*_{\underline{E}}$  which constitute a separating set for  $\underline{E}$ .

## §16. TOPOLOGIES

Let  $\underline{E}$  be a topos,  $(\Omega, \tau)$  its subobject classifier.

16.1 DEFINITION A Lawyere-Tierney topology on  $\underline{E}$  is a morphism  $j:\Omega \rightarrow \Omega$  in  $\underline{E}$  with the following properties.

(1) j ∘ т = т.
(3) j ∘ ∩ = ∩ ∘ (j × j).
(2) j ∘ j = j.

16.2 EXAMPLE  $id_{\Omega}: \Omega \rightarrow \Omega$  is a Lawvere-Tierney topology on  $\underline{E}$ .

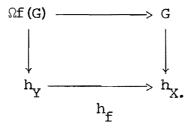
- 16.3 EXAMPLE  $\tau_{\Omega}: \Omega \rightarrow \Omega$  is a Lawvere-Tierney topology on <u>E</u>.
- 16.4 EXAMPLE  $\circ$   $: \Omega \rightarrow \Omega$  is a Lawvere-Tierney topology on  $\underline{E}$ .

16.5 THEOREM Let <u>C</u> be a small category -- then there is a one-to-one correspondence between the set of Grothendieck topologies on <u>C</u> and the set of Lawvere-Tierney topologies on  $\hat{C}$ :

PROOF Recall from 7.7 that

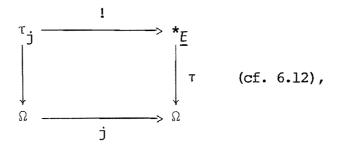
$$\Omega:\underline{\mathbf{C}}^{\mathbf{OP}} \rightarrow \underline{\mathbf{SET}}$$

is defined on an object X by letting  $\Omega X$  be the set of all subfunctors of  $h_X$  and on a morphism  $f:Y \to X$  by letting  $\Omega f:\Omega X \to \Omega Y$  operate via the pullback square



• If  $\tau$  is a Grothendieck topology on <u>C</u>, then  $\tau \in M(\Omega)$  and if  $j_{\tau} = \chi_{\tau}$ , then  $j_{\tau}$  is a Lawvere-Tierney topology on  $\hat{\underline{C}}$ .

• If  $j:\Omega \rightarrow \Omega$  is a Lawvere-Tierney topology on  $\hat{\underline{C}}$  and if



then  $\tau_i$  is a Grothendieck topology on  $\underline{C}$ .

[Note: These constructions are mutually inverse.]

16.6 EXAMPLE Let *L* be a locale --- then  $\Omega x$  is the set of all subfunctors of  $h_x$ or still,  $\Omega x$  is the set of all sieves over x. Let  $x \neq \tau_x$  be the Grothendieck topology  $\tau$  on *L* determined by the sieves that cover x (cf. 10.4) --- then  $j_{\tau}: \Omega \neq \Omega$ is the natural transformation

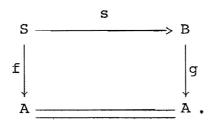
$$(j_{\tau})_{x}:\Omega x \rightarrow \Omega x,$$

where

$$(j_{\tau})_{\mathbf{x}} = \{ \mathbf{y} \le \mathbf{x} : \mathbf{y} = \mathbf{v} \quad (\mathbf{y} \land \mathbf{s}) \}$$

16.7 DEFINITION Suppose that  $j:\Omega \rightarrow \Omega$  is a Lawvere-Tierney topology on  $\underline{E}$ . Let (B,f)  $\in M(A)$  --- then (B,f) is <u>j</u>-dense in A if j  $\circ \chi_{f} = \tau_{A}$ .

16.8 EXAMPLE Let (C,  $\tau$ ) be a site and let G be a subfunctor of  $h_X$  -- then (G,i\_G) is j\_t-dense in  $h_X$  iff  $G\in\tau_X$ .



I.e.: The precomposition map

 $s^{*}:Mor(B,A) \rightarrow Mor(S,A)$ 

is bijective.

16.10 EXAMPLE Since j is idempotent and  $\underline{E}$  is finitely complete, j splits: j = i  $\circ$  r (r  $\circ$  i = id), where

But  $\Omega$  is injective (cf. 14.18), thus  $\Omega_j$  is injective (being a retract of  $\Omega$ ), and the claim is that  $\Omega_j$  is a j-sheaf. In fact, the existence of the relevant liftings is then immediate which leaves the uniqueness... .

Write  $\underline{Sh}_{1}(\underline{E})$  for the full subcategory of  $\underline{E}$  whose objects are the j-sheaves.

16.11 EXAMPLE Take  $j = id_{\Omega} - then \underline{Sh}_{i}(\underline{E}) = \underline{E}$ .

16.12 EXAMPLE Take  $j = \tau_{\Omega} - then \underline{Sh}_{j}(\underline{E})$  is the full subcategory of  $\underline{E}$  whose objects are the final objects.

16.13 THEOREM Fix a Lawvere-Tierney topology  $j:\Omega \rightarrow \Omega$  on  $\underline{E}$  — then the inclusion  $i_j:\underline{Sh}_j(\underline{E}) \rightarrow \underline{E}$  admits a left adjoint  $\underline{a}_j:\underline{E} \rightarrow \underline{Sh}_j(\underline{E})$  that preserves finite limits.

<u>N.B.</u> Let  $W_j$  be the class of morphisms in  $\underline{E}$  rendered invertible by  $\underline{a}_j$  -- then the localization  $W_j^{-1}\underline{E}$  is equivalent to  $\underline{Sh}_j(\underline{E})$  (cf. 11.20).

16.14 LEMMA Let  $f:B \rightarrow A$  be a monomorphism -- then (B,f) is j-dense in A iff  $\underline{a}_{j}f$  is an isomorphism.

16.15 SCHOLIUM Let <u>C</u> be a small category. Suppose that  $j:\Omega \rightarrow \Omega$  is a Lawvere-Tierney topology on  $\hat{\underline{C}}$  and let  $\tau_j$  be the associated Grothendieck topology on <u>C</u> (cf. 16.5) -- then

$$\underline{\operatorname{Sh}}_{j}(\widehat{\underline{C}}) = \underline{\operatorname{Sh}}_{\tau_{j}}(\underline{C})$$

[Viewing  $\underline{Sh}_{j}(\hat{C})$  as an element  $\underline{S}$  of  $\underline{S}_{\underline{C}}$  (cf. 11.9), introduce  $\tau \in \tau_{\underline{C}}$  per 11.4, thus  $\tau_{\underline{X}}$  is the set of those subfunctors  $G \xrightarrow{i_{\underline{G}}} h_{\underline{X}}$  such that  $\underline{a}_{j}i_{\underline{G}}$  is an isomorphism or still, those subfunctors  $G \xrightarrow{i_{\underline{G}}} h_{\underline{X}}$  such that  $(G, i_{\underline{G}})$  is j-dense in  $h_{\underline{X}}$  (cf. 16.14). On the other hand, a subfunctor  $G \xrightarrow{i_G} h_X$  is  $j_{\tau_j}$ -dense in  $h_X$  iff  $G \in (\tau_j)_X$ (cf. 16.8). But  $j_{\tau_j} = j$ , hence  $\tau_X = (\tau_j)_X$ , and therefore  $\tau = \tau_j$ . Since  $\underline{Sh}_j(\hat{\underline{C}}) = \underline{Sh}_{\tau}(\underline{C})$  (cf. 12.13),

it follows that

$$\underline{\mathrm{Sh}}_{j}(\underline{\widehat{C}}) = \underline{\mathrm{Sh}}_{\tau_{j}}(\underline{C}).$$

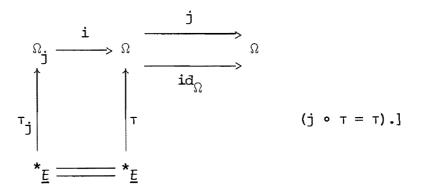
[Note: Consequently,  $\forall \ \tau \in \tau_{C'}$ 

$$\underline{\mathrm{Sh}}_{\tau}(\underline{\mathrm{C}}) = \underline{\mathrm{Sh}}_{\tau_{j_{\tau}}}(\underline{\mathrm{C}}) = \underline{\mathrm{Sh}}_{j_{\tau}}(\hat{\underline{\mathrm{C}}}).$$

16.16 REMARK Let  $\underline{E}$  be a topos -- then it can be shown that the Lawvere-Tierney topologies on  $\underline{E}$  are in a one-to-one correspondence with the reflective subcategories of  $\underline{E}$  whose reflector preserves finite limits (cf. 12.13).

16.17 THEOREM Fix a Lawvere-Tierney topology  $j:\Omega \rightarrow \Omega$  on  $\underline{E}$  — then  $\underline{Sh}_{j}(\underline{E})$  is a topos.

[Note: The pair  $(\Omega_j, \tau_j)$  is a subobject classifier for  $\underline{Sh}_j(\underline{E})$ . Here (cf. 16.10)



16.18 EXAMPLE Take j = -1  $\circ -1$  then  $\underline{Sh}$   $\underline{(\underline{E})}$  is a boolean topos.

## §17. GEOMETRIC MORPHISMS

Let C, D be finitely complete categories.

17.1 DEFINITION A geometric morphism  $f: \underline{C} \rightarrow \underline{D}$  is a pair  $(f^*, f_*)$ , where

$$f^*:\underline{D} \rightarrow \underline{C}$$

$$f_*:\underline{C} \rightarrow \underline{D}$$

are functors and

[Note: The second condition on f\* is automatic if f\* is a right adjoint.]

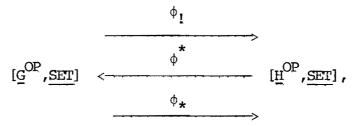
17.2 EXAMPLE Let X,Y be topological spaces and let  $f:X \rightarrow Y$  be a continuous function -- then f induces a geometric morphism  $f:\underline{Sh}(X) \rightarrow \underline{Sh}(Y)$ , where  $f_*:\underline{Sh}(X) \rightarrow \underline{Sh}(Y)$  is "direct image" and  $f^*:\underline{Sh}(Y) \rightarrow \underline{Sh}(X)$  is "inverse image".

[Note: Here  $\underline{Sh}(X)$ ,  $\underline{Sh}(Y)$  are taken per the canonical Grothendieck topology on O(X), O(Y) (cf. 11.29).]

17.3 EXAMPLE Let G,H be groups and let  $\phi: G \rightarrow H$  be a homomorphism -- then  $\phi$  induces a geometric morphism  $\phi$  from right G-sets to right H-sets, i.e.,

$$\phi: [\underline{G}^{OP}, \underline{SET}] \rightarrow [\underline{H}^{OP}, \underline{SET}] \quad (cf. 7.8).$$

[There are three functors



where

$$\phi_1 \longrightarrow \phi^* \longrightarrow \phi_*$$

• Definition of  $\phi^*$ : Given a right H-set Y,  $\phi^*(Y) = Y$  with the right G-action induced by  $\phi$ .

• Definition of  $\phi_*$ : Given a right G-set X,  $\phi_*(X) = Hom_G(H,X)$ , the set of G-equivariant functions  $H \rightarrow X$ .

• Definition of  $\phi_{!}$ : Given a right G-set X,  $\phi_{!}(X) = X \bigotimes_{G} H$ , the cartesian product X × H modulo the equivalence relation  $(x \cdot g, h) \sim (x, \phi(g) \cdot h)$ .]

17.4 EXAMPLE Take 
$$C = SISET$$
,  $D = CGH$  and consider the adjoint pair (| |,sin):  
| :SISET  $\rightarrow CGH$   
sin:CGH  $\rightarrow$  SISET.

Then | | preserves finite limits, hence (| |,sin) is a geometric morphism SISET  $\rightarrow$  CGH.

17.5 EXAMPLE Let  $\underline{E}$  be a topos that has arbitrary copowers of  $*_{\underline{E}}$ . Define a functor  $\Gamma_*:\underline{E} \rightarrow \underline{SET}$  by stipulating that

$$\Gamma_* A = Mor(*_{\underline{E}}, A)$$

and define a functor  $\Gamma^*:\underline{SET} \rightarrow \underline{E}$  by stipulating that

$$\Gamma^*S = \coprod_{S \in S} *\underline{E}$$

Then  $(\Gamma^*, \Gamma_*)$  is an adjoint pair and  $\Gamma^*$  preserves finite limits (cf. 18.2). Therefore  $(\Gamma^*, \Gamma_*)$  is a geometric morphism  $\underline{E} \rightarrow \underline{SET}$ . 17.6 EXAMPLE Let  $(\underline{C}, \tau)$  be a site -- then the adjoint pair  $(\underline{a}_{\tau}, \iota_{\tau})$  is a geometric morphism  $\underline{Sh}_{\tau}(\underline{C}) \rightarrow \hat{\underline{C}}$  (cf. 11.14).

17.7 EXAMPLE Let  $\underline{E}$  be a topos,  $j:\Omega \rightarrow \Omega$  a Lawvere-Tierney topology on  $\underline{E}$  -- then the adjoint pair  $(\underline{a}_{j}, \iota_{j})$  is a geometric morphism  $\underline{Sh}_{j}(\underline{E}) \rightarrow \underline{E}$ .

17.8 EXAMPLE Let  $\underline{E}$  be a topos. Suppose that  $f:A \rightarrow B$  is a morphism in  $\underline{E}$  -- then  $f^*:\underline{E}/B \rightarrow \underline{E}/A$  has a left adjoint  $f_1:\underline{E}/A \rightarrow \underline{E}/B$  and a right adjoint  $f_*:\underline{E}/A \rightarrow \underline{E}/B$ (cf. 15.3). Therefore the adjoint pair ( $f^*, f_*$ ) is a geometric morphism  $\underline{E}/A \rightarrow \underline{E}/B$ .

17.9 EXAMPLE Let  $\underline{I},\underline{J}$  be small categories and let  $\underline{S}$  be a complete and cocomplete category. Suppose that  $F:\underline{I} \rightarrow \underline{J}$  is a functor -- then by the theory of Kan extensions,

```
F^*:[J,S] \rightarrow [I,S]
```

has a right adjoint

$$F_*:[\underline{I},\underline{S}] \rightarrow [\underline{J},\underline{S}]$$

and a left adjoint

 $F_1:[\underline{I},\underline{S}] \rightarrow [\underline{J},\underline{S}].$ 

Therefore F\* preserves limits and the adjoint pair (F\*,F<sub>\*</sub>) is a geometric morphism  $[\underline{I},\underline{S}] \rightarrow [\underline{J},\underline{S}]$ .

17.10 EXAMPLE Let L, M be locales and let  $f: L \to M$  be a localic arrow (cf. 9.6) -then f induces a geometric morphism  $\underline{Sh}(L) \to \underline{Sh}(M)$  (taken per the canonical Grothendieck topology on L, M (cf. 11.29)), call it f to forgo any possibility of confusion.

[Proceed as follows. The functor  $f^*:M \to L$  gives rise to a functor  $f^*:\hat{L} \to \hat{M}$ (technically,  $f^{**} = ((f^*)^{OP})^*$ ), which then restricts to a functor  $f_*:\underline{Sh}(L) \to \underline{Sh}(M)$ . On the other hand, f\*\* has a left adjoint  $f_{\underline{1}}^*: \widehat{M} \to \widehat{L}$  (take  $\underline{S} = \underline{SET}$  in 17.9). Accordingly, denote the composite

$$\underline{\operatorname{Sh}}(M) \xrightarrow{\iota_{\tau}} \widehat{M} \xrightarrow{f_{!}^{\star}} \widehat{L} \xrightarrow{\underline{a}_{\tau}} \underline{\operatorname{Sh}}(L)$$

by  $f^{*}$  -- then  $f^{*}$  is a left adjoint for  $f_{\star}.$  Proof: Given  $F\in Ob\ \underline{Sh}(L)$ ,  $G\in Ob\ \underline{Sh}(M)$ ,

Mor (f\*G,F) 
$$\approx$$
 Mor ( $\underline{a}_{T} f_{!}^{*} \iota_{T} G, F$ )  
 $\approx$  Mor ( $f_{!}^{*} \iota_{T} G, \iota_{T} F$ )  
 $\approx$  Mor ( $\iota_{T} G, f^{**} \iota_{T} F$ )  
 $\approx$  Mor ( $\iota_{T} G, \iota_{T} f_{*} F$ )  
 $\approx$  Mor ( $G, f_{*} F$ ).

The final point is that f\* preserves finite limits. Since this is true of  $\iota_{\tau}$  and  $\underline{a}_{\tau}$ , matters reduce to verifying it for f<sup>\*</sup><sub>1</sub> (which is not an a priori property of Kan extensions...).]

17.11 DEFINITION Let  $f,g:\underline{C} \rightarrow \underline{D}$  be geometric morphisms --- then a geometric transformation  $\xi:f \rightarrow g$  is a natural transformation  $f^* \rightarrow g^*$ .

[Note: Since

natural transformations  $f^* \rightarrow g^*$  correspond bijectively to natural transformations  $g_* \rightarrow f_{*}$ .

## §18. GROTHENDIECK TOPOSES

Let  $\underline{E}$  be a topos.

18.1 DEFINITION  $\underline{E}$  is said to be defined over <u>SET</u> if  $\underline{E}$  admits a geometric morphism  $\underline{E} \rightarrow \underline{SET}$ .

18.2 THEOREM  $\underline{E}$  is defined over  $\underline{SET}$  iff  $\underline{E}$  has arbitrary copowers of  $*_{\underline{E}}$ .

PROOF If  $f:\underline{E} \rightarrow \underline{SET}$  is a geometric morphism, then f\* preserves finite limits, thus in particular f\*\*  $\approx *_{\overline{E}}$ . Therefore, since f\* preserves colimits, for any set S,

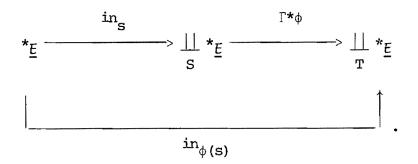
Turning to the converse, define  $\Gamma_*: \underline{E} \rightarrow \underline{SET}$  by

$$\Gamma_{\star}A = Mor(\star_{\underline{E}}, A)$$

and define  $\Gamma^*:\underline{\text{SET}} \rightarrow \underline{E}$  by

$$\Gamma^*S = \coprod_{S} *_{\underline{E}} (\Gamma^*\emptyset \approx \emptyset_{\underline{E}}).$$

Here  $\Gamma^{*}\phi$  ( $\phi: S \rightarrow T$ ) is the unique arrow in  $\underline{E}$  such that  $\forall s \in S$ ,  $\Gamma^{*}\phi \circ in_{s} = in_{\phi(s)}$ :



It is clear that  $(\Gamma^*, \Gamma_*)$  is an adjoint pair, so the issue is whether  $\Gamma^*$  preserves

finite limits and for this one need only show that  $\Gamma^*$  preserves finite products and equalizers.

• By construction,  $\Gamma^*$  sends final objects to final objects. Suppose now that S and T are sets. Distinguish two cases: (1) S is empty or T is empty; (2) S is not empty and T is not empty. If S is empty, then  $S \times T = \emptyset \times T = \emptyset$ and  $\Gamma^*(\emptyset \times T) = \Gamma^*\emptyset \approx \emptyset_{\underline{E}}$ , while  $\Gamma^*\emptyset \times \Gamma^*T \approx \emptyset_{\underline{E}} \times \Gamma^*T \approx \emptyset_{\underline{E}}$  (cf. 5.13 and 5.14). If neither S nor T is empty, then

$$\Gamma^*(S \times T) = \underbrace{\prod}_{S \times T} *\underline{E}$$

On the other hand,

$$\Gamma^{*}S \times \Gamma^{*}T = \underbrace{\prod}_{S} *\underline{E} \times \underbrace{\prod}_{T} *\underline{E}$$
$$\approx \underbrace{\prod}_{S} (\underbrace{\prod}_{T} *\underline{E})$$
$$\approx \underbrace{\prod}_{S} \times \underline{T} *\underline{E}$$

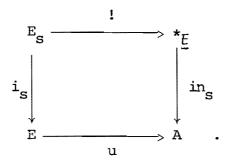
• Let  $S \xrightarrow{\phi} T$  be arrows in <u>SET</u> and let  $K = eq(\phi, \psi)$ , so  $K \xrightarrow{\kappa} S \xrightarrow{\phi} T$ .

Put A =  $\Gamma^*S$ , B =  $\Gamma^*T$ , C =  $\Gamma^*K$ , f =  $\Gamma^*\phi$ , g =  $\Gamma^*\psi$ , k =  $\Gamma^*\kappa$  -- then the claim is that

$$C \xrightarrow{k} A \xrightarrow{f} B$$

is an equalizer in E. Thus consider a morphism u:E  $\rightarrow$  A and  $\forall$  s  $\in$  S, define E by s

the pullback square



Then  $i_s$  is a monomorphism (this being the case of  $i_s$ ) and since  $\{\star_{\underline{E}} \xrightarrow{i_s} A: s \in S\}$ is a coproduct diagram in  $\underline{E}$ , the same is true of  $\{E_s \xrightarrow{i_s} E: s \in S\}$  (cf. 14.13). I.e.:

$$\mathbf{E} \approx \coprod_{\mathbf{s} \in \mathbf{S}} \mathbf{E}_{\mathbf{s}}.$$

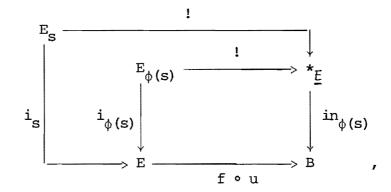
If u equalizes f and g (=> f  $\circ$  u = g  $\circ$  u), then this time

$$\mathbf{E} \approx \frac{||}{\mathbf{t} \in \mathbf{T}} \mathbf{E}_{\mathbf{t}}$$

And there are monomorphisms

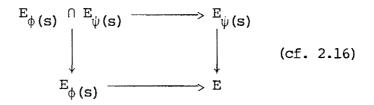
$$\begin{array}{ccc} & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ &$$

E.g.: Given the situation

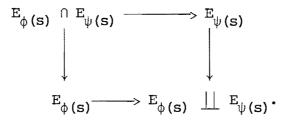


$$f \circ u \circ i_s = f \circ i_s \circ ! = i_{\phi(s)} \circ !,$$

from which a unique arrow  $\lambda_s: E_s \to E_{\phi(s)}$  such that  $i_s = i_{\phi(s)} \circ \lambda_s$ . Moreover,  $\lambda_s$  is a monomorphism (because  $i_s$  is a monomorphism). Proceeding, the intersection  $E_{\phi(s)} \cap E_{\psi(s)}$  is officially defined by the pullback square



but the answer is the same if instead we use the pullback square



The data provides us with a monomorphism

$$E_{s} \stackrel{\rightarrow}{\rightarrow} E_{\phi(s)} \cap E_{\psi(s)}$$
 (s  $\in$  S)

and if  $\phi(s) \neq \psi(s)$ , then  $E_{\phi(s)} \cap E_{\psi(s)} \approx \emptyset_{\underline{E}}$ , hence  $E_{s} \approx \emptyset_{\underline{E}}$ . Consequently,

$$E \approx \prod_{s \in K} E_s$$

and  $u: E \rightarrow A$  factors through k (uniquely).

[Note: The geometric morphism  $(\Gamma^*, \Gamma_*)$  extends to a geometric morphism

$$\underline{\text{SIE}} = [\underline{\Delta}^{\text{OP}}, \underline{E}] \rightarrow [\underline{\Delta}^{\text{OP}}, \underline{\text{SET}}] = \underline{\text{SISET}}$$

denoted by the same symbol.

Define

$$\Gamma^*: SISET \rightarrow SIE$$

by

$$(\Gamma^{*K})_{n} = \frac{\prod}{K_{n}} * \underline{\underline{E}}$$

• Define

$$\Gamma_*:\underline{SIE} \rightarrow \underline{SISET}$$

by

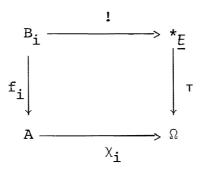
$$(\Gamma_*X)_n = \operatorname{Mor}(*_{\underline{E}}, X_n).$$

18.3 LEMMA Suppose that  $\underline{E}$  has arbitrary copowers of  $*_{\underline{E}}$ . Let  $A \in Ob \underline{E}$  and let  $\{B_{\underline{i}} \xrightarrow{f_{\underline{i}}} A: i \in I\} \subset M(A) \longrightarrow then \underset{i \in I}{\coprod} B_{\underline{i}} exists.$ PROOF First of all, the copower  $\coprod A$  exists, In fact,

$$A \times \coprod_{I} *_{\underline{E}} \approx \coprod_{I} A \times *_{\underline{E}} \approx \coprod_{I} A.$$

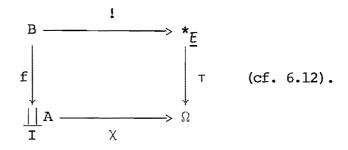
Ι

Next, for each  $i \in I$ , let  $\chi_i$  be the classifying arrow of  $(B_i, f_i)$  in A:

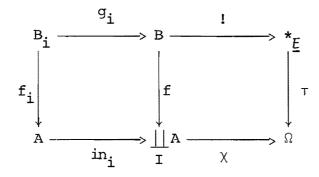


Determine  $\chi: \coprod A \rightarrow \Omega$  via the  $\chi_i$  ( $\chi \circ in_i = \chi_i$ ) and form the pullback square I

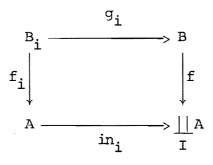
5.



Then for each  $i \in I$ , there is a unique arrow  $g_i:B_i \rightarrow B$  such that the diagram



commutes (so  $g_i$  is necessarily a monomorphism). Inspection of the rectangle and the right hand square then implies that the left hand square



is a pullback. Since {A  $\longrightarrow$   $\coprod$  A:i  $\in$  I} is a coproduct diagram, the same is I

true of  $\{B_{i} \xrightarrow{g_{i}} B: i \in I\}$  (cf. 14.13), hence  $\coprod_{i \in I} B_{i}$  exists.

18.4 APPLICATION Under the preceding hypotheses, the copower  $\coprod_I$  A exists (sic), as does the power  $\coprod_T$  A:

$$A \xrightarrow{\prod} {}^{*}\underline{E} = \prod_{I} {}^{$$

18.5 EXAMPLE Suppose that  $\underline{E}$  has arbitrary copowers of  $*_{\underline{E}}$  — then it does not follow that  $\underline{E}$  has coproducts.

[Let  $\underline{E}$  be the full subcategory of  $[Z^{OP}, \underline{SET}]$  whose objects are the right Z-sets S with the property that multiplication by n is the identity on S for some positive integer n -- then  $\underline{E}$  is a topos and has arbitrary copowers of  $*_{\underline{E}}$  but  $\underline{E}$  does not have coproducts (e.g., one cannot construct  $\underset{n\geq 1}{\coprod} Z/nZ$ ).]

18.6 DEFINITION Let  $\underline{E}$  be a topos -- then  $\underline{E}$  is said to be a <u>Grothendieck topos</u> if  $\underline{E}$  is cocomplete and has a separating set.

[Note: In general, a cocomplete topos need not admit a separating set.]

18.7 EXAMPLE Let  $(\underline{C}, \tau)$  be a site — then the topos  $\underline{Sh}_{\tau}(\underline{C})$  (cf. 14.4) is a Grothendieck topos (cf. 13.1 and 13.15).

18.8 DEFINITION Let  $\underline{E}$  be a topos -- then a <u>subseparator</u> is an object  $\Gamma$  in  $\underline{E}$  with the property that  $M(\Gamma)$  contains a separating set.

18.9 LEMMA Suppose that  $\underline{E}$  is a Grothendieck topos -- then  $\underline{E}$  has a subseparator. PROOF If U is a separating set, let

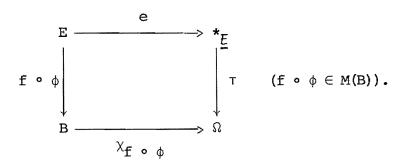
$$\Gamma = \coprod_{U \in \mathcal{U}} U.$$

Then  $\Gamma$  is a subseparator.

18.10 RAPPEL An object X in a category <u>C</u> is a <u>coseparator</u> if for every pair  $f,g:A \rightarrow B$  of distinct morphisms in <u>C</u>, there exists a morphism  $\sigma:B \rightarrow X$  such that  $\sigma \circ f \neq \sigma \circ g$ .

18.11 LEMMA Let  $\underline{E}$  be a topos. Assume:  $\Gamma$  is a subseparator -- then  $\Omega^{\Gamma}$  is a coseparator.

[Consider the simplest possibility, viz. when  $\Gamma = *_{\underline{E}} \iff \Omega = \infty^* \cong \Omega$ ). Let  $f,g:A \rightarrow B$  be morphisms such that for any  $\sigma:B \rightarrow \Omega$ ,  $\sigma \circ f = \sigma \circ g$ . Claim: f = g. To see this, let  $e:E \rightarrow *_{\underline{E}}$  be a subfinal object and given a morphism  $\phi:E \rightarrow A$ , pass to the pullback square



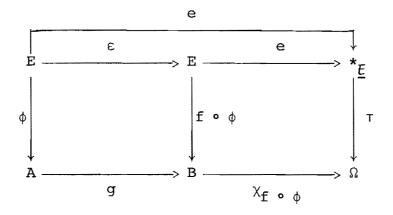
Since  $\chi_{f \circ \varphi} \in Mor(B, \Omega)$ , from the assumptions

$$\chi_{f \circ \phi} \circ f = \chi_{f \circ \phi} \circ g,$$

thus

$$\tau_{\mathbf{E}} = \chi_{\mathbf{f} \circ \phi} \circ \mathbf{f} \circ \phi = \chi_{\mathbf{f} \circ \phi} \circ \mathbf{g} \circ \phi,$$

so there exists a unique morphism  $\varepsilon: E \rightarrow E$  rendering the diagram



commutative. But Mor(E,E) =  $\{id_E\}$ , hence  $\varepsilon = id_E$ , which implies that  $f \circ \phi = g \circ \phi$ . Therefore f = g (E and  $\phi$  being arbitrary).]

[Note: In general,  $\Omega$  is not a coseparator but if  $\Omega$  is a coseparator, it does not follow that  $*_E$  is a subseparator.]

18.12 REMARK Let  $\underline{E}$  be a Grothendieck topos -- then  $\underline{E}$  satisfies the axiom of choice iff  $\underline{E}$  is a boolean topos and  $*_{\underline{E}}$  is a subseparator.

[E.g.: If  $\underline{E}$  satisfies the axiom of choice, then  $\underline{E}$  is a boolean topos (cf. 15.24) and  $*_{\underline{E}}$  is a subseparator (cf. 15.28).]

18.13 LEMMA A topos  $\underline{E}$  is a Grothendicek topos iff it is defined over <u>SET</u> and has a subseparator.

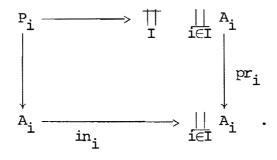
PROOF That the conditions are necessary is implied by 18.2 and 18.9. As for the sufficiency, since a topos is finitely cocomplete (cf. 14.5), to finish the proof it suffices to show that  $\underline{E}$  has coproducts. For this purpose, note first that  $\underline{E}$  has arbitrary powers of objects (cf. 18.4) and has a coseparator, call if X (cf. 18.11). Suppose now that  $\{A_i : i \in I\}$  is a set-indexed collection of objects of  $\underline{E}$ . Choose a set S such that  $\forall i \in I$ , Mor $(A_i, X) \in S$  and put  $B = \prod_{S} X$  — then the monomorphism

$$A_{i} \longrightarrow \prod X$$
Mor (A<sub>i</sub>, X)

leads to a monomorphism  $A_i \rightarrow B$ . Therefore  $\underset{i \in I}{\underset{I}{\Vdash}} A_i$  can be constructed as an element of  $M(\underset{I}{\underset{I}{\sqcup}} B)$ .

18.14 LEMMA Every Grothendieck topos  $\underline{E}$  is complete.

PROOF Given a set-indexed collection of objects  $\{A_i: i \in I\}$  of  $\underline{E}$ , define  $\underline{P}_i$  by the pullback square



Then

$$\bigcap_{i\in I} P_i = \prod_{i\in I} A_i.$$

18.15 LEMMA If  $\underline{E}$  is a Grothendieck topos, then  $\forall A \in Ob \underline{E}$ , the topos  $\underline{E}/A$  (cf. 15.1) is a Grothendieck topos.

PROOF As a category,  $\underline{E}/A$  is cocomplete ( $\underline{E}$  being cocomplete). This said, let  $U = \{U\}$  be a separating set (per  $\underline{E}$ ) and put

$$U/A = \{f: U \rightarrow A, U \in U\}.$$

Then U/A is a separating set (per E/A).

18.16 THEOREM If  $\underline{E}$  is a cocomplete topos, then for any small category  $\underline{I}$ , the functor category  $[\underline{I}, \underline{E}]$  is a cocomplete topos.

[Note: If  $\underline{E}$  is a topos (hence finitely cocomplete (cf. 14.5), then for any finite category  $\underline{I}$ , the functor category  $[\underline{I}, \underline{E}]$  is a topos.]

18.17 LEMMA If  $\underline{E}$  is a Grothendieck topos, then for any small category  $\underline{I}$ , the functor category  $[\underline{I}, \underline{E}]$  is a Grothendieck topos.

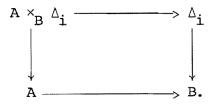
**PROOF** If  $U = \{U\}$  is a separating set for E, then

$$\{F_{U,i}: U \in U, i \in Ob \underline{I}\}$$

is a separating set for  $[\underline{I}, \underline{E}]$ , where

$$F_{U,i}(j) = \coprod_{Mor(i,j)} U \quad (j \in Ob \underline{I}).$$

Let  $\underline{E}$  be a Grothendieck topos,  $\underline{I}$  a small category, and  $\Delta: \underline{I} \rightarrow \underline{E}$  a functor. Put  $B = \operatorname{colim}_{\underline{I}} \Delta$  and let  $A \rightarrow B$  be a morphism --- then  $\forall i \in Ob \underline{I}$ , there is a pullback square



18.18 LEMMA The canonical arrow

$$\operatorname{colim}_{\underline{I}} (i \to A \times_{B} \Delta_{i}) \to A$$

is an isomorphism.

Given a set  $\{X_i : i \in I\}$  of objects in  $\underline{E}$ , put

$$X = \prod_{i \in I} X_i$$
.

18.19 EXAMPLE Let  $Y \rightarrow X$  be a morphism -- then the canonical arrow

$$\lim_{\mathbf{i}\in\mathbf{I}} x_{\mathbf{i}} \times_X Y \neq Y$$

is an isomorphism.

18.20 EXAMPLE Let  $Y \in Ob \underline{E}$  -- then

$$\lim_{i \in I} (X_i \times Y) \approx X \times Y \quad (cf. 5.8).$$

[This is a special case of 18.19: Replace Y by X  $\times$  Y, consider the projection X  $\times$  Y  $\rightarrow$  X, and note that

$$X_i \times (X \times Y) \approx X_i \times Y.$$

The following result is Giraud's "recognition principle".

18.21 THEOREM Suppose that  $\underline{E}$  is a Grothendieck topos — then there exists a site ( $\underline{C}, \tau$ ) such that  $\underline{E}$  is equivalent to  $\underline{Sh}_{T}(\underline{C})$ .

[Here is a sketch of the proof. Take for <u>C</u> the small full subcategory of <u>E</u> whose objects are a separating set. Given  $X \in Ob \underline{C}$ , let  $\tau_X$  be the set of subfunctors  $G \rightarrow h_X$  such that the arrow

$$\underbrace{\prod}_{X \in Ob \ C} \underbrace{\prod}_{g \in GX} Y \to X$$

is an epimorphism -- then the assignment  $X \rightarrow \tau_X$  defines a Grothendieck topology on <u>C</u>.

Next,  $\forall A \in Ob \underline{E}$ , the presheaf  $h_A | \underline{C}^{OP}$  is a  $\tau$ -sheaf  $(h_A = Mor(-,A))$  and the specification  $A \rightarrow h_A | \underline{C}^{OP}$  defines a functor  $\underline{E} \rightarrow \underline{Sh}_{\tau}(\underline{C})$  which at length can be shown to be an equivalence of categories.]

[Note: Making a simple expansion, one can always arrange that  $\underline{C}$  is finitely complete.]

13.22 REMARK The Grothendieck topology figuring in 13.21 is subcanonical. However, it is possible to enlarge <u>C</u> so as to replace "subcanonical" by "canonical". Thus let  $\mathcal{U} = \{U\}$  be a separating set and for each  $U \in \mathcal{U}$ , let  $\{U_i : i \in I_U\}$  be a set of representatives for  $\operatorname{Sub}_{\underline{E}} U$  ( $\underline{E}$  is wellpowered (cf. 6.13)). Perform the construction of 18.21 on the full subcategory of  $\underline{E}$  generated by the  $U_i$  ( $i \in I_U$ ,  $U \in \mathcal{U}$ ) -then the resulting " $\tau$ " is canonical.

18.23 LEMMA Every Grothendieck topos E is presentable (cf. 13.16).

18.24 LEMMA Every Grothendieck topos E is cowellpowered (cf. 13.17).

18.25 CRITERION Let  $\underline{E}, \underline{F}$  be Grothendieck toposes — then any functor  $\underline{F} \rightarrow \underline{E}$ which preserves colimits has a right adjoint  $\underline{E} \rightarrow \underline{F}$ .

[The categories involved are cocomplete, cowellpowered, and have separating sets. Now quote the appropriate "adjoint functor theorem".]

18.26 NOTATION Given Grothendieck toposes  $\underline{E}, \underline{F}$ , write  $[\underline{E}, \underline{F}]_{gro}$  for the metacategory whose objects are the geometric morphisms  $\underline{E} \rightarrow \underline{F}$  and whose morphisms are the geometric transformations. 18.27 LEMMA Let  $\underline{E}, \underline{F}$  be Grothendieck toposes -- then  $[\underline{E}, \underline{F}]_{geo}$  is a category. [In other words, if  $f, g: \underline{E} \rightarrow \underline{F}$  are geometric morphisms, then there is but a set of natural transformations  $f^* \rightarrow g^*$ .]

18.28 LEMMA Let  $\underline{E}, \underline{F}$  be Grothendieck toposes and suppose that  $f:\underline{E} \rightarrow \underline{F}$  is a geometric morphism -- then the following conditions are equivalent.

- (1) f\* is faithful;
- (2) f\* reflects isomorphisms;
- (3) f\* reflects epimorphisms;
- (4) f\* reflects monomorphisms.

18.29 THEOREM Let  $\underline{E}$  be a Grothendieck topos -- then there is a Grothendieck topos  $\underline{B}$  satisfying the axiom of choice and a geometric morphism  $f:\underline{B} \rightarrow \underline{E}$  such that  $f^*$  is faithful.

## §19. POINTS

Let  $\underline{E}$  be a Grothendieck topos.

19.1 DEFINITION A point of  $\underline{E}$  is a geometric morphism  $f:\underline{SET} \rightarrow \underline{E}$ .

<u>N.B.</u> Alternatively, a point of  $\underline{E}$  is a functor  $p:\underline{E} \rightarrow \underline{SET}$  which preserves colimits and finite limits (cf. 18.15).

19.2 EXAMPLE Let X be a nonempty topological space — then each  $x \in X$  determines a point  $p_{X}:\underline{Sh}(X) \rightarrow \underline{SET}$ , where  $\underline{Sh}(X)$  is computed per the canonical Grothendieck topology on O(X).

[Apply 17.2 to the continuous function  $\{*\} \xrightarrow{X} X$ , thus  $p_X: \underline{Sh}(X) \rightarrow \underline{Sh}(\{*\}) = \underline{SET}$  sends F to its stalk  $F_X$  at x.]

[Note: If X is sober, then this construction is exhaustive, i.e., up to natural isomorphism, every point  $\underline{Sh}(X) \rightarrow \underline{SET}$  is a "p<sub>x</sub>". In general, the full subcategory of <u>TOP</u> whose objects are the sober topological spaces is reflective with arrow of reflection  $X \rightarrow \text{sob } X$ . But

 $O(X) \iff O(sob X)$  (cf. 9.26),

hence

$$Sh(X) \iff Sh(sob X)$$
.

Therefore the points of sob X "parameterize" the points of  $\underline{Sh}(X)$ : If  $f:\underline{SET} \rightarrow \underline{Sh}(X)$ is a point, let U be the union of all open  $V \subset X$  such that  $f*V = \emptyset$  -- then X - U is an irreducible closed subset of X, thus is a point of sob X. Conversely, ....]

19.3 REMARK If X is empty, then Sh(X) is the full subcategory of SET whose

1.

objects are the final objects so there is no functor  $p:\underline{Sh}(X) \rightarrow \underline{SET}$  which preserves colimits and finite limits. Proof: All objects in  $\underline{Sh}(X)$  are both initial and final.

19.4 EXAMPLE Let X be a nonempty Hausdorff topological space in which no singletons are open -- then

$$\frac{\mathrm{Sh}}{\mathrm{m}} = (\mathrm{Sh}(\mathrm{X})) \quad (\mathrm{cf. 16.18})$$

has no points.

19.5 NOTATION Given a Grothendieck topos E, let

$$\underline{PT}(\underline{E}) = [\underline{SET}, \underline{E}]_{qeo} \quad (cf. 18.26).$$

N.B. PT(E) is a category (cf. 18.27).

[Note: It is not necessarily true that  $\underline{PT}(\underline{E})$  is equivalent to a small category (e.g., there are  $\underline{E}$  for which  $\underline{PT}(\underline{E})$  is equivalent to SET).]

19.6 RAPPEL Let <u>C</u> be a small category — then the functor  $Y_{\underline{C}}^*: [\hat{\underline{C}}, \underline{SET}] \rightarrow [\underline{C}, \underline{SET}]$ has a left adjoint that sends  $T \in Ob[\underline{C}, \underline{SET}]$  to  $\Gamma_{\underline{T}} \in Ob[\underline{\hat{C}}, \underline{SET}]$ .

[Note:  $\Gamma_{T}$  is the <u>realization functor</u>; it is a left adjoint for the <u>singular</u> <u>functor</u>  $\sin_{T}: \underline{SET} \rightarrow \hat{\underline{C}}$  which is defined by the prescription

$$(\sin_m Y)X = Mor(TX,Y).]$$

19.7 LEMMA Let <u>C</u> be a small category. Suppose that  $f:\underline{SET} \rightarrow \hat{\underline{C}}$  is a point -then there exists a functor  $T:\underline{C} \rightarrow \underline{SET}$  such that  $f^*$  is naturally isomorphic to  $\Gamma_{\overline{T}}$ . 19.8 DEFINITION Let <u>C</u> be a small category — then a functor  $T:\underline{C} \rightarrow \underline{SET}$  is said to be <u>flat</u> if  $\Gamma_{T}$  preserves finite limits.

So, if T is flat, then the adjoint pair  $(\Gamma_T, \sin_T)$  is a geometric morphism  $\underline{\text{SET}} \rightarrow \hat{\underline{C}}$ , i.e., is a point of  $\hat{\underline{C}}$ . Moreover, up to natural isomorphism, all points of  $\hat{\underline{C}}$  are of this form (cf. 19.7).

Write  $[\underline{C}, \underline{SET}]_{flat}$  for the full subcategory of  $[\underline{C}, \underline{SET}]$  whose objects are the flat functors.

19.9 THEOREM There is an equivalence

$$[\underline{C}, \underline{SET}]_{flat} \longleftrightarrow \underline{PT}(\underline{\hat{C}})$$

of categories.

[Send T to  $(\Gamma_{T}, sin_{T})$  and send f to f\*  $\circ Y_{C}$ .]

19.10 REMARK Let  $\tau$  be a Grothendieck topology on  $\underline{C}$  -- then  $\underline{PT}(\underline{Sh}_{\tau}(\underline{C}))$  is equivalent to the full subcategory of  $\underline{PT}(\hat{\underline{C}})$  consisting of those points that factor through  $\iota_{\tau}$ .

19.11 DEFINITION Let  $\underline{C}$  be a category. Suppose that the  $\underline{C}_i$  are categories and the  $F_i:\underline{C} \neq \underline{C}_i$  are functors -- then  $\{F_i\}$  is <u>faithful</u> if given distinct morphisms  $f,g:X \neq Y$  in  $\underline{C}$ , there exists an  $F_i$  such that  $F_if \neq F_ig$ .

19.12 EXAMPLE Take  $\underline{C} = \underline{Sh}(X)$  (X a nonempty topological space), let  $\underline{C}_{\underline{X}} = \underline{SET}$ (x  $\in X$ ), and let  $\underline{p}_{\underline{X}}:\underline{Sh}(X) \rightarrow \underline{SET}$  be as in 19.2 -- then  $\{\underline{p}_{\underline{X}}\}$  is faithful. 19.13 DEFINITION Let  $\underline{C}$  be a category. Suppose that the  $\underline{C}_i$  are categories and the  $F_i:\underline{C} \rightarrow \underline{C}_i$  are functors.

•  $\{F_i\}$  reflects isomorphisms if any  $f \in Mor \ \underline{C}$  with the property that  $F_i$  is an isomorphism for all  $F_i$  must itself be an isomorphism in  $\underline{C}$ .

•  $\{F_i\}$  reflects monomorphisms if any  $f \in Mor \ \underline{C}$  with the property that  $F_i$  is a monomorphism for all  $F_i$  must itself be a monomorphism in  $\underline{C}$ .

•  $\{F_i\}$  reflects epimorphisms if any  $f \in Mor \ \underline{C}$  with the property that  $F_if$  is an epimorphism for all  $F_i$  must itself be an epimorphism in  $\underline{C}$ .

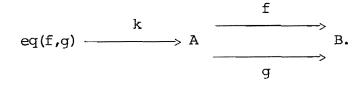
Let  $P \subset Ob PT(E)$  be a class of points.

19.14 LEMMA Suppose that P is faithful -- then P reflects isomorphisms.

PROOF It is immediate that P reflects monomorphisms and epimorphisms. But  $\underline{E}$  is balanced (cf. 14.7).

19.15 LEMMA Suppose that P reflects isomorphisms -- then P is faithful.

PROOF Let f,g:A  $\rightarrow$  B be morphisms in <u>E</u> and suppose that pf = pg for all  $p \in P$ . Form the equalizer diagram



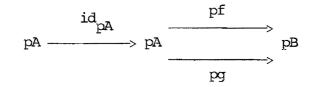
Since p preserves finite limits, it preserves equalizers:

$$p(eq(f,g)) \approx eq(pf,pg).$$

Therefore

$$p(eq(f,g)) \xrightarrow{pk} pA \xrightarrow{pf} pB$$

is an equalizer diagram. But pf = pg, thus



is also an equalizer diagram, which implies that pk is an isomorphism, hence k is an isomorphism, hence f = g ( $f \circ k = g \circ k$ ).

19.16 DEFINITION  $\underline{E}$  is said to have enough points if the class of all points of  $\underline{E}$  is faithful.

19.17 THEOREM If E has enough points, then E has a faithful set of points.

19.18 DEFINITION A weak point of  $\underline{E}$  is a functor  $p:\underline{E} \rightarrow \underline{SET}$  which preserves epimorphisms and finite limits.

N.B. Every point is a weak point.

19.19 LEMMA A class of weak points of E is faithful iff it reflects isomorphisms.

19.20 THEOREM The class of all weak points of E is faithful.

PROOF Take  $\underline{B}$  and  $f:\underline{B} \neq \underline{E}$  as in 18.29 -- then every epimorphism of  $\underline{B}$  has a section, thus  $\forall B \in Ob \underline{B}$ , the functor  $X \neq Mor(B,X)$  from  $\underline{B}$  to <u>SET</u> is a weak point of  $\underline{B}$ , so  $\forall B \in \underline{B}$ , the functor  $X \neq Mor(B,f^*X)$  from  $\underline{E}$  to <u>SET</u> is a weak point of  $\underline{E}$ 

(f\* preserves epimorphisms (being a left adjoint)). And:  $\{p_B: B \in Ob \ \underline{B}\}$  is a faithful class of weak points of  $\underline{E}$ . Proof: Bearing in mind 19.19, suppose that  $\phi: U \rightarrow V$  is a morphism in  $\underline{E}$  such that  $\forall B \in Ob \ \underline{B}$ ,

$$p_B \phi: Mor(B, f*U) \rightarrow Mor(B, f*V)$$

is bijective -- then  $f^{\phi}:f^{U} \rightarrow f^{V}$  is an isomorphism. But  $f^{*}$  reflects isomorphisms (cf. 18.28), hence  $\phi$  is an isomorphism.

19.21 LEMMA Let  $p:\underline{E} \rightarrow \underline{SET}$  be a weak point. Given a morphism  $f:A \rightarrow B$  in  $\underline{E}$ , factor it per 3.9:

$$A \longrightarrow M \longrightarrow B$$
 (f = m • k).

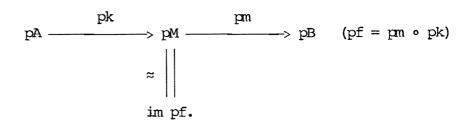
Then

 $pM \approx im pf$ 

or still,

$$p(im f) \approx im pf.$$

PROOF Since p preserves epimorphisms and monomorphisms, pk is a surjection and pm is an injection:



19.22 LEMMA Suppose that  $\{p\}$  is a faithful class of weak points of  $\underline{E}$  -- then  $\{p\}$  reflects epimorphisms.

PROOF First, f:A  $\rightarrow$  B is an epimorphism iff the canonical arrow M  $\longrightarrow$  B is

an epimorphism, then  $\forall$  p, pm is an isomorphism (cf. 19.21), hence m is an isomorphism (cf. 19.19).

19.23 SCHOLIUM A morphism f in  $\underline{E}$  is an epimorphism iff  $\forall$  weak point p, pf is an epimorphism.

19.24 LEMMA Suppose that R is an equivalence relation on X and  $p:\underline{E} \rightarrow \underline{SET}$  is a weak point --- then pR is an equivalence relation on pX and

$$pX/pR \approx p(X/R)$$
.

19.25 APPLICATION Let  $f,g \in Mor(X,Y)$  and let

$$(f,g): X \to Y \times Y.$$

Suppose that im(f,g) is an equivalence relation on Y and  $p:\underline{E} \rightarrow \underline{SET}$  is a weak point -then p(im(f,g)) ( $\approx$  im p(f,g) (cf. 19.21)) is an equivalence relation on pY and the canonical map

$$coker(pf,pg) \rightarrow p(coker(f,g))$$

is bijective.

19.26 LEMMA Let R be a relation on X. Assume:  $\forall$  weak point  $p:\underline{E} \rightarrow \underline{SET}$ , pR is an equivalence relation on pX -- then R is an equivalence relation on X, hence

$$pX/pR \approx p(X/R)$$
.

19.27 APPLICATION Let  $f,g \in Mor(X,Y)$  and let

$$(f,q): X \rightarrow Y \times Y.$$

Assume:  $\forall$  weak point p: $\underline{E} \rightarrow \underline{SET}$ , p(im(f,g)) ( $\approx$  im p(f,g) (cf. 19.21)) is an equivalence relation on pY -- then im(f,g) is an equivalence relation on Y and the canonical map

$$coker(pf,pg) \rightarrow p(coker(f,g))$$

is bijective.

# \$20. CISINSKI<sup>†</sup> THEORY

Let  $\underline{E}$  be a Grothendieck topos -- then the class  $M \subset Mor \underline{E}$  of monomorphisms is retract stable and the pair (M, RLP(M)) is a w.f.s. on E.

N.B. Elements of RLP(M) are called trivial fibrations.

20.1 THEOREM There exists a set  $M \subset M$  such that M = LLP(RLP(M)), hence M = cof M (E being presentable (cf. 18.23)).

20.2 RAPPEL Let <u>C</u> be a category,  $W \subset Mor C$  a class of morphisms -- then (<u>C</u>, W) is a category pair if W is closed under composition and contains the identities of <u>C</u>.

20.3 DEFINITION Suppose that  $(\underline{E}, W)$  is a category pair -- then W is an  $\underline{E}$ -localizer provided the following conditions are met.

(1) W satisfies the 2 out of 3 condition.

(2) W contains RLP(M).

(3)  $W \cap M$  is a stable class, i.e., is closed under the formation of pushouts and transfinite compositions.

Let  $C \subset Mor \not E$  -- then the  $\not E$ -localizer generated by C, denoted W(C), is the intersection of all the  $\not E$ -localizers containing C. The minimal  $\not E$ -localizer is

 $W(\emptyset)$  ( $\emptyset$  the empty set of morphisms).

[Note: Let  $C_1, C_2 \subset Mor \ge --$  then

$$\mathbb{W}(C_1 \cup C_2) = \mathbb{W}(\mathbb{W}(C_1) \cup \mathbb{W}(C_2)).]$$

20.4 DEFINITION An E-localizer is admissible if it is generated by a set of

<sup>†</sup> Asterisque <u>308</u> (2006); see also Faisceaux Localement Aspheriques (2003) (preprint).

morphisms of  $\underline{E}$ .

20.5 EXAMPLE Mor E is an admissible E-localizer. In fact,

$$W(\{\emptyset_{\underline{E}} \rightarrow *_{\underline{E}}\}) = Mor \underline{E}.$$

20.6 EXAMPLE Take  $\underline{E} = \underline{SISET}$  ( $= \hat{\Delta}$ ) and let  $W_{\infty}$  be the class of simplicial weak equivalences -- then  $W_{\infty}$  is a  $\hat{\Delta}$ -localizer.

 $\bullet~\ensuremath{\mathsf{W}}_\infty$  is generated by the projections

$$p_{\mathbf{K}}: \mathbf{K} \times \Delta[\mathbf{1}] \rightarrow \mathbf{K} \quad (\mathbf{K} \in \mathbf{Ob} \ \underline{\Delta}).$$

•  $W_{\infty}$  is generated by the maps  $\Delta[n] \rightarrow \Delta[0]$   $(n \ge 0)$ .

<u>N.B.</u> It follows from the first description that  $W_{\infty}$  is closed under the formation of products of pairs of arrows and from the second description that  $W_{\infty}$  is admissible.

[Note: In SISET, a simplicial weak equivalence is a simplicial map  $f:X \rightarrow Y$  such that  $|f|:|X| \rightarrow |Y|$  is a homotopy equivalence.]

20.7 EXAMPLE Take  $\underline{E} = \underline{\text{SET}}$  -- then  $W(\emptyset)$  is the class

$$\{ \emptyset \to \emptyset \} \cup \{ f: X \to Y \ (X \neq \emptyset) \}.$$

20.8 NOTATION Given  $C \subset Mor E$ , let cart C be the class of arrows of the form

$$f \times id_{Z}: X \times Z \rightarrow Y \times Z$$
 ( $f \in C, Z \in Ob \underline{E}$ ).

20.9 LEMMA The  $\underline{E}$ -localizer generated by cart C is closed under the formation of products of pairs of arrows and is admissible if C is a set.

20.10 APPLICATION The minimal  $\underline{E}$ -localizer  $W(\emptyset)$  is closed under the formation of products of pairs of arrows.

[Note: This is one way to distinguish a generic E-localizer W from  $W(\emptyset)$ .]

20.11 DEFINITION A cofibrantly generated model structure on  $\underline{E}$  is said to be a Cisinski structure if the cofibrations are the monomorphisms.

[Note: The acyclic fibrations of a Cisinski structure are the trivial fibrations.]

20.12 THEOREM Suppose that  $(\underline{E}, W)$  is a category pair — then W is an admissible <u>E</u>-localizer iff there exists a cofibrantly generated model structure on <u>E</u> whose class of weak equivalences are the elements of W and whose cofibrations are the monomorphisms.

20.13 SCHOLIUM The map

## $W \rightarrow W, M, RLP(W \cap M)$

induces a bijection between the class of admissible  $\underline{E}$ -localizers and the class of Cisinski structures on  $\underline{E}$ .

20.14 REMARK The stable class  $W \cap M$  is retract stable. In addition, W is necessarily saturated, i.e.,  $W = \overline{W}$ .

20.15 LEMMA Let W be an admissible  $\underline{E}$ -localizer --- then the cofibrantly generated model structure on  $\underline{E}$  determined by W is left proper.

20.16 EXAMPLE Take  $\underline{E} = \underline{\text{SISET}}$  and let W be the class of categorical weak equivalences -- then W is a  $\hat{\Delta}$ -localizer. As such, it is generated by the maps  $I[n] \rightarrow \Delta[n]$  ( $n \ge 0$ ), hence W is admissible. The resulting cofibrantly generated model

3.

structure on SISET is the Joyal structure. It is left proper but not right proper.

[Note: In SISET, a categorical weak equivalence is a simplicial map  $f:X_1 \rightarrow X_2$ such that for every weak Kan complex Y, the arrow

$$c_0 \max(X_2, Y) \rightarrow c_0 \max(X_1, Y)$$

is bijective.]

N.B. Every categorical weak equivalence is a simplicial weak equivalence.

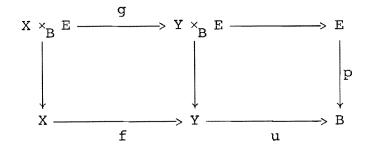
20.17 CRITERION Let  $S \subset Mor \not E$  be a set -- then the cofibrantly generated model structure on  $\not E$  corresponding to W(S) is right proper iff

- $\forall$  arrow f:X  $\rightarrow$  Y in S,
- $\forall$  fibration p:E  $\rightarrow$  B with B fibrant,
- $\forall \text{ arrow } u: Y \rightarrow B,$

the induced arrow

$$g: X \times_{\mathbf{R}} E \rightarrow Y \times_{\mathbf{R}} E$$

per



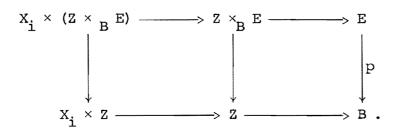
is in W(S).

[Note: One can replace the set S by a class C provided that W(C) is admissible.]

<u>N.B.</u> Take  $S = \emptyset$  to see that the Cisinski structure on  $\underline{E}$  corresponding to  $W(\emptyset)$  is right proper.

20.18 LEMMA If  $X_i$  ( $i \in I$ ) is a set of objects of  $\underline{E}$ , then the  $\underline{E}$ -localizer generated by the projections  $X_i \times Z \rightarrow Z$  for all i and Z is admissible (cf. 20.9) and the associated Cisinski structure is right proper (hence proper (cf. 20.15)). [To infer right proper, apply 20.17 and consider

or still,



But the arrow

 $X_{i} \times (Z \times_{B} E) \rightarrow Z \times_{B} E$ 

is in our generating class.]

20.19 EXAMPLE Take <u>SISET</u> in its Kan structure -- then this model structure is proper.

[Since all objects are cofibrant, left proper is an application of standard generalities while classically, right proper lies deeper in that it uses the fact that the geometric realization of a Kan fibration is a Serre fibration. But, as has been noted in 20.6,  $W_{\infty}$  is generated by the projections

$$\mathbf{p}_{\mathbf{K}}:\mathbf{K}\times\Delta[\mathbf{1}]\rightarrow\mathbf{K}\qquad(\mathbf{K}\in\mathsf{Ob}\ \underline{\widehat{\Delta}}).$$

Therefore right proper is immediate (cf. 20.18).

20.20 LEMMA Let  $S_1, S_2 \subset Mor \not E$  be sets. Suppose that the Cisinski structures corresponding to  $W(S_1), W(S_2)$  are right proper -- then the Cisinski structure corresponding to  $W(S_1 \cup S_2)$  is right proper.

[To infer right proper, apply 20.17, noting that every fibration per  $W(S_1 \cup S_2)$  is a fibration per  $W(S_1)$  and  $W(S_2).]$ 

20.21 NOTATION Given an admissible <u>E</u>-localizer W and a small category <u>I</u>, denote by  $W_{\underline{I}} \subset Mor[\underline{I},\underline{E}]$  the class of morphisms  $\Xi:F \to G$  such that  $\forall i \in Ob \underline{I}$ ,  $\Xi_i:Fi \to Gi$  is in W.

N.B. Recall that [I, E] is a Grothendieck topos (cf. 18.17).

20.22 LEMMA  $W_{I}$  is an admissible [I, E]-localizer.

[Note: Therefore 20.12 is applicable with  $\underline{E}$  replaced by  $[\underline{I}, \underline{E}]$  and W replaced by  $W_{\underline{I}}$ .]

## APPENDIX

What follows is a summary of some basic facts from model category theory.

Let C be a model category.

DEFINITION <u>C</u> is <u>combinatorial</u> if <u>C</u> is cofibrantly generated and presentable. EXAMPLE If W is an admissible <u>E</u>-localizer, then <u>E</u> in the Cisinski structure corresponding to W is combinatorial (recall that  $\underline{E}$  is presentable (cf. 18.23)).

Fix a small category I.

-----

DEFINITION Let <u>C</u> be a model category and suppose that  $\Xi \in Mor[\underline{I},\underline{C}]$ , say  $\Xi: F \to G$ .

• E is a <u>levelwise weak equivalence</u> if  $\forall i \in Ob \ \underline{I}, E_i:Fi \rightarrow Gi is a weak equivalence in C.$ 

•  $\Xi$  is a <u>levelwise fibration</u> if  $\forall i \in Ob \ \underline{I}, \ \Xi_{\underline{i}}:Fi \rightarrow Gi$  is a fibration in <u>C</u>.

• E is a projective cofibration if it has the LLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise fibration.

DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise fibrations, and projective cofibrations is called the projective structure on  $[\underline{I},\underline{C}]$ .

THEOREM Suppose that  $\underline{C}$  is a combinatorial model category — then for every  $\underline{I}$ , the projective structure on  $[\underline{I},\underline{C}]$  is a model structure that, moreover, is combinatorial.

DEFINITION Let <u>C</u> be a model category and suppose that  $\Xi \in Mor[\underline{I},\underline{C}]$ , say  $\Xi: F \to G$ .

• E is a <u>levelwise weak equivalence</u> if  $\forall i \in Ob \ \underline{I}, E_i:Fi \rightarrow Gi$  is a weak equivalence in <u>C</u>.

• E is a <u>levelwise cofibration</u> if  $\forall i \in Ob \ \underline{I}, \ \Xi_{\underline{i}}:Fi \rightarrow Gi$  is a cofibration in <u>C</u>.

• E is an <u>injective fibration</u> if it has the RLP w.r.t. those morphisms which are simultaneously a levelwise weak equivalence and a levelwise cofibration.

DEFINITION The triple consisting of the classes of levelwise weak equivalences, levelwise cofibrations, and injective fibrations is called the <u>injective structure</u> on [I,C].

THEOREM Suppose that  $\underline{C}$  is a combinatorial model category — then for every  $\underline{I}$ , the injective structure on  $[\underline{I},\underline{C}]$  is a model structure that, moreover, is combinatorial.

### REMARK

• Every projective cofibration is necessarily levelwise, hence is a cofibration in the injective structure.

• Every injective fibration is necessarily levelwise, hence is a fibration in the projective structure.

EXAMPLE If W is an admissible  $\underline{E}$ -localizer, then the Cisinski structure on  $[\underline{I},\underline{E}]$  corresponding to  $W_{\underline{I}}$  (cf. 20.22) is the injective structure (monomorphisms are levelwise).

[Note: Of course one can also equip [I, E] with its projective structure.]

LEMMA Suppose that C is combinatorial -- then

<u>C</u> left proper => [I,C] (Projective Structure) left proper [I,C] (Injective Structure)

8.

REMARK If W is an admissible <u>E</u>-localizer, then the Cisinski structure on  $[\underline{I},\underline{E}]$  corresponding to  $W_{\underline{I}}$  (cf. 20.22) is left proper (cf. 20.15) and is right proper if the Cisinski structure on <u>E</u> corresponding to W is right proper.

Let C and C' be model categories.

DEFINITION A left adjoint functor  $F: \underline{C} \rightarrow \underline{C}'$  is a <u>left model functor</u> if F preserves cofibrations and acyclic cofibrations.

DEFINITION A right adjoint functor  $F':\underline{C}' \rightarrow \underline{C}$  is a <u>right model functor</u> if F' preserves fibrations and acyclic fibrations.

LEMMA Suppose that

$$F:\underline{C} \to \underline{C}'$$

$$F':\underline{C}' \to \underline{C}$$

are an adjoint pair -- then F is a left model functor iff F' is a right model functor.

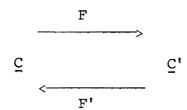
DEFINITION A model pair is an adjoint situation (F,F'), where F is a left model functor and F' is a right model functor.

LEMMA The adjoint situation (F,F') is a model pair iff F preserves cofibrations and F' preserves fibrations.

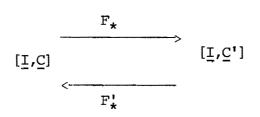
9.

LEMMA The adjoint situation (F,F') is a model pair iff F preserves acyclic cofibrations and F' preserves acyclic fibrations.

REMARK If C and C' are combinatorial and if



is a model pair, then composition with F and F' determines a model pair



w.r.t. either the projective structure or the injective structure.

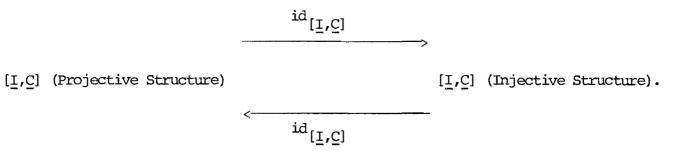
If the adjoint situation (F,F') is a model pair, then the derived functors

$$LF:\underline{HC} \rightarrow \underline{HC'}$$
$$RF':\underline{HC'} \rightarrow \underline{HC}$$

exist and are an adjoint pair.

DEFINITION A model pair (F,F') is a model equivalence if the adjoint pair (LF,RF') is an adjoint equivalence of homotopy categories.

LEMMA Suppose that C is combinatorial and consider the setup



Then  $(id_{[\underline{I},\underline{C}]}, id_{[\underline{I},\underline{C}]})$  is a model equivalence.

#### **§21.** SIMPLICIAL MACHINERY

Let  $\underline{C}$  be a category.

21.1 NOTATION <u>SIC</u> is the functor category  $[\Delta^{OP}, \underline{C}]$  and a <u>simplicial object</u> X in <u>C</u> is an object in <u>SIC</u>.

21.2 RAPPEL Assume: C has coproducts. Define X | = | K by

$$(X \mid \underline{k})_n = K_n \cdot X_n (= \coprod_{K_n} X_n).$$

Then

$$|$$
:SIC × SISET + SIC

is a simplicial action, the canonical simplicial action.

[Note: Therefore

$$X = (K \times L) \approx (X = K) = L$$

and

 $x = |\Delta[0] \approx x$ ,

subject to the usual assumptions.]

N.B. Take  $\underline{C} = \underline{SET} - - \underline{then}$ 

 $X = |K \approx X \times K.$ 

In fact,

$$(X \times K)_n \approx X_n \times K_n \approx K_n \times X_n \approx K_n \cdot X_n.$$

21.3 REMARK Thus there is an S-category  $|\_|SIC$  such that <u>SIC</u> is isomorphic to the underlying category  $U|\_|SIC$ .

[Recall the construction: Put  $0 = Ob \underline{SIC}$  and assign to each ordered pair

 $X, Y \in O$  the simplicial set HOM(X,Y) defined by

$$HOM(X,Y)_{n} = Mor(X|_{[n],Y}) \quad (n \ge 0).$$

21.4 LEMMA Assume: C has coproducts -- then  $\forall X \in Ob \ \underline{SIC}$ , the functor

 $X|\_|-:\underline{SISET} \rightarrow \underline{SIC}$ 

has a right adjoint, viz. the functor

$$HOM(X, --): SIC \rightarrow SISET.$$

21.5 LEMMA Assume: C has coproducts and is complete -- then  $\forall \ K \in Ob \ \hat{\underline{\Delta}}$ , the functor

$$--|$$
\_|K:SIC  $\rightarrow$  SIC

has a right adjoint, denoted by

$$X \rightarrow hom(K,X)$$
.

N.B. In terms of SIC,

$$\begin{array}{|c|c|c|c|c|} & \operatorname{Mor}(X \mid \underline{\ } \mid K, Y) \approx \operatorname{Mor}(K, \operatorname{HOM}(X, Y)) \\ & \operatorname{Mor}(X \mid \underline{\ } \mid K, Y) \approx \operatorname{Mor}(X, \operatorname{hom}(K, Y)), \end{array}$$

and in terms of |-|SIC|,

HOM(X|
$$[K,Y] \approx map(K,HOM(X,Y))$$
  
HOM(X| $[K,Y] \approx HOM(X,hom(K,Y)).$ 

[Note: Here is another point. On the one hand,

$$Mor(X | - | (K \times L), Y) \approx Mor(X, hom(K \times L, Y)),$$

while on the other hand,

$$Mor(X| [] (K \times L), Y) \approx Mor((X| [K) [] L, Y)$$

$$\approx Mor(X|_{K,hom(L,Y)})$$
  
$$\approx Mor(X,hom(K,hom(L,Y))).$$

Therefore

$$hom(K \times L, Y) \approx hom(K, hom(L, Y)).$$

21.6 LEMMA Assume: <u>C</u> has coproducts and is complete. Suppose that  $K \approx \text{colim}_i K_i$  -- then  $\forall X, Y \in Ob \underline{SIC}$ ,

Mor(X, hom(colim<sub>i</sub> K<sub>i</sub>, Y)) 
$$\approx$$
 lim<sub>i</sub> Mor(X, hom(K<sub>i</sub>, Y)).

PROOF

LHS 
$$\approx$$
 Mor(X|\_|colim<sub>i</sub> K<sub>i</sub>,Y)  
 $\approx$  Mor(colim<sub>i</sub> X|\_|K<sub>i</sub>,Y)  
 $\approx$  lim<sub>i</sub> Mor(X|\_|K<sub>i</sub>,Y)  $\approx$  RHS.

21.7 NOTATION Let  $\underline{C}$  be a complete category. Given a simplicial object X in  $\underline{C}$  and a simplicial set K, put

$$X \ \ \kappa = \int_{[n]} (X_n)^{K_n},$$

an object in  $\underline{C}$ .

21.8 EXAMPLE Take K =  $\Delta[n]$  -- then it follows from the integral Yoneda lemma that

$$X \ \ \land \ \ \bigtriangleup[n] \approx X_n.$$

Let K be a simplicial set. Assume:  $\underline{C}$  has coproducts — then K determines a functor

$$K \cdot -: \underline{C} \rightarrow SIC$$

by writing

$$(K \cdot X) [n] = K_n \cdot X.$$

21.9 LEMMA Assume: C has coproducts and is complete -- then K  $\cdot$  -- is a left adjoint for

$$\uparrow$$
 K:SIC  $\rightarrow$  C.

21.10 LEMMA Assume: C has coproducts and is complete. Suppose that  $K \approx \text{colim}_i K_i -- \text{ then } \forall X \in Ob \text{ <u>SIC</u>,}$ 

$$X \ h \ K \approx \lim_{i} X \ h \ K_{i}.$$

PROOF Given  $A \in Ob \ \underline{C}$ , let  $\underline{A} \in Ob \ \underline{SIC}$  be the constant simplicial object determined by A, thus

$$Mor (A, X \ \ \ K) \approx Mor (K \cdot A, X)$$
$$\approx Mor (\underline{A} | - | K, X)$$
$$\approx Mor (colim_{i} \underline{A} | - | K_{i}, X)$$
$$\approx lim_{i} Mor (\underline{A} | - | K_{i}, X)$$
$$\approx lim_{i} Mor (\underline{A} | - | K_{i}, X)$$
$$\approx lim_{i} Mor (K_{i} \cdot A, X)$$
$$\approx lim_{i} Mor (A, X \ \ K_{i})$$
$$\approx Mor (A, lim_{i} X \ \ K_{i}).$$

21.11 LEMMA Assume: C has coproducts and is complete -- then  $\forall X \in Ob \underline{SIC}$ ,

$$\hom(\mathbf{K},\mathbf{X})_{\mathbf{n}} \approx \mathbf{X} \mathrel{\Uparrow} (\mathbf{K} \times \Delta[\mathbf{n}]).$$

PROOF Write

$$K \times \Delta[n] \approx \operatorname{colim}_{i} \Delta[n_{i}].$$

Then

$$X \oint (K \times \Delta[n]) \approx \lim_{i} X \oint \Delta[n_{i}]$$
$$\approx \lim_{i} X_{n_{i}} (cf. 21.8)$$
$$\approx \hom(K, X)_{n}.$$

[Note: The not so obvious final point is implicit in the proof of 21.5 (which was omitted).]

21.12 EXAMPLE Take n = 0 to get

$$\hom(K,X)_0 \approx X \Uparrow K$$

and then replace K by  $\Delta[n]$  to get

$$\hom(\Delta[n], X)_0 \approx X \ \varphi \ \Delta[n] \approx X_n.$$

[Note: Accordingly,

$$hom(K,X)_{n} \approx hom(\Delta[n], hom(K,X))_{0}$$
$$\approx hom(K \times \Delta[n],X)_{0}.$$

21.13 LEMMA Assume:  $\underline{C}$  has coproducts and is complete -- then  $\forall K, L \in Ob \stackrel{\wedge}{\underline{\Delta}}$ ,

hom(K,X) 
$$\oint L \approx X \oint (K \times L)$$
.

21.14 RAPPEL A simplicial set K is  $\underline{finite}$  if it has a finite number of nondegenerate simplexes.

21.15 FACT Suppose that K is finite — then there exists a finite category I and a functor  $\Phi: I \rightarrow \Delta$  such that

$$\mathbf{K} \approx \operatorname{colim} \mathbf{Y}_{\underline{\boldsymbol{\Delta}}} \circ \Phi$$

or still,

$$K \approx \operatorname{colim}_{i} \Delta[n_{i}] \quad (i \in Ob \ \underline{I}, \ \Phi i = \Delta[n_{i}]).$$

21.16 THEOREM Let C, C' be categories. Assume: C, C' have coproducts and are complete. Suppose that  $F:C \rightarrow C'$  is a functor which preserves finite limits -- then

$$\mathbf{F}_{\star}: [\Delta^{\mathrm{OP}}, \underline{\mathbf{C}}] \rightarrow [\Delta^{\mathrm{OP}}, \underline{\mathbf{C}}']$$

and  $\forall X \in Ob \ \underline{SIC}$  and every finite  $K \in Ob \ \underline{\hat{\Delta}}$ , the canonical arrow

$$F_{\star}hom(K,X) \rightarrow hom(K,F_{\star}X)$$

is an isomorphism.

PROOF Since

$$\hom(K,X)_{n} \approx \hom(K \times \Delta[n],X)_{0} \quad (cf. 21.12)$$

and since  $K \, \times \, \Delta[n]$  is finite, it will be enough to verify that

$$(\mathbf{F}_{\star} \operatorname{hom}(\mathbf{K}, \mathbf{X}))_{0} = \operatorname{Fhom}(\mathbf{K}, \mathbf{X})_{0} \approx \operatorname{hom}(\mathbf{K}, \mathbf{F}_{\star} \mathbf{X})_{0}.$$

Per 21.15, write

$$K \approx \operatorname{colim}_{i} \Delta[n_{i}].$$

Then

From (K,X)<sub>0</sub> ~ From (colim<sub>i</sub> 
$$\Delta[n_i]$$
,X)<sub>0</sub>  
~ F(X  $\uparrow$  colim<sub>i</sub>  $\Delta[n_i]$ )  
~ F(lim<sub>i</sub> X  $\uparrow$   $\Delta[n_i]$ ) (cf. 21.10)  
~ lim<sub>i</sub> F(X  $\uparrow$   $\Delta[n_i]$ )  
~ lim<sub>i</sub> FX<sub>n<sub>i</sub></sub> (cf. 21.8)  
~ lim<sub>i</sub> (F<sub>\*</sub>X)<sub>n<sub>i</sub></sub>

 $\approx \lim_{i} F_{*}X \oint \Delta[n_{i}]$   $\approx F_{*}X \oint \operatorname{colim}_{i} \Delta[n_{i}]$   $\approx F_{*}X \oint K$   $\approx \operatorname{hom}(K, F_{*}X)_{0}.$ 

21.17 APPLICATION Let  $\underline{E}$  be a Grothendieck topos. Suppose that  $p:\underline{E} \rightarrow \underline{SET}$  is a weak point — then for every simplicial object X in  $\underline{E}$  and for every finite simplicial set K, the canonical arrow

$$p_{\star}hom(K,X) \rightarrow hom(K,p_{\star}X)$$

is an isomorphism.

-----

#### §22. LIFTING

Let E be a Grothendieck topos.

[Note:  $\underline{E}$  is cocomplete (by definition), hence has coproducts, and is complete (cf. 18.14). Therefore the technology developed in §21 is applicable.]

22.1 DEFINITION A geometric family is a class 4 of monomorphisms of finite simplicial sets.

.

22.2 EXAMPLE The inclusions

$$\Delta[n] \rightarrow \Delta[n] (n \ge 0)$$

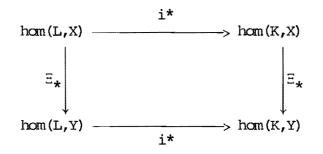
constitute a geometric family.

22.3 EXAMPLE The inclusions

$$\Lambda[k,n] \rightarrow \Delta[n] \quad (0 \le k \le n, n \ge 1)$$

constitute a geometric family.

Given an element  $i:K \rightarrow L$  of a geometric family  $\Psi$  and a morphism  $E:X \rightarrow Y$  of simplicial objects in E, there is a commutative diagram



which then leads to an arrow

 $(\Xi_*, i^*)$ :hom $(L, X) \rightarrow hom(L, Y) \times_{hom(K, Y)} hom(K, X)$ 

or, upon evaluating at 0, to an arrow

$$(\Xi_{\star}, i^{\star})_{0}: \hom(L, X)_{0} \rightarrow \hom(L, Y)_{0} \times \hom(K, Y)_{0} \hom(K, X)_{0}.$$

22.4 DEFINITION  $\Xi: X \rightarrow Y$  has the <u>local right lifting property w.r.t. 4</u> if  $\forall$  i:K  $\rightarrow$  L in 4, the arrow  $(\Xi_*, i^*)_0$  is an epimorphism in  $\underline{E}$ .

22.5 EXAMPLE Take  $\underline{E} = \underline{SET}$  — then  $\Xi: X \rightarrow Y$  has the local right lifting property w.r.t. 4 iff  $\Xi: X \rightarrow Y$  has the right lifting property w.r.t. 4.

[For simplicial sets A and B,

$$hom(A,B) = map(A,B) \implies hom(A,B)_0 = Mor(A,B).$$

22.6 NOTATION Given a geometric family 4, denote by  $LOC_{\rm q}(\underline{E})$  the class of morphisms in SIE that have the local right lifting property w.r.t. 4.

22.7 LEMMA Let  $\underline{E}$ ,  $\underline{F}$  be Grothendieck toposes and let  $f:\underline{E} \rightarrow \underline{F}$  be a geometric morphism -- then

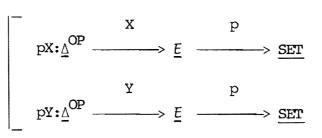
$$(f^*)_{*}LOC_{U}(\underline{F}) \subset LOC_{U}(\underline{E}).$$

[Apply 21.16 (f\* preserves finite limits).]

[Note: By definition,  $f^*: \underline{F} \rightarrow \underline{E}$ . Therefore

$$(f^*)_*: [\underline{\wedge}^{OP}, \underline{F}] \rightarrow [\underline{\wedge}^{OP}, \underline{E}].]$$

Let  $\Xi: X \to Y$  be a morphism of simplicial objects in  $\underline{E}$ . Suppose that  $p:\underline{E} \to \underline{SET}$  is a weak point of  $\underline{E}$  -- then the compositions



are simplicial sets and

 $p\Xi:pX \rightarrow pY$ 

is a simplicial map.

[Note: Here,  $\forall$  n

$$[ (pX)_n = pX_n, E_n: X_n \to Y_n, (pY)_n = pY_n ]$$

and  $(pE)_n = pE_n$ , thus

$$pX_n \xrightarrow{pE_n} pY_n.$$

22.8 CRITERION  $E:X \rightarrow Y$  has the local right lifting property w.r.t. 4 iff for every weak point  $p:\underline{E} \rightarrow \underline{SET}$ ,  $pE:pX \rightarrow pY$  has the right lifting property w.r.t. 4.

It is obvious that  $LOC_{U}(\underline{E})$  contains the isomorphisms.

22.9 LEMMA The class  $LOC_{q}(\underline{E})$  is composition stable, pullback stable, and closed under the formation of retracts.

Let I be a small category — then  $[I, \underline{E}]$  is a Grothendieck topos (cf. 18.17) and epimorphisms are levelwise. N.B. There is an identification

$$[\underline{\Delta}^{\mathrm{OP}}, [\underline{\mathtt{I}}, \underline{\mathtt{E}}]] \approx [\underline{\mathtt{I}}, [\underline{\Delta}^{\mathrm{OP}}, \underline{\mathtt{E}}]].$$

22.10 LEMMA Denote by  $LOC_{\mathbf{U}}(\underline{E})_{\underline{I}}$  the class of morphisms  $E: F \to G$  such that  $\forall i \in Ob \underline{I}, E_{\underline{i}}: Fi \to Gi is in <math>LOC_{\mathbf{U}}(\underline{E})$  -- then

$$\operatorname{LOC}_{\operatorname{q}}(\underline{E})_{\underline{I}} = \operatorname{LOC}_{\operatorname{q}}([\underline{I},\underline{E}]).$$

22.11 LEMMA The class  $LOC_{\rm U}(\underline{E})$  is closed under the formation of filtered colimits.

[If I is filtered, then the functor

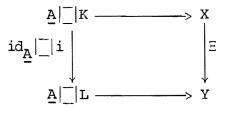
$$\operatorname{colim}_{\underline{I}}: [\underline{I}, \underline{E}] \to \underline{E}$$

preserves finite limits. But colim\_ has a right adjoint, viz. the constant diagram functor. In other words, the data provides us with a geometric morphism  $\underline{E} \rightarrow [\underline{I}, \underline{E}]$ . Now quote 22.7 (modulo 22.10).]

22.12 LEMMA  $\exists: X \rightarrow Y$  has the local right lifting property w.r.t. 4 if it has the right lifting property w.r.t. the arrows

$$\mathrm{id}_{\underline{A}}|_{-}^{-}|\mathrm{i}:\underline{A}|_{-}^{-}|\mathrm{K} \neq \underline{A}|_{-}^{-}|\mathrm{L},$$

where A runs through the objects of  $\underline{E}$  and i:K  $\rightarrow$  L runs through the elements of 4, i.e., if every commutative diagram



admits a filler.

N.B. The arrow

$$\underline{\mathbf{A}}[\underline{\ }|\mathbf{K} \rightarrow \underline{\mathbf{A}}[\underline{\ }|\mathbf{L}$$

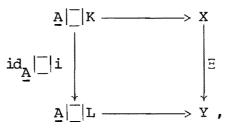
is a monomorphism.

[From the definitions,

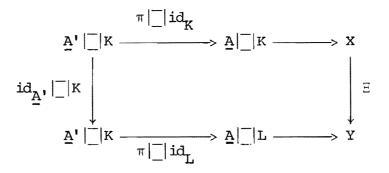
$$(\underline{A}|_{-}|K)_{n} = \underbrace{||}_{K} A$$
$$(\underline{A}|_{-}|L)_{n} = \underbrace{||}_{L} A,$$

and  $K_n$  injects into  $L_n$ .]

22.13 REMARK There is a characterization, namely  $\Xi: X \rightarrow Y$  has the local right lifting property w.r.t.  $\Psi$  iff for every  $A \in Ob \underline{E}$ , for every  $i: K \rightarrow L$  in  $\Psi$ , and for every commutative diagram



one can find an  $A' \in Ob \underline{E}$  and an epimorphism  $\pi: A' \rightarrow A$  with the property that the commutative diagram



admits a filler.

# §23. LOCALIZERS OF DESCENT

Let  $\underline{E}$  be a Grothendieck topos.

23.1 DEFINITION Let  $E:X \to Y$  be a morphism of simplicial objects in  $\underline{E}$  — then E is said to be a <u>hypercovering</u> of <u>SIE</u> if it has the local right lifting property w.r.t. the inclusions  $\hat{\Delta}[n] \to \Delta[n]$  ( $n \ge 0$ ).

[Note: Recall that

$$\begin{array}{c} hom(\Delta[n],X)_{0} \approx X_{n} \\ (cf. 21.12). \\ hom(\Delta[n],Y)_{0} \approx Y_{n} \end{array}$$

On the other hand,

and

$$\begin{bmatrix} X \ & \dot{\Delta}[n] \approx M_{n} X \\ Y \ & \dot{\Delta}[n] \approx M_{n} Y, \end{bmatrix}$$

the symbols on the right standing for the matching object of "Reedy theory", thus

$$\begin{bmatrix} M_{n}X(=M_{n}]X) = (\cos k^{(n-1)}X)_{n} \\ M_{n}Y(=M_{n}]Y) = (\cos k^{(n-1)}Y)_{n}, \end{bmatrix}$$

the matching morphisms being the canonical arrows

$$\begin{bmatrix} & x_n \neq M_n X \\ & y_n \neq M_n Y. \end{bmatrix}$$

Therefore the demand is that  $\forall n \ge 0$ , the arrow

$$X_n \rightarrow Y_n \times_{M_n Y} M_n X$$

is an epimorphism in E.]

23.2 NOTATION HR(E) is the class of hypercoverings of SIE, so

$$HR(\underline{E}) = LOC \qquad (\underline{E}) .$$
$$\{ \Delta[n] \rightarrow \Delta[n] \quad (n \ge 0) \}$$

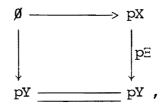
[Note: The stability properties formulated in 22.9 are in force here.]

23.3 EXAMPLE Take  $\underline{E} = \underline{SET}$  -- then in this situation,  $HR(\underline{E})$  is the class of acyclic Kan fibrations (cf. 22.5).

23.4 LEMMA Every hypercovering  $\exists: X \rightarrow Y$  is an epimorphism.

PROOF Since epimorphisms in <u>SIE</u> are levelwise, it suffices to prove that  $\forall n$ ,  $E_n:X_n \neq Y_n$  is an epimorphism in <u>E</u>. To this end, let  $p:\underline{E} \neq \underline{SET}$  be a weak point -then  $pE:pX \neq pY$  has the right lifting property w.r.t. the  $\dot{\Delta}[n] \neq \Delta[n]$  ( $n \ge 0$ ) (cf. 22.8), hence is an acyclic Kan fibration, hence is an epimorphism (see below). But  $pE_n = (pE)_n$  is an epimorphism in <u>SET</u>, thus one can quote 19.23.

[Note: In SISET, all objects are cofibrant, so in the commutative diagram



there is an arrow w:pY  $\rightarrow$  pX such that pE  $\circ$  w = id<sub>pY</sub>, which implies that pE is an epimorphism.]

23.5 LEMMA The hypercoverings are closed under the formation of products of pairs of arrows.

PROOF Suppose that

$$= \underbrace{ \begin{array}{c} \Xi_1 : X_1 \rightarrow Y_1 \\ \Xi_2 : X_2 \rightarrow Y_2 \end{array} }$$

are hypercoverings -- then for any weak point  $p:\underline{E} \rightarrow \underline{SET}$ ,

$$p(\Xi_1 \times \Xi_2) \approx p(\Xi_1) \times p(\Xi_2).$$

But  $\begin{bmatrix} pE_1 \\ are acyclic Kan fibrations and the product of two acyclic Kan fibrations pE_2 \end{bmatrix}$ 

is an acyclic Kan fibration. Now apply 22.8.

23.6 DEFINITION The <u>SIE</u>-localizer of descent is the <u>SIE</u>-localizer generated by  $HR(\underline{E})$ , i.e.,

 $W(HR(\underline{E}))$ .

N.B. The elements of  $W(HR(\underline{E}))$  are called the weak equivalences of descent.

23.7 EXAMPLE Take  $\underline{E} = \underline{SET} - - \text{then}$ 

$$W(HR(E)) = W(\emptyset),$$

the minimal  $\hat{\Delta}$ -localizer.

[Since HR( $\underline{E}$ ) is the class of acyclic Kan fibrations (cf. 23.3), if W is a  $\hat{\Delta}$ -localizer, then

$$W \Rightarrow \operatorname{RLP}(M) = \operatorname{RLP}(\{\Delta[n] \Rightarrow \Delta[n] \ (n \ge 0)\})$$
$$= \operatorname{HR}(\underline{E}).$$

$$W \supset W(HR(E))$$
.]

23.8 LEMMA W(HR(E)) is admissible.

Consequently, <u>SIE</u> admits a cofibrantly generated model structure whose class of weak equivalences are the elements of  $W(HR(\underline{E}))$  and whose cofibrations are the monomorphisms (cf. 20.12).

23.9 REMARK The foregoing model structure on <u>SIE</u> is left proper (cf. 20.15) and right proper (use 20.17 (the elements of  $HR(\underline{E})$  are pullback stable)).

<u>N.B.</u>  $W(HR(\underline{E}))$  is closed under the formation of products of pairs of arrows (use 20.9 (cf. 23.5)).

23.10 RAPPEL The geometric morphism ( $\Gamma^*, \Gamma_*$ ) of 18.2 extends to a geometric morphism SIE  $\rightarrow$  SISET denoted by the same symbol. In particular:

$$\Gamma^*:SISET \rightarrow SIE$$

is defined by the prescription

$$(\Gamma^{*K})_{n} = \prod_{K} * \underline{\underline{E}}.$$

So  $\forall X \in Ob SIE$ ,

$$(X \times \Gamma^{*}K)_{n} = X_{n} \times (\Gamma^{*}K)_{n}$$
$$= X_{n} \times (\underbrace{||}_{K_{n}} * \underline{\underline{E}})$$
$$\approx \underbrace{||}_{K_{n}} X_{n} \times * \underline{\underline{E}} \quad (cf. 18.20)$$

$$= \coprod_{K_n} X_n = (X|_{K_n})_n \quad (cf. 21.2).$$

Therefore

$$X = K \approx X \times \Gamma^* K$$

23.11 NOTATION Given  $X \in Ob E$ , X is the constant simplicial object in <u>SIE</u>.

23.12 DEFINITION Let W be a  $\hat{\Delta}$ -localizer -- then the <u>SIE-localizer</u> of W-descent, denoted W<sub>E</sub>, is the <u>SIE-localizer</u> generated by HR(<u>E</u>) and by the morphisms

$$\operatorname{id}_{\underline{X}} \left[ - \left[ f: \underline{X} \right] - \left[ K \to \underline{X} \right] \right] L,$$

where  $X \in Ob \not E$  and  $f: K \rightarrow L$  is an arrow in W.

<u>N.B.</u> The elements of  $W_{\underline{E}}$  are called the weak equivalences of W-descent.

23.13 LEMMA Suppose that W = W(C) ( $C \subset Mor \hat{\Delta}$ ) -- then  $W_{\underline{E}}$  is generated by HR( $\underline{E}$ ) and by the morphisms

$$\operatorname{id}_{\underline{X}} = |f:\underline{X}| = |K \to \underline{X}| = |L,$$

where  $X \in Ob \not E$  and  $f: K \rightarrow L$  is an arrow in C.

PROOF Letting  $W_{\underline{E},C}$  be the <u>SIE</u>-localizer generated by the morphisms in question, it is clear that  $W_{\underline{E},C} \subset W_{\underline{E}}$ . To go the other way, given  $X \in Ob \underline{E}$ , let

$$F_X:\underline{\widehat{\Delta}} \to \underline{SIE}$$

be the functor that sends K to  $\underline{X}|_{-}^{-}|K (\approx \underline{X} \times \Gamma^*K)$  — then  $F_{\underline{X}}^{-1}W_{\underline{E},C}$  is a  $\hat{\Delta}$ -localizer (cf. infra) and

$$C \subset F_X^{-1} W_{\underline{E},C} \Longrightarrow W \subset F_X^{-1} W_{\underline{E},C}$$

Since this is true of all  $X \in Ob \ \underline{E}$ , it follows that  $W_{\underline{E}} \subset W_{\underline{E},C}$ .

[Note: The claim is that  $F_X^{-1}W_{\underline{E},C}$  satisfies the three conditions of 20.3. E.g., to check condition (2), let  $f:K \rightarrow L$  be an acyclic Kan fibration -- then  $\Gamma^*f:\Gamma^*K \rightarrow \Gamma^*L$  is a hypercovering (cf. 22.7), thus the same is true of

$$\operatorname{id}_{\underline{X}} \times \Gamma^* f: \underline{X} \times \Gamma^* K \to \underline{X} \times \Gamma^* L \quad (cf. 23.5).$$

I.e.:

$$\operatorname{id}_{\underline{X}} \times \Gamma^* f \in \operatorname{HR}(\underline{E}).$$

Therefore  $F_X^{-1}W_{E,C}$  contains the class of acyclic Kan fibrations, as claimed.]

N.B. The SIE-localizer of  $W(\emptyset)$ -descent is the SIE-localizer of descent.

23.14 EXAMPLE Consider the SIE-localizer generated by  $HR(\underline{E})$  and by the morphisms

$$\mathrm{id}_{\underline{X}}[\underline{\ }]p_{K}:\underline{X}[\underline{\ }](K \times \Delta[1]) \to \underline{X}[\underline{\ }]K \ (K \in \mathrm{Ob} \ \underline{\widehat{\Delta}})$$

Then this is the SIE-localizer of  $W_{m}$ -descent (cf. 20.6).

23.15 LEMMA If W is admissible, then  $\mathrm{W}_{\mathrm{E}}$  is admissible.

23.16 THEOREM If W is admissible, then <u>SIE</u> admits a cofibrantly generated model structure whose class of weak equivalences are the elements of  $W_{\underline{E}}$  and whose co-fibrations are the monomorphisms (cf. 20.12).

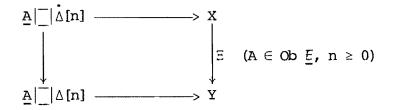
[Note: If the Cisinski structure on  $\hat{\underline{\Delta}}$  per W is proper, then the Cisinski structure on <u>SIE</u> per W<sub>E</sub> is proper.]

23.17 SCHOLIUM SIE admits a cofibrantly generated proper model structure whose

class of weak equivalences are the elements of  $(W_{\infty})_{\underline{E}}$  and whose cofibrations are the monomorphisms.

23.18 LEMMA Every trivial fibration  $\Xi: X \rightarrow Y$  is a hypercovering.

PROOF By definition,  $\Xi \in RLP(M)$ , where  $M \subset Mor \underline{SIE}$  is the class of monomorphisms. Accordingly, every commutative diagram



admits a filler. Therefore  $\Xi$  has the local right lifting property w.r.t. the inclusions  $\hat{\Delta}[n] \rightarrow \Delta[n]$  ( $n \ge 0$ ) (cf. 22.12). And this just means that  $\Xi$  is a hypercovering.

Let  $\underline{E}$ ,  $\underline{F}$  be Grothendieck toposes and let  $f:\underline{E} \rightarrow \underline{F}$  be a geometric morphism -then f induces a geometric morphism si  $f:\underline{SIE} \rightarrow \underline{SIF}$ , thus there is an adjoint pair (si f\*, si f\_\*) and si f\* preserves finite limits.

[Note:  $si f^* = (f^*)_* (cf. 22.7).$ ]

23.19 LEMMA Suppose that W is admissible -- then

si 
$$f^*W_F \subset W_E$$
.

PROOF Applying 22.7 (and bearing in mind 23.18), it follows that (si  $f^*$ )<sup>-1</sup> $W_{\underline{F}}$ is a <u>SIF</u>-localizer which contains the hypercoverings. On the other hand, if  $Y \in Ob \underline{F}$  and  $f: K \rightarrow L$  is an arrow in W, then

$$(\text{si f})^*(\operatorname{id}_{\underline{Y}}|_{-}^{-}|f) \approx \operatorname{id}_{\underline{f^*Y}}|_{-}^{-}|f.$$

Therefore

$$W_{\underline{F}} \subset (\text{sift})^{-1} W_{\underline{E}}$$

or still,

si f\*W<sub>F</sub> 
$$\subset$$
 W<sub>E</sub>.

23.20 THEOREM Suppose that W is admissible -- then the adjoint situation

si 
$$f^*:\underline{SIF} \rightarrow \underline{SIE}$$
  
si  $f_*:\underline{SIE} \rightarrow \underline{SIF}$ 

is a model pair.

PROOF In fact, si f\* preserves finite limits, hence preserves cofibrations (these being the monomorphisms). Meanwhile, thanks to 23.19, si f\* sends weak equivalences to weak equivalences.

Let  $\underline{I}$  be a small category -- then  $[\underline{I},\underline{E}]$  is a Grothendieck topos (cf. 18.17) and

$$\underline{\mathrm{SI}}[\underline{\mathrm{I}},\underline{\mathrm{E}}] \;=\; [\underline{\mathrm{A}}^{\mathrm{OP}},[\underline{\mathrm{I}},\underline{\mathrm{E}}]] \;\approx\; [\underline{\mathrm{I}},[\underline{\mathrm{A}}^{\mathrm{OP}},\underline{\mathrm{E}}]] \;=\; [\underline{\mathrm{I}},\underline{\mathrm{SIE}}] \,.$$

Let W be an admissible  $\Delta$ -localizer -- then  $W_{\underline{E}}$  is an admissible <u>SIE</u>-localizer (cf. 23.15), so it makes sense to form  $(W_{\underline{E}})_{\underline{I}}$  (cf. 20.21), which is an admissible [I,SIE]-localizer (cf. 20.22).

23.21 LEMMA In [1,SIE],

$$W_{[\underline{I},\underline{E}]} = (W_{\underline{E}})\underline{I}.$$

Therefore the Cisinski structure on  $[\underline{I}, \underline{SIE}]$  per  $W_{[\underline{I}, \underline{E}]}$  is the injective structure on  $[\underline{I}, \underline{SIE}]$  w.r.t. the Cisinski structure on  $\underline{SIE}$  per  $W_{\underline{E}}$ .

## §24. LOCAL FIBRATIONS AND LOCAL WEAK EQUIVALENCES

Let  $\underline{E}$  be a Grothendieck topos.

24.1 DEFINITION Let  $\exists : X \neq Y$  be a morphism of simplicial objects in  $\underline{E}$  — then  $\exists$  is said to be a <u>local fibration</u> if it has the local right lifting property w.r.t. the inclusions  $\Lambda[k,n] \neq \Lambda[n]$  ( $0 \le k \le n, n \ge 1$ ).

24.2 LEMMA  $E:X \rightarrow Y$  is a local fibration iff for every weak point  $p:\underline{E} \rightarrow \underline{SET}$ , p $E:pX \rightarrow pY$  is a Kan fibration (cf. 22.8).

N.B. Therefore the hypercoverings are local fibrations.

24.3 LEMMA Let  $E:X \rightarrow Y$  be a local fibration and let  $i:K \rightarrow L$  be a monomorphism of finite simplicial sets -- then the arrow

$$(\Xi_*, i^*)$$
: hom  $(L, X) \rightarrow hom (L, Y) \times_{hom (K, Y)} hom (K, X)$ 

is a local fibration which is a hypercovering if E is a hypercovering or i is a simplicial weak equivalence.

[Note: These conditions are reminiscent of those figuring in the definition of "simplicial model category".]

24.4 DEFINITION Consider <u>SIE</u> in its Cisinski structure per an admissible  $W \in Mor \hat{\Delta}$  (cf. 23.16) --- then the elements of

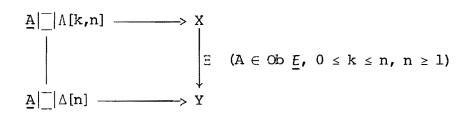
$$\operatorname{RLP}(W_{\underline{E}} \cap M)$$

are called the fibrations of W-descent.

24.5 EXAMPLE Take W = W<sub> $\infty$ </sub> -- then every fibration  $\Xi:X \rightarrow Y$  of W<sub> $\infty$ </sub>-descent is a

local fibration.

[In view of 22.12, it suffices to show that every commutative diagram



admits a filler. But this is plain: The arrow

 $\underline{\mathbf{A}} | \underline{\ } | \boldsymbol{\Lambda} [ \mathbf{k}, \mathbf{n} ] \longrightarrow \underline{\mathbf{A}} | \underline{\ } | \boldsymbol{\Lambda} [ \mathbf{n} ]$ 

is both a weak equivalence of  $W_{m}$ -descent and a monomorphism.]

24.6 REMARK Suppose that  $\underline{E}$  satisfies the axiom of choice — then in this case, the fibrations of  $W_{\infty}$ -descent are precisely the local fibrations (Rezk<sup>†</sup>).

24.7 DEFINITION A simplicial object X in  $\underline{E}$  is said to be <u>locally fibrant</u> if the arrow X  $\rightarrow *_{SIE}$  is a local fibration.

24.8 LEMMA X is locally fibrant iff for every weak point  $p:\underline{E} \rightarrow \underline{SET}$ , pX is a Kan complex.

24.9 EXAMPLE If X is locally fibrant and if K is a finite simplicial set, then hom(K,X) is locally fibrant.

[In fact,  $\forall$  weak point  $p: \underline{E} \rightarrow \underline{SET}$ ,

$$p_{\star}hom(K,X) \approx hom(K,p_{\star}X)$$
 (cf. 21.17)  
 $\equiv map(K,p_{\star}X)$ 

<sup>†</sup> arXiv:math/9811038

or still, dropping the sub-\*,

phom(K,X) 
$$\approx$$
 map(K,pX).

But

$$pX Kan => map(K, pX) Kan.$$
]

24.10 EXAMPLE If X is locally fibrant, then hom( $\Delta$ [1],X) is locally fibrant and there is a local fibration

hom  $(\Delta[1], X) \rightarrow X \times X$ .

[In 24.3, let  $K = \Delta[0] \mid \Delta[0], L = \Delta[1]$ .]

24.11 NOTATION Let  $\underline{SIE}_{loc}$  be the full subcategory of  $\underline{SIE}$  whose objects are locally fibrant.

24.12 DEFINITION Let  $E:X \rightarrow Y$  be a morphism of locally fibrant simplicial objects in  $\underline{E}$  -- then E is said to be a <u>local weak equivalence</u> if for every weak point  $p:\underline{E} \rightarrow SET$ ,  $pE:pX \rightarrow pY$  is a simplicial weak equivalence, i.e.,  $pE \in W_{\infty}$ .

[Note: Take  $\underline{E} = \underline{SET}$  -- then it is true but not obvious that "local weak equivalence" coincides with "simplicial weak equivalence" (cf. 24.23).]

24.13 RAPPEL Consider a triple ( $\underline{C}$ , W, fib), where  $\underline{C}$  is a category with a final object \* and

are two composition closed classes of morphisms termed

<u>weak equivalences</u> <u>fibrations</u>, the acyclic fibrations being the elements of

 $W \cap fib.$ 

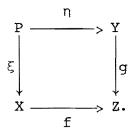
Then  $\underline{C}$  is said to be a category of fibrant objects provided that the following axioms are satisfied.

(FIB-1) For every object X in C, the arrow  $X \rightarrow *$  is a fibration.

(FIB-2) All isomorphisms are weak equivalences and all isomorphisms are fibrations.

(FIB-3) Given composable morphisms f,g, if any two of f,g,g  $\circ$  f are weak equivalences, so is the third.

(FIB-4) Every 2-sink X  $\xrightarrow{f}$  Z  $< \xrightarrow{g}$  Y, where g is a fibration (acyclic fibration), admits a pullback X  $< \xrightarrow{\xi}$  P  $\xrightarrow{n}$  Y, where  $\xi$  is a fibration (acyclic fibration):



(FIB-5) Every morphism in  $\underline{C}$  can be written as the composite of a weak equivalence and a fibration.

24.14 THEOREM Take  $\underline{C} = \underline{SIE}_{loc}$  and let

W = the local weak equivalences fib = the local fibrations.

Then the triple  $(\underline{C}, W, fib)$  is a category of fibrant objects and the acyclic fibrations

are the hypercoverings.

[Note: Given an arrow  $\Xi$  in  $\underline{SIE}_{loc}$ , one can write  $\Xi = q \circ j$ , where q is a local fibration and j is a local weak equivalence with the property that it has a left inverse r which is a hypercovering  $(r \circ j = id)$ .]

24.15 LEMMA Suppose that  $\exists: X \rightarrow Y$  is a local weak equivalence — then  $\exists$  is a weak equivalence of descent.

PROOF Write  $\Xi = q \circ j$  per supra -- then q is a local weak equivalence (this being the case of  $\Xi$  and j). But q is also a local fibration, thus q is a hypercovering, thus q is a weak equivalence of descent. As for j, it too is a weak equivalence of descent. To see this, recall that  $W(HR(\underline{E}))$  is the class of weak equivalences for a model structure on <u>SIE</u>, hence is saturated:

$$\overline{W(HR(\underline{E}))} = W(HR(\underline{E})).$$

Therefore any arrow whose image in the homotopy category is an isomorphism is necessarily in  $W(HR(\underline{E}))$ . But  $r \circ j = id$  and  $r \in HR(\underline{E})$ , hence is invertible in the homotopy category, hence the same holds for j, i.e., j is a weak equivalence of descent.

The functor  $\underline{E} \rightarrow \underline{SIE}$  that sends X to X (cf. 23.11) has a left adjoint  $\pi_0: \underline{SIE} \rightarrow \underline{E}$  that sends X to the coequalizer of the arrows

$$\begin{bmatrix} d_0: X_1 \rightarrow X_0 \\ d_1: X_1 \rightarrow X_0, \end{bmatrix}$$

SO

[Note: Take  $\underline{E} = \underline{SET}$  -- then in the context of simplicial sets,  $\pi_0$  preserves finite products and  $\pi_0 X$  can be identified with the set of components of X.]

24.16 LEMMA Suppose that X is locally fibrant -- then for every weak point  $p: \underline{E} \rightarrow \underline{SET}$ , the canonical map

$$\pi_0 p_X \longrightarrow p_{\pi_0} X$$

is bijective.

PROOF Let R be the image of the arrow

$$(\mathbf{d}_0, \mathbf{d}_1) : \mathbf{X}_1 \to \mathbf{X}_0 \times \mathbf{X}_0.$$

Then R is a relation on  $X_0$  and  $\forall$  weak point  $p: \underline{E} \rightarrow \underline{SET}$ , pX is a Kan complex and pR is an equivalence relation on  $pX_0$ . Therefore R is an equivalence relation on  $X_0$  and the canonical map

$$\pi_0 p_X \longrightarrow p\pi_0^X$$

is bijective (cf. 19.27).

24.17 RAPPEL The class of all weak points of  $\underline{E}$  is faithful (cf. 19.20), hence reflects isomorphisms (cf. 19.19).

24.18 LEMMA The restriction of  $\pi_0$  to <u>SIE</u><sub>loc</sub> preserves finite products.

PROOF To check that the canonical arrow

$$\pi_0 (X \times Y) \longrightarrow \pi_0 X \times \pi_0 Y$$

is an isomorphism, let  $p: \underline{E} \rightarrow \underline{SET}$  be a weak point and note that

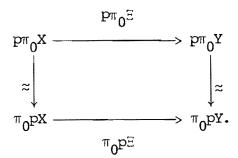
$$p\pi_0^{}(X \times Y) \approx \pi_0^{} p(X \times Y)$$
$$\approx \pi_0^{}(pX \times pY)$$

$$\approx \pi_0 p X \times \pi_0 p Y$$
$$\approx p \pi_0 X \times p \pi_0 Y$$
$$\approx p (\pi_0 X \times \pi_0 Y)$$

[Note: It is clear that  $\pi_0$  preserves final objects.]

24.19 LEMMA Let  $\Xi: X \to Y$  be a local weak equivalence -- then  $\pi_0 \Xi: \pi_0 X \to \pi_0 Y$  is an isomorphism.

PROOF Take a weak point  $p: \underline{E} \rightarrow \underline{SET}$  and consider the commutative diagram



Since pE is a simplicial weak equivalence, the arrow

$$\pi_0^{p\Xi}:\pi_0^{pX} \rightarrow \pi_0^{pY}$$

is bijective. Therefore the arrow

$$p\pi_0 \Xi: p\pi_0 X \rightarrow p\pi_0 Y$$

is bijective.

The preceding considerations can be extended from  $\pi_0$  to  $\pi_n$  (n ≥ 1) but before doing this it will be best to review how things go for simplicial sets (i.e., the case  $\underline{E} = \underline{SET}$ ). Thus given a Kan complex X, let

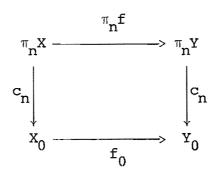
$$\pi_n^X = \coprod_{\mathbf{x}_0 \in X_0} \pi_n^{(X,\mathbf{x}_0)}.$$

Then there is a map  $c_n: \pi_n X \to X_0$  and  $\pi_n X$  is a group object in <u>SET</u>/X<sub>0</sub> (abelian if  $n \ge 2$ ).

[Note: The construction  $X \to \pi_n X$  is functorial in X and natural w.r.t.  $c_n$ .] <u>N.B.</u> Denote by  $\alpha^n X$  the n<sup>th</sup> loop space of X -- then  $\alpha^n X$  is a Kan complex and

$$\pi_0 \Omega^n X = \pi_n X.$$

24.20 THEOREM Let X and Y be Kan complexes,  $f:X \rightarrow Y$  a simplicial map -- then f is a simplicial weak equivalence iff  $\pi_0 f: \pi_0 X \rightarrow \pi_0 Y$  is bijective and  $\forall n \ge 1$ , the commutative diagram

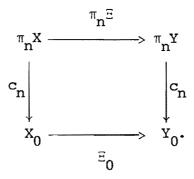


is a pullback square.

While I shall omit the particulars, the story for an arbitrary  $\underline{E}$  is analogous: One can assign to each locally fibrant X its n<sup>th</sup> loop space  $\Omega^n X$ , a locally fibrant simplicial object in  $\underline{E}$ , and

$$\pi_0 \Omega^n \mathbf{X} = \pi_n \mathbf{X}.$$

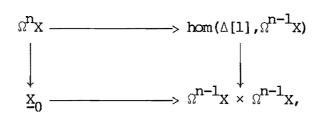
<u>N.B.</u> There is a map  $c_n:\pi_nX \to X_0$  and for any  $E:X \to Y$ , there is a commutative diagram



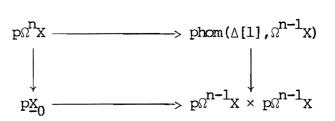
24.21 LEMMA Let  $p:\underline{E} \rightarrow \underline{SET}$  be a weak point — then

$$p\Omega^n X \approx \Omega^n p X.$$

PROOF The formalities give rise to a pullback square



the vertical arrow on the RHS being an instance of 24.10. Now apply p -- then the commutative diagram



is a pullback square in SISET. Proceeding inductively, it can be assumed that

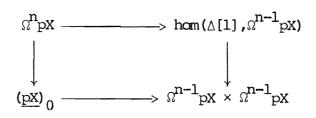
$$p\Omega^{n-1}X \approx \Omega^{n-1}pX$$

Here 
$$p\underline{x}_0 = (\underline{p}\underline{x})_0$$
 and  

$$phom(\Delta[1], \Omega^{n-1}\underline{x}) \approx hom(\Delta[1], \Omega^{n-1}\underline{x}) \quad (cf. 21.17)$$

$$\approx hom(\Delta[1], \Omega^{n-1}\underline{p}\underline{x}).$$

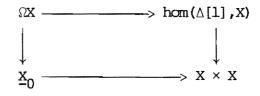
But the commutative diagram



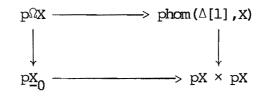
is also a pullback square in SISET. Therefore

 $p\Omega^n x \approx \Omega^n p x.$ 

[Note: If n = 1, then there is a pullback square



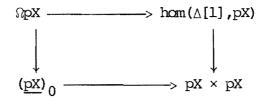
from which a pullback square



in SISET. But

$$phom(\Delta[1],X) \approx hom(\Delta[1],pX)$$
 (cf. 21.17)

and the commutative diagram



is also a pullback square in SISET. Therefore

$$p\Omega X \approx \Omega p X.$$
]

24.22 LEMMA Let  $p: \underline{E} \rightarrow SET$  be a weak point --- then

$$\pi_{n} p X \approx p \pi_{n} X.$$

PROOF In fact,

$$\pi_{n} p X = \pi_{0} \Omega^{n} p X$$

$$\approx \pi_{0} p \Omega^{n} X \quad (cf. 24.21)$$

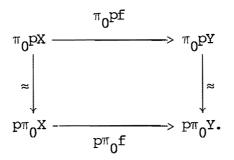
$$\approx p \pi_{0} \Omega^{n} X \quad (cf. 24.16)$$

$$= p \pi_{n} X.$$

24.23 THEOREM Let X and Y be Kan complexes,  $f:X \rightarrow Y$  a simplicial map -- then f is a local weak equivalence iff f is a simplicial weak equivalence.

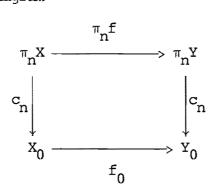
PROOF The nontrivial claim is that if f is a simplicial weak equivalence, then for any weak point  $p:\underline{SET} \rightarrow \underline{SET}$ ,  $pf:pX \rightarrow pY$  is a simplicial weak equivalence, and to establish this, we shall apply 24.20.

• Consider the commutative diagram

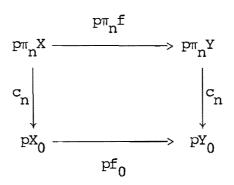


Then  $\pi_0 f$  is bijective, hence  $p\pi_0 f$  is bijective, hence  $\pi_0 pf$  is bijective.

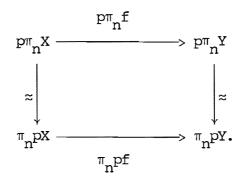
• The commutative diagram



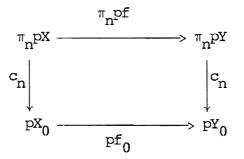
is a pullback square, thus the commutative diagram



is a pullback square. But

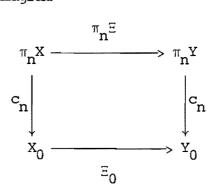


Therefore the commutative diagram



is a pullback square.

24.24 THEOREM Let  $\Xi: X \to Y$  be a morphism of locally fibrant simplicial objects in  $\underline{E}$  -- then  $\Xi$  is a local weak equivalence iff  $\pi_0 \Xi: \pi_0 X \to \pi_0 Y$  is an isomorphism and  $\forall n \ge 1$ , the commutative diagram

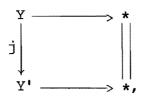


is a pullback square.

Every local weak equivalence is a weak equivalence of descent (cf. 24.15), hence is a weak equivalence of  $W_{\infty}$ -descent. When  $\underline{E} = \underline{SET}$ , this can be turned around: Every weak equivalence of  $W_{\infty}$ -descent (a.k.a. simplicial weak equivalence) is a local weak equivalence (cf. 24.23), a conclusion that persists to an arbitrary  $\underline{E}$ .

24.25 LEMMA Let  $\Xi: X \to Y$  be a morphism of locally fibrant simplicial objects in <u>E</u>. Assume:  $\Xi$  is a weak equivalence of  $W_{\infty}$ -descent -- then  $\Xi$  is a local weak equivalence.

[The full proof is lengthy and technical but here is the strategy. First treat the case when Y = \* and use it to treat the case when in addition the arrow  $Y \rightarrow *$  is a fibration of  $W_{\infty}$ -descent. This done, factor  $Y \rightarrow *$  as



where j is an acyclic cofibration (thus a weak equivalence of  $W_{\infty}$ -descent) and Y'  $\rightarrow$  \* is a fibration of  $W_{\infty}$ -descent. Consider

$$X \xrightarrow{\Xi} Y \xrightarrow{j} Y'$$

Then j is a local weak equivalence and  $j \circ E$  is a local weak equivalence. Therefore E is a local weak equivalence.

[Note: Another approach is to use 24.6 and prove it initially under the assumption that  $\underline{E}$  satisfies the axiom of choice. To proceed in general, take  $f:\underline{B} \rightarrow \underline{E}$  as in 18.29 -- then si  $f^{*}\underline{E}$  is a weak equivalence of  $W_{\infty}$ -descent (cf. 23.19), hence is a local weak equivalence. And from there it is not difficult to see that  $\underline{E}$  is a local weak equivalence.]

Using standard methods, one can introduce a functor

$$Ex^{\tilde{}}:SIE \rightarrow SIE$$

and a natural transformation

$$e^{\tilde{}}:id_{\underline{SIE}} \to Ex^{\tilde{}}$$

with the property that if X is a locally fibrant simplicial object in  $\underline{E}$ , then  $\underline{E}_X^{\infty}$  is a locally fibrant simplicial object in  $\underline{E}$  and the arrow  $\underline{e}_X^{\infty}: X \to \underline{E}_X^{\infty} X$  is a local weak equivalence.

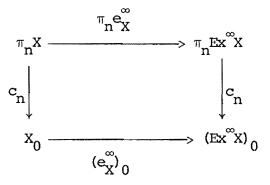
24.26 LEMMA If X is a locally fibrant simplicial object in  $\underline{E}$ , then the arrow  $e_{X}^{\infty}:X \rightarrow Ex^{\infty}X$  induces an isomorphism

$$\pi_0 X \rightarrow \pi_0 E x^{\infty} X \quad (cf. 24.19)$$

and  $\forall n \ge 1$ ,

$$\pi_n X \approx \pi_n E X X.$$

PROOF The commutative diagram



is a pullback square (cf. 24.24). But  $(e_X^{\infty})_0$  is an isomorphism and the pullback of an isomorphism is an isomorphism. Therefore  $\pi_n e_X^{\infty}$  is an isomorphism.

24.27 LEMMA If X is a simplicial object in  $\underline{E}$ , then  $\underline{Ex^{\infty}X}$  is a locally fibrant simplicial object in  $\underline{E}$  and the arrow  $\underline{e_X^{\infty}}: X \to \underline{Ex^{\infty}X}$  is a weak equivalence of  $W_{\infty}$ -descent.

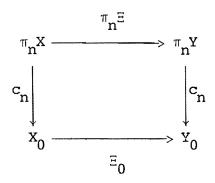
24.28 DEFINITION Given  $X \in Ob$  SIE, put

$$\pi_n X = \pi_n E x^{\infty} X \quad (n \ge 1).$$

[Note: Up to isomorphism, matters are consistent when  $X \in Ob \underbrace{SIE}_{loc}$  (cf. 24.26).]

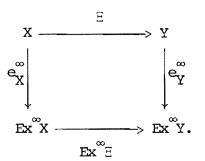
24.29 THEOREM Let  $\Xi: X \rightarrow Y$  be a morphism of simplicial objects in  $\underline{E}$  -- then the following conditions are equivalent.

- (1)  $\Xi$  is a weak equivalence of  $W_{\rm m}{\rm -descent}.$
- (3)  $\operatorname{Ex}^{\infty} \Xi$  is a local weak equivalence.
- (4)  $\pi_0 \Xi: \pi_0 X \rightarrow \pi_0 Y$  is an isomorphism and  $\forall n \ge 1$ , the commutative diagram

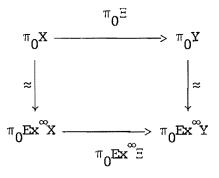


is a pullback square.

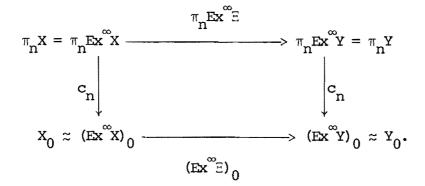
PROOF Taking into account 24.27, the equivalence of (1) and (2) results upon inspection of the commutative diagram



Next, since  $Ex^{\infty}X$  and  $Ex^{\infty}Y$  are locally fibrant, the equivalence of (2) and (3) follows from 24.25. Finally, in view of 24.24, the equivalence of (3) and (4) can be read off from consideration of



and



Let

$$W_{\infty}$$
 = the local weak equivalences  
 $(W_{\infty})_{\underline{E}}$  = the weak equivalences of  $W_{\infty}$ -descent.

24.30 LEMMA The arrow of inclusion

$$i_{loc}: \underline{SIE}_{loc} \rightarrow \underline{SIE}$$

is a morphism of category pairs (cf. 25.9) and the induced functor

$$\underbrace{\mathbf{u}_{loc}}^{\mathbf{u}}: \mathcal{W}_{\infty}^{-1} \underline{\mathbf{SIE}}_{loc} \rightarrow (\mathcal{W}_{\infty}) \underline{\underline{E}}^{-1} \underline{\mathbf{SIE}}$$

is an equivalence of categories.

[Use  $Ex^{\infty}$  to construct a functor in the opposite direction.]

24.31 NOTATION Put

$$\underline{H}_{\infty}\underline{SIE} = (W_{\infty}) \underline{\underline{E}}^{-1} \underline{SIE}.$$

24.32 LEMMA The arrow

$$\underline{E} \rightarrow \underline{H}_{\infty} \underline{SIE}$$

that sends X to the image of  $\underline{X}$  in the homotopy category is fully faithful.

## §25. COMPARISON PRINCIPLES

Let C be a small category -- then

$$\underline{SIC} = [\underline{\Delta}^{OP}, [\underline{C}^{OP}, \underline{SET}]]$$
$$\approx [\underline{C}^{OP}, [\underline{\Delta}^{OP}, \underline{SET}]]$$
$$= [\underline{C}^{OP}, \underline{SISET}].$$

25.1 LEMMA Let W be an admissible  $\Delta$ -localizer — then the elements of W, are C levelwise the elements of W.

PROOF In 23.21, let  $\underline{I} = \underline{C}^{OP}$  and  $\underline{E} = \underline{SET}$ .

25.2 REMARK Since

$$\underline{\operatorname{SIC}} \approx [\underline{\operatorname{C}}^{\operatorname{OP}}, \underline{\operatorname{SISET}}],$$

it follows that if W is an admissible  $\underline{\hat{\Delta}}$ -localizer and if the Cisinski structure on <u>SISET</u> determined by W is proper, then the Cisinski structure on <u>SIC</u> determined by W<sub>2</sub> is proper.

Let C be a small category,  $\tau$  a Grothendieck topology on C.

25.3 RAPPEL The inclusion  $\iota_{\tau}: \underline{Sh}_{\tau}(\underline{C}) \rightarrow \underline{\hat{C}}$  admits a left adjoint  $\underline{a}_{\tau}: \underline{\hat{C}} \rightarrow \underline{Sh}_{\tau}(\underline{C})$  that preserves finite limits (cf. 11.14).

Abusing the notation, we shall use the same symbols  $\begin{bmatrix} a_{\tau} \\ t_{\tau} \end{bmatrix}$  for the induced adjoint pair

$$\stackrel{=}{\underline{\operatorname{SIC}}} \xrightarrow{} \underline{\operatorname{SISh}}_{\tau} (\underline{C})$$

$$\underbrace{\operatorname{SISh}}_{\tau} (\underline{C}) \xrightarrow{} \underline{\operatorname{SIC}}.$$

25.4 DEFINITION Let  $E:X \to Y$  be a morphism of simplicial objects in  $\hat{\underline{C}}$  -- then E is said to be a <u>t</u>-hypercovering if its image  $\underline{a}_{T}E$  is a hypercovering of  $\underline{SISh}_{T}(\underline{C})$ .

25.5 DEFINITION Let W be a  $\underline{\hat{\Delta}}$ -localizer — then the <u>SIC</u>-localizer of (W, \tau)-descent, denoted W<sub>C</sub>(\tau), is the <u>SIC</u>-localizer generated by the  $\tau$ -hypercoverings and by the

morphisms

$$\operatorname{id}_{\underline{X}} = \operatorname{f:}_{\underline{X}} = \operatorname{K} \to \underline{X} = \operatorname{L},$$

where  $X \in Ob \; \hat{\underline{C}}$  and  $\texttt{f:K} \not \rightarrow \texttt{L}$  is an arrow in W.

N.B. The elements of  $W_{\lambda}(\tau)$  are called the weak equivalences of  $(W,\tau)$ -descent  $\underline{C}$  and the elements of

are called the fibrations of  $(W, \tau)$ -descent.

25.6 EXAMPLE Take for  $\tau$  the minimal Grothendieck topology on <u>C</u> (cf. 11.11) -then  $\underline{Sh}_{\tau}(\underline{C}) = \hat{\underline{C}}$  and  $\underline{W}_{\hat{\underline{C}}}(\tau) = \underline{W}_{\hat{\underline{C}}}$ .

25.7 LEMMA If X is a simplicial object in  $\hat{\underline{C}}$ , then the canonical arrow  $X \rightarrow \iota_{\tau} \underline{a}_{\tau} X$  is a weak equivalence of  $(W, \tau)$ -descent.

25.8 THEOREM Let W be a  $\hat{\Delta}$ -localizer -- then

$$= \underbrace{\mathbf{a}_{\tau}^{-1} \mathsf{W}_{\underline{\mathbf{Sh}}_{\tau}}(\underline{\mathbf{C}})}_{\boldsymbol{\Gamma}_{\tau}^{-1} \mathsf{W}_{\underline{\mathbf{C}}}(\tau)} = \underbrace{\mathsf{W}_{\underline{\mathbf{C}}}(\tau)}_{\underline{\mathbf{C}}}$$

$$= \underbrace{\mathsf{v}_{\tau}^{-1} \mathsf{W}_{\underline{\mathbf{C}}}(\tau)}_{\underline{\mathbf{C}}} = \underbrace{\mathsf{W}_{\underline{\mathbf{Sh}}_{\tau}}(\underline{\mathbf{C}})}_{\underline{\mathbf{Sh}}_{\tau}}.$$

PROOF The pair  $(\underline{a}_{\tau}, \iota_{\tau})$  defines a geometric morphism  $\underline{Sh}_{\tau}(\underline{C}) \rightarrow \underline{\hat{C}}$  and  $\underline{a}_{\tau}^{-1} W \underline{Sh}_{\tau}(\underline{C})$ is a  $\underline{SI\hat{C}}$ -localizer which contains  $W_{\hat{C}}$  (cf. 23.19). In particular: The

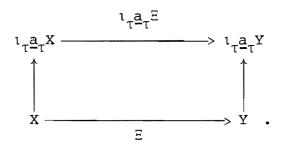
$$\operatorname{id}_{\underline{X}}|_{-}^{-}| \mathtt{f} \in \underline{\mathtt{a}}_{\tau}^{-1} \mathtt{W}_{\underline{Sh}_{\tau}}(\underline{\mathtt{C}}).$$

But the  $\tau$ -hypercoverings are also in  $\underline{a}_{\tau}^{-1} W_{\underline{Sh}_{\tau}(\underline{C})}$ , thus

$$\underline{\underline{a}}_{\tau}^{-1} \mathbb{W}_{\underline{Sh}_{\tau}}(\underline{C}) \stackrel{\supset}{\to} \underbrace{\mathbb{W}}_{\underline{C}}(\tau).$$

As for  $\iota_{\tau}^{-1}W_{\hat{C}}(\tau)$ , it is a <u>SISh</u><sub> $\tau$ </sub>(<u>C</u>)-localizer and  $\iota_{\tau}^{-1}W_{\hat{C}}(\tau) \supset W_{\underline{Sh}_{\tau}}(\underline{C})$ .

• Let  $E:X \to Y$  be an element of  $\underline{a}_T^{-1} W_{\underline{Sh}_T}(\underline{C})$  -- then the claim is that  $E \in W_{\widehat{C}}(\tau)$ . To see this, consider the commutative diagram



Here

$$\underline{a}_{\tau}^{\Xi} \in \underline{W}_{\underline{Sh}_{\tau}}(\underline{C}) \stackrel{c}{\to} \iota_{\tau}^{-1} \underline{W}_{\underline{C}}(\tau)$$
$$=> \\ \iota_{\tau} \underline{a}_{\tau}^{\Xi} \in \underline{W}_{\underline{C}}(\tau).$$

On the other hand, the vertical arrows are weak equivalences of  $(W, \tau)$ -descent

-----

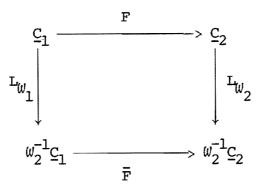
(cf. 25.7). But  $W_{\uparrow}(\tau)$  satisfies the 2 out of 3 condition. Therefore  $\Xi \in W_{\uparrow}(\tau)$ . <u>C</u>

• Let  $E: X \to Y$  be an element of  $\iota_{\tau}^{-1} W_{\hat{C}}(\tau)$  — then the claim is that  $E \in W_{\underline{Sh}_{\tau}}(\tau)$ . Proof:  $\iota_{\tau} E \in W_{\hat{C}}(\tau) \implies \underline{a}_{\tau} \iota_{\tau} E \in W_{\underline{Sh}_{\tau}}(\underline{C})$  $\Longrightarrow E \in W_{\underline{Sh}_{\tau}}(\underline{C}) \quad (\underline{a}_{\tau} \circ \iota_{\tau} = id).$ 

25.9 RAPPEL A morphism

$$\mathrm{F} \colon (\underline{\mathrm{C}}_1, \boldsymbol{\mathcal{W}}_1) \ \to \ (\underline{\mathrm{C}}_2, \boldsymbol{\mathcal{W}}_2)$$

of category pairs is a functor  $F:\underline{C}_1 \to \underline{C}_2$  such that  $FW_1 \subset W_2$ , thus there is a unique functor  $\overline{F}:W_1^{-1}\underline{C}_1 \to W_2^{-1}\underline{C}_2$  for which the diagram



commutes.

Take

$$\begin{array}{c} \underline{C}_{1} = \underline{SIC} \\ \underline{C}_{2} = \underline{SISh}_{T}(\underline{C}) \end{array} \right| \begin{array}{c} w_{1} = W_{\underline{C}}(\tau) \\ w_{2} = W_{\underline{Sh}_{T}}(\underline{C}) \end{array}$$

5.

and let

 $F = \underline{a}_{\tau}$ .

Then  $\underline{a}_{\tau}:\underline{C}_1 \rightarrow \underline{C}_2$  is a morphism of category pairs, so

$$\overline{\underline{a}_{\tau}}: \mathscr{W}_{1}^{-1}\underline{C}_{1} \to \mathscr{W}_{2}^{-1}\underline{C}_{2}.$$

Take

$$\begin{array}{c} \underline{C}_{1} = \underline{SISh}_{T}(\underline{C}) \\ \underline{C}_{2} = \underline{SIC} \end{array} , \qquad \begin{array}{c} w_{1} = w_{\underline{Sh}_{T}}(\underline{C}) \\ w_{2} = w_{\underline{C}}(\tau) \\ \underline{C}_{2} \end{array}$$

and let

 $F = \iota_T$ .

Then  $\iota_{\tau}: \underline{C}_1 \rightarrow \underline{C}_2$  is a morphism of category pairs, so

$$\overline{\mathfrak{l}_{\tau}}: \mathscr{W}_{1}^{-1} \underline{\mathsf{C}}_{1} \to \mathscr{W}_{2}^{-1} \underline{\mathsf{C}}_{2}.$$

25.10 THEOREM The functors  $\begin{bmatrix} a_{\tau} \\ a_{\tau} \end{bmatrix}$  are an adjoint pair and induce an adjoint  $a_{\tau}$ 

equivalence of metacategories.

[The arrows of adjunction are natural isomorphisms.]

25.11 CRITERION Let  $\underline{E}_1$ ,  $\underline{E}_2$  be Grothendieck toposes, let  $\Phi:\underline{E}_1 \rightarrow \underline{E}_2$  be a functor, and let  $W_2$  be an admissible  $\underline{E}_2$ -localizer. Assume that  $\Phi$  preserves colimits and finite limits and that  $\Phi^{-1}W_2$  is an  $\underline{E}_1$ -localizer -- then  $\Phi^{-1}W_2$  is admissible.

25.12 LEMMA If W is admissible, then  $W_{\hat{L}}(\tau)$  is admissible.  $\underline{\underline{C}}$ 

PROOF In 25.11, let  $\underline{E}_1 = \underline{SIC}$ ,  $\underline{E}_2 = \underline{SISh}_{\tau}(\underline{C})$ ,  $\Phi = \underline{a}_{\tau}$ ,  $W_2 = W_{\underline{Sh}_{\tau}(\underline{C})}$  -- then  $W_{\underline{Sh}_{\tau}(\underline{C})}$  is admissible (cf. 23.15) and

$$\underline{\underline{a}}_{\tau}^{-1} \mathbb{W}_{\underline{\mathrm{Sh}}_{\tau}}(\underline{C}) = \mathbb{W}_{\hat{C}}(\tau) \quad (\text{cf. 25.8}).$$

25.13 REMARK Since  $W_{\hat{L}}(\tau)$  is admissible if W is admissible,  $\underline{SI\hat{L}}$  admits a  $\hat{\underline{C}}$  cofibrantly generated model structure whose class of weak equivalences are the elements of  $W_{\hat{L}}(\tau)$  and whose cofibrations are the monomorphisms (cf. 20.12).  $\underline{C}$  Accordingly, in 25.10, the data gives rise to an adjoint equivalence of homotopy categories.

[Note: If <u>C</u> is a model category, then <u>HC</u> (=  $W^{-1}C$ ) is a category (and not just a metacategory).]

25.14 LEMMA Suppose that W is admissible and that the Cisinski structure on  $\hat{\Delta}$  per W is proper -- then the Cisinski structure on <u>SIC</u> per W<sub> $\hat{\lambda}$ </sub>( $\tau$ ) is proper.

PROOF To begin with, this is the case if  $\tau$  is the minimal Grothendieck topology on C (cf. 25.1 and 25.6). In general, there are two points.

(1) Since  $\underline{a}_{\tau}$  preserves finite limits, hence preserves pullbacks, the  $\tau$ -hypercoverings are pullback stable (cf. 22.9).

(2) Every fibration of W-descent per  $W_{\lambda}(\tau)$  is a fibration of W-descent per  $\underbrace{C}_{C}$ 

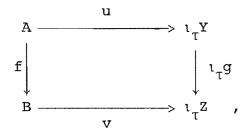
```
Now quote 20.17.
```

[Note: As always, it is right proper which is at issue (cf. 20.15).]

25.15 LEMMA Suppose that W is admissible and that the Cisinski structure on  $\hat{\underline{\Delta}}$  per W is proper -- then the Cisinski structure on  $\underline{SISh}_{T}(\underline{C})$  per  $\underline{W}_{\underline{Sh}_{T}}(\underline{C})$  is proper.

PROOF Fibrations in  $\underline{SISh}_{T}(\underline{C})$  "are" fibrations in  $\underline{SIC}$  and pullbacks in  $\underline{SISh}_{T}(\underline{C})$  "are" pullbacks in SIC.

[To provide a modicum of detail, suppose that  $g: Y \rightarrow Z$  is a fibration of W-descent per  $\underline{SISh}_{T}(\underline{C})$  -- then  $\iota_{T}g$  is a fibration of W-descent per  $\underline{SIC}$ . Thus consider the lifting problem



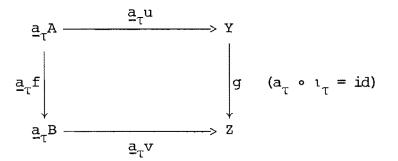
where f is an acyclic cofibration -- then

 $f \in W_{\underline{C}}(\tau) \Rightarrow \underline{a}_{T}f \in W_{\underline{Sh}_{T}}(\underline{C})$  (cf. 25.8).

But  $\underline{a}_{\tau}$  preserves monomorphisms, hence

$$\underline{a}_{\tau} \mathbf{f} : \underline{a}_{\tau} \mathbf{A} \neq \underline{a}_{\tau} \mathbf{B}$$

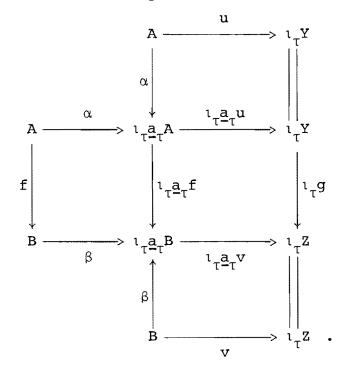
is an acyclic cofibration. Therefore the commutative diagram



has a filler  $w:\underline{a}_{\tau}B \rightarrow Y$ , i.e.,

$$\begin{bmatrix} w \circ \underline{a}_{\tau} f = \underline{a}_{\tau} u \\ g \circ w = \underline{a}_{\tau} v. \end{bmatrix}$$

Now form the commutative diagram



Then  $\iota_{\tau} w \circ \beta: B \rightarrow \iota_{\tau} Y$  is a solution to our lifting problem:

$$\begin{bmatrix} \iota_{\tau} \mathbf{w} \circ \beta \circ \mathbf{f} = \iota_{\tau} \mathbf{w} \circ \iota_{\tau} \mathbf{a}_{\tau} \mathbf{f} \circ \alpha = \iota_{\tau} \mathbf{a}_{\tau} \mathbf{u} \circ \alpha = \mathbf{u} \\ \iota_{\tau} \mathbf{g} \circ \iota_{\tau} \mathbf{w} \circ \beta = \iota_{\tau} \mathbf{a}_{\tau} \mathbf{v} \circ \beta = \mathbf{v}. \end{bmatrix}$$

25.16 SCHOLIUM (cf. 23.17) Fix  $\tau \in \tau_{C}^{}$  and take W = W\_{\infty}^{} -- then

$$\begin{bmatrix} \underline{SIC} \\ \underline{SISh}_{T} (\underline{C}) \end{bmatrix}$$

admit a cofibrantly generated proper model structure whose class of weak equivalences are the elements of

$$= (W_{\infty})_{\underline{C}} (\tau)$$

$$= (W_{\infty})_{\underline{Sh}_{\tau}} (\underline{C})$$

and whose cofibrations are the monomorphisms.

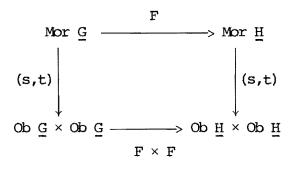
[Note: Here there is present an additional item of structure, viz. that these model categories are simplicial model categories.]

## INTERNAL AFFAIRS

IA-1 NOTATION <u>GRD</u> is the full subcategory of <u>CAT</u> whose objects are the groupoids (the morphisms are functors).

IA-2 LEMMA Let  $G, H \in Ob \ \underline{GRD}$  and suppose that  $F: G \rightarrow H$  is a functor.

• F is fully faithful iff the diagram



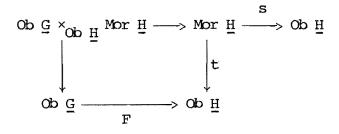
is a pullback in SET.

• F has a representative image iff the composite

$$Ob \subseteq \times_{Ob \underbrace{H}} Mor \underbrace{H} \longrightarrow Mor \underbrace{H} \longrightarrow Ob \underbrace{H}$$

is surjective.

[Note: Here



<u>N.B.</u> These points characterize an equivalence between groupoids and provide the motivation for the notion of "internal equivalence" infra. IA-3 THEOREM <u>GRD</u> is a model category if weak equivalence = equivalence and the cofibrations are those functors  $F: G \rightarrow H$  such that the map

$$\begin{array}{c} \text{Ob } \underline{G} \neq \text{Ob } \underline{H} \\ \\ X \neq FX \end{array}$$

is injective.

[Note: All objects are fibrant and cofibrant.]

IA-4 LEMMA Let  $G, H \in Ob \underline{GRD}$ ,  $F: G \rightarrow H$  a functor -- then F is an equivalence iff the induced simplicial map ner F:ner  $G \rightarrow$  ner H of nerves is a simplicial weak equivalence.

IA-5 LEMMA Let  $G, H \in Ob \underline{GRD}$ ,  $F: G \rightarrow H$  a functor -- then F is a fibration iff the induced simplicial map ner F:ner  $G \rightarrow$  ner H of nerves is a Kan fibration.

IA-6 LEMMA Let X,Y be simplicial sets and let  $f:X \rightarrow Y$  be a simplicial map.

• If f is a simplicial weak equivalence, then the induced morphism  $If: IX \rightarrow IIY$  of fundamental groupoids is an equivalence.

• If f is a cofibration, then the induced morphism  $\Pi f: \Pi X \rightarrow \Pi Y$  of fundamental groupoids is injective on objects.

IA-7 REMARK Since

## $\Pi:\texttt{SISET} \rightarrow \texttt{GRD}$

is a left adjoint for

## ner:GRD $\rightarrow$ SISET,

it follows from the lemmas that  $\Pi$  is a left model functor, i.e., preserves cofibrations and acyclic cofibrations, and ner is a right model functor, i.e., preserves fibrations and acyclic fibrations.

[Note: Here the underlying model structure on <u>SISET</u> is, of course, the Kan structure. To get a model equivalence, simply replace it by its truncation at level 1 (thus now the weak equivalences are the 1-equivalences (so the arrows are isomorphisms at  $\pi_0$  and  $\pi_1$ )).]

Let  $\underline{E}$  be a Grothendieck topos — then  $\underline{E}$  is complete so the formalism of internal category theory is applicable. And, as will be seen below, the results outlined above for the case  $\underline{E}$  = SET actually go through in general.

IA-8 NOTATION <u>GRD(E)</u> is the full subcategory of <u>CAT(E)</u> whose objects are the groupoids in <u>E</u> (the morphisms are internal functors).

[Note: Recall that an object  $\underline{G}$  of  $\underline{GRD}(\underline{E})$  is a pair  $(\underline{G}_0, \underline{G}_1)$  of objects of  $\underline{E}$  together with a battery of morphisms satisfying the usual axioms.]

IA-9 EXAMPLE Let C be a small category -- then

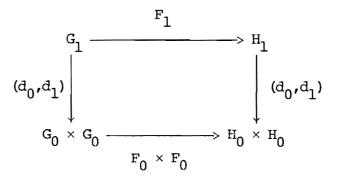
$$\underline{\operatorname{GRD}}(\widehat{\underline{C}}) \approx [\underline{\underline{C}}^{\operatorname{OP}}, \underline{\operatorname{GRD}}].$$

IA-10 DEFINITION Let  $\underline{G},\underline{H} \in Ob \underline{GRD}(\underline{E})$  and suppose that  $F:\underline{G} \rightarrow \underline{H}$  is an internal functor, hence  $F = (F_0,F_1)$ , where

are morphisms in E (subject to ...) - then F is said to be an internal equivalence if

3.

(1) The diagram



is a pullback in  $\underline{E}$  and

(2) The composite

$$G_0 \times_{H_0} H_1 \longrightarrow H_1 \xrightarrow{d_0} H_0$$

is an epimorphism.

[Note: Here

IA-11 THEOREM <u>GRD(E)</u> is a model category if weak equivalence = internal equivalence and the cofibrations are those internal functors  $F: G \rightarrow H$  such that the arrow

$$F_0:G_0 \rightarrow H_0$$

is a monomorphism.

<u>N.B.</u> Take  $\underline{E} = \underline{SET}$  to recover IA-3.

IA-12 RAPPEL Every category <u>C</u> in <u>E</u> gives rise to a simplicial object ner <u>C</u> in <u>E</u> by letting ner<sub>0</sub><u>C</u> = <u>C</u><sub>0</sub>, ner<sub>1</sub><u>C</u> = <u>C</u><sub>1</sub>, and

$$\operatorname{ner}_{n-}^{C} = C_{1} \times_{C_{0}} \cdots \times_{C_{0}}^{C_{1}} (n \text{ factors}).$$

[Note: An internal functor  $\underline{C} \rightarrow \underline{C}'$  induces a morphism ner  $\underline{C} \rightarrow$  ner  $\underline{C}'$  of simplicial objects.]

IA-13 LEMMA Let  $G, H \in Ob \underline{GRD}(\underline{E})$ ,  $F:\underline{G} \to \underline{H}$  an internal functor — then F is an internal equivalence iff ner F:ner  $\underline{G} \to ner \underline{H}$  is a weak equivalence of  $W_{\infty}$ -descent.

IA-14 REMARK The functor

$$\operatorname{ner}: \operatorname{GRD}(\underline{E}) \to \operatorname{SIE}$$

has a left adjoint

$$\Pi: SIE \rightarrow GRD(E).$$

Working with the model structure on <u>SIE</u> per 23.17 (the weak equivalences thus being the weak equivalences of  $W_{\infty}$ -descent), what was said in IA-7 can be said again. In particular: If  $\underline{G} \in Ob \ GRD(\underline{E})$  is fibrant, then ner  $\underline{G}$  is fibrant.

Let <u>C</u> be a small category,  $\tau$  a Grothendieck topology on <u>C</u> -- then <u>SIC</u> admits a cofibrantly generated proper model structure whose class of weak equivalences are the elements of

and whose cofibrations are the monomorphisms (cf. 25.16).

[Note: If  $\tau$  is the minimal Grothendieck topology on C, then

$$(W_{\infty})_{\hat{\underline{C}}}(\tau) = (W_{\infty})_{\hat{\underline{C}}}$$

and the elements of  $(W_{\infty})_{\hat{C}}$  are levelwise the elements of  $W_{\infty}$  (cf. 25.1). Therefore

in this case the model structure on

$$\underline{\operatorname{SIC}} \approx [\underline{\operatorname{C}}^{\operatorname{OP}}, \underline{\operatorname{SISET}}]$$

is the injective structure.]

• If  $G:\underline{C}^{OP} \rightarrow \underline{GRD}$ , then • If  $G, H:\underline{C}^{OP} \rightarrow \underline{GRD}$  and if  $E:G \rightarrow H$ , then

ner  $\Xi$ :ner  $G \rightarrow$  ner H.

IA-15 THEOREM [ $\underline{C}^{OP}$ , <u>GRD</u>] is a model category if the weak equivalences are the E:G  $\rightarrow$  H such that ner E is a weak equivalence of  $(W_{\infty}, \tau)$ -descent and the fibrations are the E:G  $\rightarrow$  H such that ner E is a fibration of  $(W_{\infty}, \tau)$ -descent.

For ease of reference, call the objects of  $[\underline{C}^{OP}, \underline{SISET}]$  simplicial presheaves and the objects of  $[\underline{C}^{OP}, \underline{GRD}]$  simplicial groupoids.

IA-16 DEFINITION A <u>fibrant model</u> for a simplicial presheaf X is a fibrant simplicial presheaf  $X_f$  and a weak equivalence of  $(W_{\infty}, \tau)$ -descent  $X \rightarrow X_f$ .

IA-17 DEFINITION A simplicial presheaf X is said to <u>satisfy descent</u> if for some fibrant model  $X_{f}$ , the arrow

is a simplicial weak equivalence  $\forall U \in Ob C$ .

IA-18 LEMMA If A and B are fibrant simplicial presheaves and if  $f:A \rightarrow B$  is a weak equivalence of  $(W_{\infty}, \tau)$ -descent, then  $\forall U \in Ob \underline{C}$ , the arrow AU  $\rightarrow$  BU is a simplicial weak equivalence.

IA-19 APPLICATION If X is a simplicial presheaf, if  $X_f$  and  $X'_f$  are fibrant models for X, and if  $\forall U \in Ob C$ , the arrow

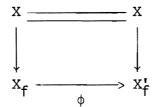
 $XU \rightarrow X_{f}U$ 

is a simplicial weak equivalence, then  $\forall \ U \in \mbox{Ob}\ \underline{C},$  the arrow

 $XU \rightarrow X_{f}^{\dagger}U$ 

is a simplicial weak equivalence.

[Choose  $\phi: X_f \to X_f'$  such that the diagram



commutes -- then  $\phi$  is a weak equivalence of  $(W_{\infty}, \tau)$ -descent (by the 2 out of 3 condition), hence  $\forall U \in Ob C$ , the arrow

$$X_{f}U \rightarrow X_{f}'U$$

is a simplicial weak equivalence, from which the assertion.]

Consequently, the notion of "descent" is independent of the choice of a fibrant model.

IA-20 DEFINITION Let G be a simplicial groupoid -- then G is said to be a

stack if ner G satisfies descent.

IA-21 DEFINITION A stack completion of a presheaf of groupoids G is a weak equivalence  $G \rightarrow G'$ , where G' is a stack.

It is a fact that a stack completion for a given G always exists. E.g.: One possibility is to take G' = G-tors<sub>d</sub> (Jardine's "discrete G-torsors").

IA-22 REMARK The definition of stack is a moving target.