The Selberg Trace Formula IX:
Contribution from the Conjugacy Classes (The Regular Case)

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* Research supported in part by the National Science Foundation

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§1. Introduction

This is the ninth in a projected series of papers in which we plan to come to grips with the Selberg trace formula, the ultimate objective being a reasonably explicit expression. In our last publication [2-(h)], we isolated the contribution to the trace arising from the continuous spectrum, call it

$$\mathbf{Con}-\mathbf{Sp}(\alpha:\Gamma).$$

Here, we shall initiate the study of the contribution to the trace arising from the conjugacy classes, call it

$$\mathbf{Con}-\mathbf{Cl}(\alpha:\Gamma).$$

Thus, in the usual notation, one has

$$\operatorname{tr}(L_{G/\Gamma}^{\mathrm{dis}}(\alpha)) = \mathbf{Con}-\mathbf{Cl}(\alpha:\Gamma)+\mathbf{Con}-\mathbf{Sp}(\alpha:\Gamma).$$

And, thanks to Theorem 4.3 of [2-(h)], there is a formula for

$$\mathbf{Con}-\mathbf{Sp}(\alpha:\Gamma)$$

involving familiar ingredients, namely $\mathbf{c}$-functions, exponentials, and **Ind**-functions. As for

$$\mathbf{Con}-\mathbf{Cl}(\alpha:\Gamma),$$

it will turn out that

$$\mathbf{Con}-\mathbf{Cl}(\alpha:\Gamma)=\sum_{\mathcal{C}_0}\sum_{\mathcal{C}\succeq\mathcal{C}_0}\mathbf{Con}-\mathbf{Cl}(\alpha:\Gamma_{\mathcal{C},\mathcal{C}_0}),$$

reflecting the partition

$$\Gamma=\coprod_{\mathcal{C}_0}\coprod_{\mathcal{C}\succeq\mathcal{C}_0}\Gamma_{\mathcal{C},\mathcal{C}_0}$$

explained in §2. The objective then is to find a formula for

$$\mathbf{Con}-\mathbf{Cl}(\alpha:\Gamma_{\mathcal{C},\mathcal{C}_0})$$

involving familiar ingredients, namely orbital integrals (weighted or not) and their variants.

[Note: Strictly speaking (and in complete analogy with the conclusions of [2-(h)]), the formula will also contain a parameter $\mathbf{H}$ from the truncation space that, however, we shall ignore for the purposes of this Introduction.]

The simplest case is when $\mathcal{C} = \mathcal{C}_0$, $\Gamma_{\mathcal{C}_0,\mathcal{C}_0}$ being what we like to think of as the $\mathcal{C}_0$-regular elements of Γ. Each such element is necessarily semisimple and the identity component of its centralizer is contained in the Γ-Levi subgroup minimal with respect to "$\in$", although this need not be true of the full centralizer, a complicating circumstance. The situation is therefore similar to that considered by Arthur [1-(b), §8], although a little more general due to the last mentioned point. The basis for our analysis will be the machinery set down in [2-(f)], applied to the case at hand. Its consistent use serves to smooth out most of the technical wrinkles, permitting the exposition to proceed in a systematic fashion.

We begin in §2 with a classification of the elements of Γ, introducing in particular the $\Gamma_{\mathcal{C},\mathcal{C}_0}$. This material generalizes the rank-1 considerations of [2-(a), §5]. In §3, the fine structure of $\Gamma_{\mathcal{C}_0,\mathcal{C}_0}$ is examined. §4 is a brief exposition of the "big picture" and may be regarded as a supplement to this Introduction. The study of

$$\mathbf{Con} - \mathbf{Cl}(\alpha : \Gamma_{\mathcal{C}_0,\mathcal{C}_0})$$

is taken up in §5, the main result being Theorem 5.4, which then, in §6, is recast inductively as Theorem 6.3.

Finally, in what follows, the abbreviation TES will refer to our monograph, *The Theory of Eisenstein Systems*, Academic Press, N.Y., 1981.

§2. Classification of the Elements of Γ

The purpose of this § is to devise a delineation of the elements of Γ suitable for the calculations which are to follow (here and elsewhere).

Given $\gamma \in \Gamma$, let us agree to write $\{\gamma\}_G$ (respectively $\{\gamma\}_\Gamma$) for its G-conjugacy class (respectively Γ-conjugacy class), G_γ (respectively Γ_γ) for its G-centralizer (respectively Γ-centralizer).

Put

$$Z_\Gamma = \text{ center of } \Gamma.$$

Proposition 2.1. *Suppose that $\gamma \in Z_\Gamma$—then γ is semisimple. Furthermore, Γ_γ is a nonuniform lattice in G_γ.*

[To prove this, one need only repeat the discussion on p. 19 of [2-(a)], the assumption there that $\text{rank}(\Gamma) = 1$ being of no relevance at all.]

An element $\gamma \in \Gamma$ is said to be G-regular provided that γ belongs to no proper Γ-cuspidal parabolic subgroup of G. Denote by Γ_G the set of such—then Γ_G is obviously invariant under Γ-conjugacy. Calling $[\Gamma_G]$ some choice of representatives for the Γ-conjugacy classes in Γ_G, form

$$\coprod_{\gamma \in [\Gamma_G]} (G/\Gamma_\gamma) \times \{\gamma\}$$

and let Φ be the canonical map from this set to G, viz.

$$\Phi(x\Gamma_\gamma, \gamma) = x\gamma x^{-1}.$$

Proposition 2.2. *Φ is a proper map.*

We shall need a preliminary result.

Lemma 2.3. *Let P be a Γ-percuspidal parabolic subgroup of G with split component A. Suppose that*

$$\begin{cases} \{a_n\} \in A[t] \\ \{\gamma_n\} \in \Gamma \end{cases}$$

are sequences such that

$$a_n^\lambda \to -\infty (\lambda \in \Sigma_P^0(\mathfrak{g}, \mathfrak{a}))$$

and

$$a_n \gamma_n a_n^{-1} \quad \textit{stays bounded.}$$

Then eventually

$$\gamma_n \in P_F, \quad F = \Sigma_P^0(\mathfrak{g}, \mathfrak{a}) - \{\lambda\}.$$

Proof. Let $\eta \in \Gamma \cap N_F$—then

$$a_n \eta a_n^{-1} \to 1$$

and so

$$(a_n \gamma_n a_n^{-1}) a_n \eta a_n^{-1} (a_n \gamma_n a_n^{-1})^{-1} \to 1$$

or still

$$a_n \gamma_n \eta \gamma_n^{-1} a_n^{-1} \to 1.$$

Thanks to the lemma on p. 47 of TES,

$$\gamma_n \eta \gamma_n^{-1} \in N$$

eventually. Consequently, upon taking generators for $\Gamma \cap N_F$,

$$\gamma_n (\Gamma \cap N_F) \gamma_n^{-1} \subset N$$

eventually. This implies that for all sufficiently large n,

$$\mathrm{Ad}(\gamma_n) \mathfrak{n}_F \subset \mathfrak{n},$$

from which, passing to orthocomplements,

$$\mathrm{Ad}(\gamma_n) \mathfrak{p}_F \supset \mathfrak{p},$$

i.e.,

$$\gamma_n P_F \gamma_n^{-1} \supset P.$$

Therefore, $\forall n >> 0$,

$$\gamma_n P_F \gamma_n^{-1} = P_F \Rightarrow \gamma_n \in P_F,$$

as desired.

Proof of Proposition 2.2. Let C_G be a compact subset of G—then we must show that $\Phi^{-1}(C_G)$ is a compact subset of

$$\coprod_{\gamma \in [\Gamma_G]} (G/\Gamma_\gamma) \times \{\gamma\}.$$

For this purpose, let

$$\{(x_n \Gamma_{\gamma_n}, \gamma_n)\}$$

be a sequence in $\Phi^{-1}(C_G)$. Since G can be covered by finitely many $\mathfrak{S} \bullet \Gamma$, $\mathfrak{S}$ a Siegel domain relative to a Γ-percuspidal P, there is no loss of generality in supposing to begin with that $x_n \in \mathfrak{S} \bullet \Gamma \ \forall n$. That being, write as usual

$$x_n = k_n a_n s_n \delta_n.$$

Because

$$k_n a_n s_n a_n^{-1}$$

stays bounded, the same holds for

$$a_n \delta_n \gamma_n \delta_n^{-1} a_n^{-1}.$$

But

$$\delta_n \gamma_n \delta_n^{-1} \notin P_F \qquad \forall F \neq \Sigma_P^0(\mathfrak{g}, \mathfrak{a})$$

and so, on the basis of the foregoing lemma, a_n must stay bounded too. Therefore, after passing to a subsequence if necessary, we may assume that k_n, a_n, and s_n are all convergent, along with $x_n \gamma_n x_n^{-1}$. Accordingly, $\delta_n \gamma_n \delta_n^{-1}$ is convergent, hence is eventually constant. Thus, by definition of $[\Gamma_G]$, $\forall n >> 0$, $\gamma_n = \gamma$ and $\delta_n \Gamma_\gamma = \delta \Gamma$, implying that $\lim x_n \Gamma_\gamma$ exists. This establishes the compactness in

$$\coprod_{\gamma \in [\Gamma_G]} (G/\Gamma_\gamma) \times \{\gamma\}$$

of $\Phi^{-1}(C_G)$.

The following points are immediate corollaries.

(1) $\forall \gamma \in \Gamma_G$, γ is semisimple and Γ_γ is a uniform lattice in G_γ.

(2) $\forall$ compactum $C_G \subset G$,

$$\#(\{\gamma \in [\Gamma_G] : \{\gamma\}_G \cap C_G \neq \emptyset\}) < +\infty.$$

(3) $\forall$ compactum $C_G \subset G$ and $\forall \gamma \in \Gamma_G$,

$$\{x \in G : x \gamma x^{-1} \in C_G\}$$

is compact mod G_γ.

(4) $\forall \gamma \in \Gamma_G$,

$$\#\Big(\{\gamma' \in [\Gamma_G] : \{\gamma'\}_G = \{\gamma\}_G\}\Big) < +\infty.$$

Observe that no Γ-central element can belong to Γ_G. Indeed, if $\gamma \in Z_\Gamma$, then Γ_γ is a nonuniform lattice in G_γ, whereas if $\gamma \in \Gamma_G$, then Γ_γ is a uniform lattice in G_γ.

Suppose now that P is a Γ-cuspidal parabolic subgroup of G—then an element $\gamma \in \Gamma$ is said to be P-regular if $\gamma \in P$ but $\gamma \notin P'$ for all $P' \prec P$. Denote by Γ_P the set of such—then

$$\Gamma = \cup \Gamma_P,$$

although, of course, there is overlap in the union on the right.

Proposition 2.4. *If*

$$\Gamma_{P_1} \cap \Gamma_{P_2} \neq \emptyset,$$

then P_1 and P_2 are associate.

Admitting this momentarily, given an association class $\mathcal{C}$, put

$$\Gamma_{\mathcal{C}} = \cup_{P \in \mathcal{C}} \Gamma_P.$$

Since

$$\gamma \Gamma_P \gamma^{-1} = \Gamma_{\gamma P \gamma^{-1}},$$

it is clear that $\Gamma_{\mathcal{C}}$ is invariant under Γ-conjugacy and, by the above,

$$\mathcal{C}' \neq \mathcal{C}'' \Rightarrow \Gamma_{\mathcal{C}'} \cap \Gamma_{\mathcal{C}''} = \emptyset,$$

so

$$\Gamma = \coprod_{\mathcal{C}} \Gamma_{\mathcal{C}}.$$

Needless to say,

$$\Gamma_{\{G\}} = \Gamma_G.$$

In general, if P is a Γ-cuspidal parabolic subgroup of G with unipotent radical N, then a Levi subgroup L of P is a closed reductive subgroup with the property that the multiplication $L \times N \rightarrow P$ is an isomorphism of analytic manifolds (cf. TES, p. 31). To reflect Γ-cuspidality, it will be best to specialize this notion, using the

$$\begin{cases} (G^*, \Gamma^*) \\ (G^\#, \Gamma^\#) \end{cases}$$

formalism in TES (pp. 40-41), putting for $1 \leq i \leq r$

$$\begin{cases} G_i^{\#} = G^{\#} / \prod\limits_{j \neq i} G_j^* \\ \Gamma_i^{\#} = \Gamma^{\#} \bullet \prod\limits_{j \neq i} G_j^* / \prod\limits_{j \neq i} G_j^* \end{cases}$$

and

$$x_i^{\#} = (xZ \bullet G_{\mathrm{c,ss}}) \bullet \prod_{j \neq i} G_j^* \quad (x \in G).$$

By a Γ-Levi subgroup L of P we shall then understand a Levi subgroup L of P such that

(i) When $i \leq r_1, L_i^{\#} = G_i^{\#}$.

(ii) When $r_1 < i \leq r_2$, $L_i^{\#}$ is a Levi subgroup of $P_i^{\#}$ (per $\Gamma_i^{\#}$).

(iii) When $r_2 < i \leq r$, $\mathrm{Ad}(L_i^{\#})$ is a $\mathbf{Q}$-Levi subgroup of $\mathrm{Ad}(P_i^{\#})$ (per $\mathrm{Ad}(\Gamma_i^{\#})$).

Generically, let

$$\gamma = \gamma_{\mathrm{s}} \gamma_{\mathrm{u}}$$

be the Jordan decomposition of γ. Note that

$$\gamma_{\mathrm{s}} \in P \ \text{ iff } \ \forall i, \ (\gamma_{\mathrm{s}}^{\#})_i \in P_i^{\#}.$$

Moreover, if L is a Γ-Levi subgroup of P, then

$$\gamma_{\mathrm{s}} \in L \ \text{ iff } \ \forall i, \ (\gamma_{\mathrm{s}}^{\#})_i \in L_i^{\#}.$$

Lemma 2.5. *Suppose that $\gamma \in \Gamma_P$—then there exists a Γ-Levi subgroup $L = M \bullet A$ of P such that $\gamma_{\mathrm{s}} \in M$. In addition, P is minimal with respect to "$\gamma_{\mathrm{s}} \in P$".*

Proof. In view of the reductions outlined above, we can deal with each possibility separately, mentally making the necessary changes in the notation. Start off with any M and let $\{\delta\} = M \cap \gamma N$—then $\delta \in \Gamma_M$ and we claim that δ is M-regular, hence semisimple. For otherwise,

$$\delta \in {}'P \Rightarrow \delta N \subset P' \Rightarrow \gamma \in P',$$

an impossibility. Let

$$\mathfrak{n}(\delta) = \mathrm{Im}(\mathrm{Ad}(\delta) - 1)$$

and use the surjection

$$\phi : \begin{cases} \mathfrak{n}(\delta) \times N_\delta \rightarrow N \\ (X, n) \rightarrow \delta^{-1} \exp(X)(\delta n) \exp(-X) \end{cases}$$

to write

$$\gamma = \delta\phi(X, n) = \exp(X)(\delta n) \exp(-X).$$

Since δ and n commute,

$$\gamma = [\exp(X)\delta \exp(-X)] \bullet [\exp(X)n \exp(-X)]$$

is the Jordan decomposition of γ. In particular,

$$\gamma_s \in \exp(X)M \exp(-X),$$

leading to the first assertion. As for the second, if the contrary were true, then the M-regularity of γ_s would be violated.

In passing, observe that L is actually minimal with respect to "$\gamma_s \in L$".

Proof of Proposition 2.4. Let

$$\gamma \in \Gamma_{P_1} \cap \Gamma_{P_2}.$$

To prove that P_1 and P_2 are associate, we need only produce an $L = M \bullet A$ with $\gamma_s \in M$ that is simultaneously a Γ-Levi subgroup for both P_1 and P_2. To this end, we shall proceed on a case-by-case basis. If $i \leq r_1$, then there is nothing to prove. If $r_1 < i \leq r_2$, then either $P_1 = P_2$ or $P_1 \neq P_2$ and in the latter situation

$$P_1 \cap P_2 = L = M \bullet A \Rightarrow \gamma \in M \Rightarrow \gamma = \gamma_s.$$

Finally, if $r_2 < i \leq r$, then L is minimal with respect of "$\gamma_s \in L$" iff A is maximal with respect to $A \subset G^0_{\gamma_s}$, meaning that A is a maximal $\mathbf{Q}$-split torus in $G^0_{\gamma_s}$. Any two such are conjugate. Because

$$P_1 \cap G^0_{\gamma_s} \text{ and } P_2 \cap G^0_{\gamma_s}$$

are minimal $\mathbf{Q}$-parabolic subgroups, they share a split component and we can take for L its G-centralizer.

Having decomposed Γ as a disjoint union

$$\coprod_{\mathcal{C}} \Gamma_{\mathcal{C}},$$

the next step will be to fix a $\mathcal{C}_0$ and decompose $\Gamma_{\mathcal{C}_0}$ still further in terms of the $\mathcal{C} \succeq \mathcal{C}_0$.

Lemma 2.6. *Let $L = M \bullet A$ be a Γ-Levi subgroup of P. Suppose that*

$$\gamma \in P \text{ and } \gamma_s \in M.$$

Then the following conditions are equivalent:

$$\begin{array}{ll} \text{(i) } N_\gamma = \{1\}; & \text{(iii) } P_{\gamma_s} \subset L \\ \text{(ii) } N_{\gamma_s} = \{1\}; & \text{(iv) } G^0_{\gamma_s} \subset L. \end{array}$$

[The verification is straightforward, hence can be omitted.]

Remark. As can be seen by example, one cannot improve (iv) to read $G_{\gamma_s} \subset L$. For instance, take

$$G = \mathbf{PSL}_2(\mathbf{C}), \Gamma = \mathbf{PSL}_2\left(\mathbf{Z}[\sqrt{-1}]\right)$$

and consider

$$\gamma = \begin{pmatrix} 0 & \pm\sqrt{-1} \\ \mp\sqrt{-1} & 0 \end{pmatrix}.$$

Proposition 2.7. *If P_1 and P_2 are minimal with respect to "$\gamma \in P$ and $N_\gamma = \{1\}$", then P_1 and P_2 are associate.*

Proof. Once again, we shall examine cases. If $i \leq r_1$, then there is nothing to prove. If $r_1 < i \leq r_2$, then either γ and $G^0_{\gamma_s}$ are contained in a proper Γ-cuspidal or they are not. Finally, if $r_2 < i \leq r$, then γ_s lies in $L_{\mathbf{Q}}$ and

$$\begin{cases} G^0_{\gamma_s} \subset L_1 \\ G^0_{\gamma_s} \subset L_2 \,. \end{cases}$$

Denote by H the reductive algebraic $\mathbf{Q}$-group generated by γ_s and $G^0_{\gamma_s}$—then $H \subset G_{\gamma_s}$ and L_1 and L_2 are minimal with respect to "$H \subset L$", thus are the centralizers of maximal $\mathbf{Q}$-split tori in $C_G(H)^0$ (which is reductive), so are conjugate.

Fix an association class $\mathcal{C}_0$. Given a $\mathcal{C} \succeq \mathcal{C}_0$, call

$$\Gamma_{\mathcal{C},\mathcal{C}_0}$$

the set of $\gamma \in \Gamma_{\mathcal{C}_0}$ for which there exists a $P \in \mathcal{C}$ minimal with respect to "$\gamma \in P$ and $N_\gamma = \{1\}$". On the basis of the proposition supra,

$$\mathcal{C}' \neq \mathcal{C}'' \Rightarrow \Gamma_{\mathcal{C}',\mathcal{C}_0} \cap \Gamma_{\mathcal{C}''\mathcal{C}_0} = \emptyset,$$

hence

$$\Gamma_{\mathcal{C}_0} = \coprod_{\mathcal{C} \succeq \mathcal{C}_0} \Gamma_{\mathcal{C},\mathcal{C}_0}.$$

In particular,

$$Z_\Gamma \subset \Gamma_{\{G\},\mathcal{C}_0},$$

$\mathcal{C}_0$ the association class made up of the Γ-percuspidals.

Heuristically,

$$\Gamma_{\mathcal{C}_0,\mathcal{C}_0}$$

are the $\mathcal{C}_0$-regular elements of Γ (cf. [2-(a)] when rank $(\Gamma) = 1$).

Remark. Because

$$\begin{cases} \gamma \Gamma_{\mathcal{C},\mathcal{C}_0} \gamma^{-1} = \Gamma_{\mathcal{C},\mathcal{C}_0} \quad (\gamma \in \Gamma) \\ \gamma \in \Gamma_{\mathcal{C},\mathcal{C}_0} \cap P \Rightarrow \gamma \bullet (\Gamma \cap N) \subset \Gamma_{\mathcal{C},\mathcal{C}_0}, \end{cases}$$

the partition

$$\Gamma = \coprod_{\mathcal{C}_0} \coprod_{\mathcal{C} \succeq \mathcal{C}_0} \Gamma_{\mathcal{C},\mathcal{C}_0}$$

satisfies the general assumptions laid down on pp. 1376–1377 of [2-(f)]. The associated Γ-compatible families are therefore estimable (cf. [2-(f), p. 1413]).

§3. The Structure of $\Gamma_{\mathcal{C}_0,\mathcal{C}_0}$

The purpose of this § is to look more closely at the structure of $\Gamma_{\mathcal{C}_0,\mathcal{C}_0}$, the results obtained being essential preparation for the analysis carried out in §5 infra.

We shall begin with a series of simple observations.

Put $\Delta = \Gamma_{\mathcal{C}_0,\mathcal{C}_0}$ and decompose Δ into Γ-conjugacy classes Δ_i:

$$\Delta = \coprod \Delta_i.$$

Fix an index i, say $i = 0$, and fix a $\gamma_0 \in \Delta_0$.

Let P_0 be an element of $\mathcal{C}_0$ minimal with respect to "$\gamma_0 \in P$ and $N_{\gamma_0} = \{1\}$". Choose, as is possible (cf. Lemma 2.5), a Γ-Levi subgroup L_0 of P_0 such that $\mathrm{ss}(\gamma_0) \in L_0$— then the centralizer of $\mathrm{ss}(\gamma_0)$ in N_0 is necessarily trivial (cf. Lemma 2.6), thus actually $\gamma_0 = \mathrm{ss}(\gamma_0)$ and so γ_0 is semisimple.

Accordingly, L_0 is minimal with respect to "$\gamma_0 \in L$" and is in fact the only Γ-Levi subgroup with this property. To see this, examine cases, the nontrivial one being the algebraic situation. But, from the proof of Proposition 2.4, L_0 is minimal with respect to "$\gamma_0 \in L$" iff L_0 is the G-centralizer of a maximal $\mathbf{Q}$-split torus in $G^0_{\gamma_0}$, itself unique as $G^0_{\gamma_0} \subset L_0$ (cf. Lemma 2.6), from which the uniqueness of L_0. Of course, there is exactly one splitting $L_0 = M_0 \bullet A_0$ with $\gamma_0 \in M_0$.

Denote by

$$\mathcal{P}_0(L_0)$$

the set of parabolics having L_0 for a Levi factor—then all such are Γ-cuspidal. To see this, again examine cases, the nontrivial one being the rank-1 situation, covered, fortunately, by Lemma 9.3 in [2-(a)]. Obviously, therefore,

$$\mathcal{P}_0(L_0) = \{P_0 \in \mathcal{C}_0 : \gamma_0 \in P_0\}.$$

If now by

$$\mathcal{P}(L_0)$$

we understand

$$\{P \in \mathcal{C}_\Gamma : \gamma_0 \in P\},$$

then it is clear that

$$\mathcal{P}(L_0) = \Big\{P \in \mathcal{C}_\Gamma : P \succeq P_0(\exists P_0 \in \mathcal{P}_0(L_0))\Big\}.$$

Evidently,

$$\forall P \in \mathcal{P}(L_0) : \Gamma_{\gamma_0} \cap P = \Gamma_{\gamma_0} \cap L,$$

$L \supset L_0$ a Γ-Levi subgroup of P.

Given

$$\begin{cases} P \in \mathcal{P}(L_0) \\ Q \in \mathcal{C}_\Gamma, \end{cases}$$

let

$$\Delta_0(P)_Q = \{\gamma\gamma_0\gamma^{-1} : \gamma P \gamma^{-1} = Q\}.$$

Then

$$\Delta_0(P)_Q \neq \emptyset$$

iff P and Q are Γ-conjugate. Moreover,

$$\Delta_0(P)_P$$

is the $\Gamma \cap P$-conjugacy class of γ_0.

Lemma 3.1. $\forall \gamma \in \Gamma$,

$$\Delta_0(P)_{\gamma Q \gamma^{-1}} = \gamma \Delta_0(P)_Q \gamma^{-1}.$$

[The verification is immediate.]

Let Q be an element of $\mathcal{C}_\Gamma$ with the property that

$$\Delta_0 \cap Q \neq \emptyset.$$

Let E be the set of all $P \in \mathcal{P}(L_0)$ that are Γ-conjugate to Q—then we claim that E is a Γ-conjugacy class in $\mathcal{P}(L_0)$. To check that this is the case, we need only convince ourselves that a Γ-conjugate of Q is in $\mathcal{P}(L_0)$. But, for some $\gamma \in \Gamma$,

$$\begin{gathered} \gamma\gamma_0\gamma^{-1} \in \Delta_0 \cap Q \\ \Rightarrow \gamma_0 \in \gamma^{-1} Q \gamma \Rightarrow \gamma^{-1} Q \gamma \in \mathcal{P}(L_0), \end{gathered}$$

as desired. Consequently,

$$\Delta_0 \cap Q \subset \cup_{P \in E} \Delta_0(P)_Q.$$

On the other hand, from the defintions,

$$\forall P \in E, \quad \Delta_0(P)_Q \subset \Delta_0 \cap Q.$$

Therefore

$$\Delta_0 \cap Q = \cup_{P \in E} \Delta_0(P)_Q.$$

To render this union disjoint, let $E(\gamma_0)$ stand for a set of representatives from E per Γ_{γ_0}-conjugacy—then the lemma infra implies that

$$\Delta_0 \cap Q = \coprod_{P \in E(\gamma_0)} \Delta_0(P)_Q.$$

Lemma 3.2. *Suppose that*

$$P_1, P_2 \in \mathcal{P}(L_0)$$

are Γ*-conjugate to a* $Q \in \mathcal{C}_\Gamma$*—then the following are equivalent:*

$$\begin{array}{ll} (i) & \Delta_0(P_1)_Q = \Delta_0(P_2)_Q; \\ (ii) & \Delta_0(P_1)_Q \cap \Delta_0(P_2)_Q \neq \emptyset; \\ (iii) & P_1 \text{ and } P_2 \text{ are } \Gamma_{\gamma_0}\text{-conjugate.} \end{array}$$

[The verification is immediate.]

Up until this point, we have worked with a fixed γ_0. For the remainder of this §, it will be necessary to work with a variable γ, the corresponding notational changes being

$$M_0 \bullet A_0 = L_0 \to L_0(\gamma) = M_0(\gamma) \bullet A_0(\gamma)$$

$$\begin{cases} \mathcal{P}_0(L_0) \to \mathcal{P}_0(L_0(\gamma)) & \mathcal{P}_0(\gamma) \\ \mathcal{P}(L_0) \to \mathcal{P}(L_0(\gamma)) & \mathcal{P}(\gamma) \end{cases}$$

to signify dependence on γ.

Proposition 3.3. *Let C be a compact subset of G—then*

$$\#\Big(\{\{\gamma\}_\Gamma : \gamma \in \Gamma_{\mathcal{C}_0, \mathcal{C}_0} \ \& \ \{\gamma\}_G \cap C \neq \emptyset\}\Big) < +\infty.$$

[Note: When rank $(\Gamma) = 1$, this is Proposition 5.15 in [2-(a)].]

To get at this, some preparation will be required. We can certainly assume that $K \bullet C \bullet K = C$. Given now a $P \in \mathcal{C}_\Gamma$, put

$$C_P = C \bullet N \cap M(\sim (C \cap S) \bullet N/N),$$

a compact subset of M.

Lemma 3.4. *Let $\gamma \in \Gamma_{C_0,C_0}$. Suppose that $P_0(\gamma) \in \mathcal{P}_0\big(L_0(\gamma)\big)$—then the following are equivalent:*

$$(i) \quad \{\gamma\}_G \cap C \neq \emptyset;$$

$$(ii) \quad \{\gamma\}_{M_0(\gamma)} \cap C_{P_0(\gamma)} \neq \emptyset.$$

[Note: To avoid any possibility of confusion, the relevant Langlands decomposition of $P_0(\gamma)$ is $M_0(\gamma) \bullet A_0(\gamma) \bullet N_0(\gamma)$, where $\gamma \in M_0(\gamma)$.]

Proof. $(i) \Rightarrow (ii)$. By hypothesis, there exists an $x \in G$ such that $x\gamma x^{-1} \in C$. Per $P_0(\gamma)$, write, as usual, $x = kmna$. Because $a\gamma a^{-1} = \gamma$, we have

$$\begin{aligned} & x\gamma x^{-1} = kmn\gamma n^{-1}m^{-1}k^{-1} \in C \\ & \Rightarrow mn\gamma n^{-1}m^{-1} \in K \bullet C \bullet K = C \\ & \Rightarrow m\gamma m^{-1} \bullet m(\gamma^{-1}n\gamma n^{-1})m^{-1} \in C \\ & \Rightarrow m\gamma m^{-1} \in C \bullet N_0(\gamma) \\ & \Rightarrow \{\gamma\}_{M_0(\gamma)} \cap C_{P_0(\gamma)} \neq \emptyset. \end{aligned}$$

$(ii) \Rightarrow (i)$. By hypothesis, there exists an $m \in M_0(\gamma)$ such that $m\gamma m^{-1} \in C_{P_0(\gamma)}$. Accordingly, $m\gamma m^{-1} = cn$. Since

$$\det\,(\mathrm{Ad}\,(m\gamma m^{-1}|\mathfrak{n}_0(\gamma) - 1) \neq 0,$$

there exists an $n_0 \in N_0(\gamma)$ such that

$$n_0(m\gamma m^{-1})n_0^{-1} = m\gamma m^{-1}n^{-1} \in C$$

$$\Rightarrow \{\gamma\}_G \cap C \neq \emptyset.$$

Remark. The second criterion of the lemma does not depend on the choice of $P_0(\gamma) \in \mathcal{P}_0\big(L_0(\gamma)\big)$. Moreover, the use of $M_0(\gamma)$ is not crucial: One can move to the ambient special "M" provided γ is replaced by $\delta, \{\delta\} =$ "M" $\cap\, \gamma N_0(\gamma)$.

Passing to the proof of the proposition, let us proceed by contradiction and assume that the cardinality in question is infinite. There will then be infinitely many $\{\gamma\}_\Gamma(\gamma \in \Gamma_{C_0,C_0})$ having a member in some fixed $P_0(\gamma) \equiv P$ (for short), this because there are but finitely many P_{i_μ}. Working with the special "M", as always from $\gamma \in \Gamma$ we determine $\delta \in \Gamma_M$ via $\{\delta\} = M \cap \gamma N$, the Γ-conjugacy class of γ filling out the Γ_M-conjugacy class of δ. And, in the case at hand, δ is M-regular. The number of M-conjugacy classes of M-regular elements that can meet C_P is finite (cf. §2). So, there are infinitely many $\{\gamma\}_\Gamma$ $(\gamma \in \Gamma_{C_0,C_0})$ producing a fixed δ, the corresponding set of γ being precisely $\Gamma \cap \delta N$. However, we claim that $\Gamma \cap \delta N$ is the union of

$$|\det\,(\mathrm{Ad}\,(\delta)|\mathfrak{n} - 1)|$$

$\Gamma \cap N$-conjugacy classes. Granted this, we have our contradiction.

The plan will be to appeal to a generality from [2-(a)], namely:

Proposition. *Let N be a connected, simply connected nilpotent Lie group; let Γ be a lattice in N. Suppose that $\phi : N \to N$ is an automorphism of N carrying Γ into itself with* $\det\,(d\phi - 1) \neq 0$. *Put*

$$\Delta = \{n \in N : \phi(n)n^{-1} \in \Gamma\}.$$

Then Δ is a (finite) union of $|\det\,(d\phi - 1)|$ *right cosets of* Γ.

[For the details, the reader is referred to pp. 24-25 of [2-(a)]. The argument, as given there, is marginally incomplete, so we shall take this opportunity to set things straight. The difficulty is that in general the Leray-Serre spectral sequence uses local coefficients when the base space is not simply connected, a point that we had overlooked at the time. But here the local coefficients of H_* of the fiber are global. Thus, take a path in $Z \bullet \Gamma\backslash N$ and lift it to a path joining 1 and $z\gamma$—then we must show that multiplication by $z\gamma$ on

$$(Z \cap \Gamma)\backslash Z \sim \Gamma\backslash Z \bullet \Gamma$$

is the identity map on

$$H_*((Z \cap \Gamma)\backslash Z).$$

Because Z is the center of N, multiplication by γ is the identity and, as Z is arcwise connected, multiplication by z is homotopic to the identity.]

We come now to the claim. Simply fix a $\gamma \in \Gamma \cap \delta N$ and, in the notation employed above, let

$$\phi, n \to \gamma^{-1} n \gamma.$$

Conclude by observing that

$$\begin{aligned} \det\,(\mathrm{Ad}\,(\delta)|\mathfrak{n} - 1) &= \det\,(\mathrm{Ad}\,(\gamma)|\mathfrak{n} - 1) \\ &= \det\,(\mathrm{Ad}\,(\gamma)|\mathfrak{n}) \bullet \det\,(1 - \mathrm{Ad}\,(\gamma^{-1})|\mathfrak{n}), \end{aligned}$$

with

$$|\det\,(\mathrm{Ad}\,(\gamma)|\mathfrak{n})| = 1.$$

§4. Rappel

The purpose of this § is to recall the main result of [2-(h)], which will then enable us to put into perspective the central theme of the present paper.

Thus, let α be a K-central, K-finite element of $C_c^\infty(G)$—then, subject to the assumptions and conventions of [2-(g), §9],

$$L_{G/\Gamma}^{\text{dis}}(\alpha)$$

is of the trace class, its trace being equal to

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma)$$

less

$$\mathbf{Fnc}(\mathbf{H}:\mathbf{H}_{\mathrm{O}}:\alpha:\Gamma),$$

where (cf. Theorem 4.3 of [2-(h)])

$$\mathbf{Fnc}(\mathbf{H}:\mathbf{H}_{\mathrm{O}}:\alpha:\Gamma)$$

is equal to

$$\sum_{\mathcal{C}_i,\mathcal{C}_{i_0}}\sum_{\mathbf{O}_{i_0}}\sum_{w_{i_0}^\dagger}$$

$$\frac{(-1)^{\dim({}'\mathfrak{a}_{i_0}(w_{i_0}^\dagger))}}{(2\pi)^{\dim(\mathfrak{a}_{i_0}'(w_{i_0}^\dagger))}}$$

$$\times\frac{1}{|\det((1-w_{i_0}^\dagger)|\mathrm{Im}\,(1-w_{i_0}^\dagger))|}$$

$$\times\frac{1}{*({}'\mathcal{C}_{i_0}(w_{i_0}^\dagger))\bullet *(\mathcal{C}_{i_0}^{\dagger\prime})}$$

$$\times\int_{\sqrt{-1}\mathfrak{a}_{i_0}'(w_{i_0}^\dagger)}\langle\mathbf{p}(\Gamma:\mathcal{C}_i:\mathbf{H}-\mathbf{H}_{\mathrm{O}}),$$

$$\mathbf{tr}_{\mathcal{C}_i}(\sum_{j_0,{}'w_{j_0i_0}} D_*^{w_{j_0}^\dagger}[\mathbf{c}(P_{i_0}|A_{i_0}:P_{j_0}|A_{j_0}:{}'w_{j_0i_0}^{-1}:?)\bullet\mathbf{e}(\mathcal{C}_{j_0}:\mathbf{H}_{\mathrm{O}}:?)]_{{}'w_{j_0i_0}\Lambda_{i_0}'}$$

$$\times\mathbf{c}(P_{j_0}|A_{j_0}:P_{i_0}|A_{i_0}:{}'w_{j_0i_0}:\Lambda_{i_0}')$$

$$\times\mathbf{c}(P_{i_0}|A_{i_0}:P_{i_0}|A_{i_0}:w_{i_0}^\dagger:0)\bullet\mathbf{Ind}_{\mathcal{C}_{i_0}}^{G}((\mathbf{O}_{i_0},\Lambda_{i_0}'))(\alpha))\rangle|d\Lambda_{i_0}'|.$$

Upon setting $\mathbf{H}_{\mathrm{O}}=\mathbf{H}$ in

$$\mathbf{Fnc}(\mathbf{H}:\mathbf{H}_{\mathrm{O}}:\alpha:\Gamma)$$

we get a polynomial in $\mathbf{H}$,

$$\mathbf{Con}\ -\mathbf{Sp}\ (\mathbf{H}:\alpha:\Gamma),$$

that represents the contribution to the trace arising from the continuous spectrum (cf. [2-(h), §5]).

It remains to analyze

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma).$$

Referring to [2-(f), p. 1433] for its definition, break up Γ,

$$\Gamma=\coprod_{\mathcal{C}_0}\coprod_{\mathcal{C}\succeq\mathcal{C}_0}\Gamma_{\mathcal{C},\mathcal{C}_0},$$

and write, as is permissible,

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma)=\sum_{\mathcal{C}_0}\sum_{\mathcal{C}\succeq\mathcal{C}_0}\mathbf{K}(\mathbf{H}:\alpha:\Gamma_{\mathcal{C},\mathcal{C}_0}).$$

It will therefore be enough to analyze each of the

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma_{\mathcal{C},\mathcal{C}_0})$$

separately. And the easiest of these to handle is the case when $\mathcal{C}=\mathcal{C}_0$:

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma_{\mathcal{C}_0,\mathcal{C}_0}),$$

the study of which will occupy us for the remainder of the paper.

§5. Analysis of $\mathbf{K}(\mathbf{H}:\boldsymbol{\alpha}:\boldsymbol{\Gamma}_{\mathcal{C}_0,\mathcal{C}_0})$

The purpose of this § is to determine the contribution to the trace furnished by

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma_{\mathcal{C}_0,\mathcal{C}_0}).$$

As we shall see, the evaluation will ultimately be in terms of weighted orbital integrals (or just orbital integrals if $\mathcal{C}_0=\{G\}$).

Put $\Delta=\Gamma_{\mathcal{C}_0,\mathcal{C}_0}$ (cf. §3) and, as in [2-(f), §5], form

$$\Phi=\{K_{\alpha,\Delta}(P:?):P\in\mathcal{C}_\Gamma\}.$$

Then, by definition,

$$\mathbf{K}(\mathbf{H}:\alpha:\Delta)=\int_{G/\Gamma}Q(\mathbf{H}:\Phi)(x)d_G(x).$$

Thanks to what can be found in [2-(f), §7],

$$\int_{G/\Gamma}Q(\mathbf{H}:\Phi)(x)d_G(x)$$

is a polynomial in $\mathbf{H}$. Next, as in [2-(f), §5], form

$$\phi=\{k_{\alpha,\Delta}(P:?):P\in\mathcal{C}_\Gamma\}.$$

We shall prove eventually that

$$\int_{G/\Gamma}Q(\mathbf{H}:\phi)(x)d_G(x)$$

is also a polynomial in $\mathbf{H}$ if $\mathbf{H}<<\mathbf{0}$. On the other hand, Theorem 5.3 in [2-(f)] implies that the difference

$$\int_{G/\Gamma}Q(\mathbf{H}:\Phi)(x)d_G(x)-\int_{G/\Gamma}Q(\mathbf{H}:\phi)(x)d_G(x)$$

is $o(\mathbf{H})$, hence vanishes. So, it will be enough to examine

$$\int_{G/\Gamma}Q(\mathbf{H}:\phi)(x)d_G(x).$$

To this end, keep to the notation of §3 and decompose Δ into Γ-conjugacy classes Δ_i:

$$\Delta=\coprod\Delta_i.$$

Form anew

$$\phi_i = \{k_{\alpha,\Delta_i}(P:?) : P \in \mathcal{C}_\Gamma\}.$$

Then

$$k_{\alpha,\Delta} = \sum_i k_{\alpha,\Delta_i},$$

a locally finite sum, and, on the basis of Proposition 5.2 in [2-(f)],

$$\int_{G/\Gamma} Q(\mathbf{H} : \phi)(x)d_G(x) = \sum_i \int_{G/\Gamma} Q(\mathbf{H} : \phi_i)(x)d_G(x),$$

thereby reducing our study to that of

$$\int_{G/\Gamma} Q(\mathbf{H} : \phi_0)(x)d_G(x),$$

$i = 0$ being the fixed index.

An additional reduction is possible provided that we take into account the machinery from §3. Indeed, by definition, at any particular $Q \in \mathcal{C}_\Gamma$,

$$k_{\alpha,\Delta_0}(Q : x) = \sum_{\gamma \in \Delta_0 \cap Q} \alpha(x\gamma x^{-1}).$$

But (cf. §3),

$$\Delta_0 \cap Q = \coprod_{P \in E(\gamma_0)} \Delta_0(P)_Q.$$

The cardinality of the Γ_{γ_0}-equivalence class of P in $\mathcal{P}(L_0)$ is

$$[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap P].$$

It therefore follows that

$$k_{\alpha,\Delta_0}(Q : x) = \sum_{P \in \mathcal{P}(L_0)} \left(\frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap P]} \bullet \sum_{\gamma \in \Delta_0(P)_Q} \alpha\left(x\gamma x^{-1}\right)\right).$$

In this connection, bear in mind that

$$\Delta_0(P)_Q \neq \emptyset$$

iff P and Q are Γ-conjugate, so we have not really overloaded the sum. Each summand determines a Γ-compatible family of functions on G (cf. Lemma 3.1), call it $\Phi(P)$. The focal point of the analysis thus becomes

$$\int_{G/\Gamma} \sum_{P \in \mathcal{P}(L_0)} Q(\mathbf{H} : \Phi(P))(x)d_G(x).$$

Individually,

$$\int_{G/\Gamma} Q(\mathbf{H} : \Phi(P))(x) d_G(x)$$

may well diverge. To deal with this difficulty, we shall need a lemma.

Lemma 5.1. *Let f be a bounded, compactly supported, measurable function on G/Γ—then*

$$\int_{G/\Gamma} Q(\mathbf{H} : \Phi(P))(x) f(x) d_G(x)$$

is equal to

$$\frac{(-1)^{\mathbf{rank}(P)}}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_{G/\Gamma_{\gamma_0} \cap L_0} \alpha(x\gamma_0 x^{-1}) f(x)$$
$$\times \chi_{P,A:\mathscr{D}}(\mathbf{H}(P) - H_{P|A}(x)) d_G(x).$$

[Note: Here, the split component A of P is per

$$(P, S; A) \succeq (P_0, S_0; A_0) \ (P_0 \in \mathcal{P}_0(L_0)),$$

where $L_0 = M_0 \bullet A_0$ with $\gamma_0 \in M_0$ (cf. §3).]

Proof. In fact,

$$\int_{G/\Gamma} Q(\mathbf{H} : \Phi(P))(x) f(x) d_G(x)$$

$$= \frac{(-1)^{\mathbf{rank}(P)}}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap P]} \bullet \int_{G/\Gamma} \sum_{\delta \in \Gamma/\Gamma \cap P} \sum_{\gamma \in \Delta_0(P)_P}$$
$$\times \alpha(x\delta\gamma\delta^{-1}x) f(x) \chi_{P,A:\mathscr{D}}(\mathbf{H}(P) - H_{P|A}(x\gamma)) d_G(x)$$

$$= \frac{(-1)^{\mathbf{rank}(P)}}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap P]} \bullet \int_{G/\Gamma \cap P} \sum_{\gamma \in \Delta_0(P)_P}$$
$$\times \alpha(x\gamma x^{-1}) f(x) \chi_{P,A:\mathscr{D}}(\mathbf{H}(P) - H_{P|A}(x)) d_G(x)$$

$$= \frac{(-1)^{\mathbf{rank}(P)}}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap P]} \bullet \int_{G/\Gamma \cap P} \sum_{\gamma \in \Gamma \cap P/\Gamma_{\gamma_0} \cap P}$$
$$\times \alpha(x\gamma\gamma_0\gamma^{-1}x^{-1}) f(x) \chi_{P,A:\mathscr{D}}(\mathbf{H}(P) - H_{P|A}(x)) d_G(x)$$

$$= \frac{(-1)^{\mathbf{rank}(P)}}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap P]} \bullet \int_{G/\Gamma_{\gamma_0} \cap P} \alpha(x\gamma_0 x^{-1}) f(x)$$
$$\times \chi_{P,A:\mathscr{D}}(\mathbf{H}(P) - H_{P|A}(x)) d_G(x)$$

$$= \frac{(-1)^{\mathbf{rank}(P)}}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_{G/\Gamma_{\gamma_0} \cap L_0} \alpha(x \gamma_0 x^{-1}) f(x)$$
$$\times \chi_{P,A:\circlearrowleft}(\mathbf{H}(P) - H_{P|A}(x)) d_G(x),$$

as desired.

If $\mathbf{H} << \mathbf{0}$, then

$$Q(\mathbf{H} : \phi_0)$$

has bounded support (cf. Proposition 5.2 in [2-(f)]), so, if $f \equiv 1$ on a large enough set, then

$$\int_{G/\Gamma} Q(\mathbf{H} : \phi_0)(x) d_G(x)$$

is equal to

$$\frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_{G/\Gamma_{\gamma_0} \cap L_0} \alpha(x \gamma_0 x^{-1}) f(x)$$
$$\times \left(\sum_{P \in \mathcal{P}(L_0)} (-1)^{\mathbf{rank}(P)} \chi_{P,A:\circlearrowleft}(\mathbf{H}(P) - H_{P|A}(x)) \right) d_G(x)$$

or still

$$\frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_{G/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)} \alpha(x \gamma_0 x^{-1}) \int_{\mathfrak{a}_0} f(x e^H)$$
$$\times \left(\sum_{P \in \mathcal{P}(L_0)} (-1)^{\mathbf{rank}(P)} \chi_{P,A:\circlearrowleft}(\mathbf{H}(P) - H_{P|A}(x) - H) \right) dH d_G(x).$$

We shall see in a bit how the "f" can be eliminated. Anticipating this, let us consider in more detail

$$\int_{\mathfrak{a}_0} (\sum_{P \in \mathcal{P}(L_0)} (-1)^{\mathbf{rank}(P)} \chi_{P,A:\circlearrowleft}(\mathbf{H}(P) - H_{P|A}(x) - H)) dH.$$

As might be expected, the issue is primarily combinatorial in character. That being the case, our basic tools will be drawn from the repository in [2-(b), §2] and [2-(d), §3], the notation of which will be employed below without further comment.

It is well-known and familiar that there is a one-to-one correspondence between

(*i*) $\{P \in \mathcal{P}(L_0)\}$;

(*ii*) $\{\mathfrak{a}, C_{\mathfrak{a}}\}$;

(*iii*) $\{W, C_W\}$.

The following lemma gives yet another parameterization. Put $\Phi = \Sigma(\mathfrak{g}, \mathfrak{a}_0)$.

Lemma 5.2. *Fix a chamber $\mathcal{C}_O$ in $\mathfrak{a}_0$—then the set of all pairs $(W, \mathcal{C}_W)$ is in a one-to-one correspondence with the set of all pairs $(\mathcal{C}_0, F_0)$, where*

$$F_0 \subset F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O)$$

and

$$\begin{cases} W = \mathfrak{a}_0(F_0) \\ (\mathcal{C}(F_0), \mathcal{C}_W) \longleftrightarrow \mathcal{C}_0 \, . \end{cases}$$

Proof. It suffices to show that for each W there exists a unique $\mathcal{C}(W)$ such that $F_0(\mathcal{C}(W)) \subset \Phi^+(\mathcal{C}_O)$. But the projection of $\mathcal{C}_O$ onto W is a connected open set on which an element of $\Phi(W)$ is either positive or negative, hence is contained in a unique chamber $\mathcal{C}(W)$. And $\mathcal{C}(W)$ will do. If $?(W)$ is another chamber, some member of $F_0(?(W))$ must be negative on $\mathcal{C}(W)$, thus negative on the projection of $\mathcal{C}_O$ and so negative on $\mathcal{C}_O$. This means that $\mathcal{C}(W)$ is the only chamber that will do.

In the terminology of Arthur, if $\mathbf{H} << \mathbf{0}$, then

$$\{\mathbf{H}(P_0) - H_{P_0|A_0}(x) : P_0 \in \mathcal{P}_0(L_0)\}$$

is a negative A_0-orthogonal set (cf. [1-(a), p. 221]). The elements $\mathcal{C}_0$ of $\mathcal{C}_0(A_0)$ are in a one-to-one correspondence with the elements P_0 of $\mathcal{P}_0(L_0)$ (cf. TES, p. 66). Assuming that $\mathcal{C}_0 \longleftrightarrow P_0$, put

$$T_{\mathcal{C}_0}(x : \mathbf{H}) = \mathbf{H}(P_0) - H_{P_0|A_0}(x).$$

We can now get on with the manipulation. Thus

$$\begin{aligned} &\sum_{P \in \mathcal{P}(L_0)} (-1)^{\mathrm{rank}(P)} \chi_{P,A:\mathfrak{S}}(\mathbf{H}(P) - H_{P|A}(x) - H) \\ &= \sum_{\mathfrak{a}} \sum_{\mathcal{C}_\mathfrak{a}} (-1)^{\mathrm{rank}(P)} \chi^{*,F_0(\mathcal{C}_\mathfrak{a})}(T_{\mathcal{C}_0}(x : \mathbf{H}) - H) \\ &= \sum_{W, \mathcal{C}_W} (-1)^{\ell_0 - \dim(W)} \chi^{F_0(\mathcal{C}(W)), F_0(\mathcal{C}_0)}(T_{\mathcal{C}_0}(x : \mathbf{H}) - H) \\ &= \sum_{\mathcal{C}_0} \sum_{\{F : F \subset F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O)\}} (-1)^{\ell_0 - \#(F)} \chi^{F, F_0(\mathcal{C}_0)}(T_{\mathcal{C}_0}(x : \mathbf{H}) - H). \end{aligned}$$

The sum over F can be cut down considerably. Indeed, if X is a variable, then

$$\chi^{F, F_0(\mathcal{C}_0)}(X) = \begin{cases} 1 & \text{if } \lambda^i(X) > 0 \ \forall \, \lambda_i \in F_0(\mathcal{C}_0) - F \\ 0 & \text{otherwise.} \end{cases}$$

So, with

$$F_0(X) = \{\lambda_i \in F_0(\mathcal{C}_0) : \lambda^i(X) \leq 0\},$$

$$\sum_{\{F: F \subset F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O)\}} (-1)^{\ell_0 - \#(F)} \chi^{F, F_0(\mathcal{C}_0)}(X)$$

$$= \sum_{\{F: F_0(X) \subset F \subset F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O)\}} (-1)^{\ell_0 - \#(F)}$$

$$= \begin{cases} (-1)^{l_0 - \#(F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O))} & \text{if } F_0(X) = F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O) \\ 0 & \text{otherwise} \end{cases}$$

or still

$$\sum_{\{F: F \subset F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O)\}} (-1)^{\ell_0 - \#(F)} \chi^{F, F_0(\mathcal{C}_0)}(X)$$

$$= (-1)^{\ell_0 - \#(F_0(\mathcal{C}_0) \cap \Phi^+(\mathcal{C}_O))}$$

$$\times \tau_{*, F_0(\mathcal{C}_0)}(F_0(\mathcal{C}_0) - \Phi^+(\mathcal{C}_O) : X).$$

Inserting this then leads to

$$\sum_{\mathcal{C}_0} (-1)^{\#(F_0(\mathcal{C}_0) - \Phi^+(\mathcal{C}_O))}$$

$$\times \tau_{*, F_0(\mathcal{C}_0)}(F_0(\mathcal{C}_0) - \Phi^+(\mathcal{C}_O) : T_{\mathcal{C}_0}(x : \mathbf{H}) - H),$$

the characteristic function of the convex hull of the $T_{\mathcal{C}_0}(x : \mathbf{H})$ (cf. [1-(a), pp. 218–219]). Integrating over $\mathfrak{a}_0$ then gives its volume, call it

$$v_{\mathfrak{a}_0}(x : \mathbf{H}).$$

In summary, therefore, if we ignore the "f", then

$$\int_{G/\Gamma} Q(\mathbf{H} : \phi_0)(x) d_G(x)$$

is equal to

$$\frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_{G/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)} \alpha(x \gamma_0 x^{-1}) v_{\mathfrak{a}_0}(x : \mathbf{H}) d_G(x).$$

And:

Lemma 5.3. *The integral*

$$\int_{G/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)} \alpha(x \gamma_0 x^{-1}) v_{\mathfrak{a}_0}(x : \mathbf{H}) d_G(x)$$

is compactly supported.

Proof. The centralizer of γ_0 in M_0 is the same as the centralizer of γ_0 in S_0. Moreover, γ_0 is M_0-regular, thus the quotient

$$(M_0)_{\gamma_0}/\Gamma_{\gamma_0} \cap M_0$$

is compact (cf. §2 : $\Gamma \cap M_0$ has finite index in Γ_{M_0}). So,

$$\int_{G/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)} \alpha(x\gamma_0 x^{-1}) v_{\mathfrak{a}_0}(x : \mathbf{H}) d_G(x)$$

$$= \int_{S_0/\Gamma_{\gamma_0} \cap M_0} \alpha(s\gamma_0 s^{-1}) v_{\mathfrak{a}_0}(s : \mathbf{H}) d_S(s)$$

$$= \mathrm{vol}((M_0)_{\gamma_0}/\Gamma_{\gamma_0} \cap M_0)$$

$$\times \int_{S_0/(S_0)_{\gamma_0}} \alpha(s\gamma_0 s^{-1}) v_{\mathfrak{a}_0}(s : \mathbf{H}) d_S(s).$$

It remains only to note that the S_0-orbit of γ_0 is the M_0-orbit of γ_0 times N_0, which is closed.

Consequently, the "f" can in fact be dispensed with.

Remark. One would like to say that

$$\int_{G/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)} \alpha(x\gamma_0 x^{-1}) v_{\mathfrak{a}_0}(x : \mathbf{H}) d_G(x)$$

is a weighted orbital integral. This is certainly the case if G_{γ_0} is contained in L_0. But, as we have already noted in §2, this need not be true in general although $G_{\gamma_0}^0$ is always contained in L_0 (cf. Lemma 2.6). On the other hand, it is not difficult to see that

$$G_{\gamma_0}/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)$$

is at least compact, thus

$$\int_{G/A_0 \bullet (\Gamma_{\gamma_0} \cap L_0)} \alpha(x\gamma_0 x^{-1}) v_{\mathfrak{a}_0}(x : \mathbf{H}) d_G(x)$$

$$= \int_{G/G_{\gamma_0}} \alpha(x\gamma_0 x^{-1}) \mathrm{Aver}_{\gamma_0}(v_{\mathfrak{a}_0}(x : \mathbf{H})) d_G(x),$$

the density

$$\mathrm{Aver}_{\gamma_0}(v_{\mathfrak{a}_0}(x:\mathbf{H}))$$

$$= \int_{G_{\gamma_0}/A_0\bullet(\Gamma_{\gamma_0}\cap L_0)} v_{\mathfrak{a}_0}(xy:\mathbf{H})d_{G_{\gamma_0}}(y)$$

being an averaged volume element.

Having evaluated

$$\int_{G/\Gamma} Q(\mathbf{H}:\phi_0)(x)d_G(x)$$

in closed form, the next step in the analysis is to prove that it is a polynomial in $\mathbf{H}$ if $\mathbf{H} << \mathbf{0}$.

For this purpose, recall that from the

$$\begin{cases} -H_{P_0|A_0}(x) \\ \\ \mathbf{H}(P_0) - H_{P_0|A_0}(x) \end{cases} \qquad (P_0 \in \mathcal{P}_0(L_0))$$

one can manufacture Detroit families

$$\begin{cases} \mathbf{e}^{-H_?(x)} \\ \mathbf{e}^{\mathbf{H}}(?) - H_?(x) \end{cases}$$

by exponentiation (cf. [2-(d), p. 163]). Put

$$\begin{cases} v_{\mathfrak{a}_0}(x) = (-1)^{\ell_0}\text{Ш}_{\mathbf{e}^{-H_?(x)}}(0) \\ \\ v_{\mathfrak{a}_0}(x:\mathbf{H}) = (-1)^{\ell_0}\text{Ш}_{\mathbf{e}^{\mathbf{H}(?)-H_?(x)}}(0) \end{cases}$$

a permissible agreement. Owing now to Corollary 3.2 in [2-(h)], we have

$$v_{\mathfrak{a}_0}(x:\mathbf{H}) = (-1)^{\ell_0}\text{Ш}_{\mathbf{e}^{\mathbf{H}(?)-H_?(x)}}(0)$$

$$= (-1)^{\ell_0}\sum_{W,\mathcal{C}_W} \text{Ш}_{\mathbf{e}^{-H_?(x)}(\mathcal{C}_W)}(0) \bullet A_{\mathbf{e}^{\mathbf{H}},\mathcal{C}_W}(0)$$

$$= \sum_{P\in\mathcal{P}(L_0)} v_{\mathfrak{a}_0^\dagger}(m_x) \bullet p(\Gamma:P:\mathbf{H}(P)),$$

where $v_{\mathfrak{a}_0^\dagger}$ is the daggered analogue of $v_{\mathfrak{a}_0}$ and $p(\Gamma : P :?)$ is an Arthur polynomial (cf. [2-(f), p. 1429]). Accordingly,

$$\int_{G/A_0\bullet(\Gamma_{\gamma_0}\cap L_0)} \alpha(x\gamma_0x^{-1})v_{\mathfrak{a}_0}(x:\mathbf{H})d_G(x)$$

is the sum over the $P \in \mathcal{P}(L_0)$ of the

$$\int_{G/A_0\bullet(\Gamma_{\gamma_0}\cap L_0)} \alpha(x\gamma_0 x^{-1}) v_{\mathfrak{a}_0^\dagger}(m_x) d_G(x)$$

times

$$p(\Gamma : P : \mathbf{H}(P)).$$

And so

$$\int_{G/\Gamma} Q(\mathbf{H} : \phi_0)(x) d_G(x)$$

is in fact a polynomial in $\mathbf{H}$ if $\mathbf{H} << \mathbf{0}$.

While we are at it, let us also observe that

$$\frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_{G/A_0\bullet(\Gamma_{\gamma_0}\cap L_0)} \alpha(x\gamma_0 x^{-1}) v_{\mathfrak{a}_0^\dagger}(m_x) d_G(x)$$

$$= \frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap L_0]} \bullet \int_N \int_{M/A_0^\dagger(\Gamma_{\gamma_0}\cap L_0)} \alpha(nm\gamma_0 m^{-1} n^{-1}) v_{\mathfrak{a}_0^\dagger}(m) d_M(m) d_N(n)$$

$$= \frac{1}{[\Gamma_{\gamma_0} : \Gamma_{\gamma_0} \cap M]} \bullet \frac{1}{|\det(\mathrm{Ad}(\gamma_0)|\mathfrak{n} - 1)|}$$

$$\times \int_{M/A_0^\dagger\bullet(\Gamma_{\gamma_0}\cap M)} \alpha^P(m\gamma_0 m^{-1}) v_{\mathfrak{a}_0^\dagger}(m) d_M(m),$$

the last integral being the M-analogue of the integral

$$\int_{G/A_0\bullet\Gamma_{\gamma_0}} \alpha(x\gamma_0 x^{-1}) v_{\mathfrak{a}_0}(x) d_G(x)$$

on G.

Remark. When $\mathfrak{a}_0$ is special, one can interpret $v_{\mathfrak{a}_0}(x)$ geometrically in that it then gives the volume of the convex hull of the $\{-H_{P_0|A_0}(x)\}$ but this will not be true in general.

Our initial objective can now be realized. Because the support of α is compact, Proposition 3.3 guarantees us that only finitely many

$$\{\gamma\}_\Gamma : \alpha \in \Gamma_{\mathcal{C}_0,\mathcal{C}_0}$$

actually intervene, so all $\mathbf{H} << \mathbf{0}$ will work for each of them simultaneously. Hence:

Theorem 5.4. $\mathrm{K}(\mathbf{H} : \alpha : \Gamma_{\mathcal{C}_0,\mathcal{C}_0})$ *is equal to*

$$\sum_{\{\gamma\}_\Gamma : \gamma \in \Gamma_{\mathcal{C}_0,\mathcal{C}_0}} \frac{1}{[\Gamma_\gamma : \Gamma_\gamma \cap L_0(\gamma)]}$$

$$\times \int_{G/A_0(\gamma)\bullet(\Gamma_\gamma \cap L_0(\gamma))} \alpha(x\gamma x^{-1}) v_{\mathfrak{a}_0(\gamma)}(x : \mathbf{H}) d_G(x)$$

or still

$$\sum_{\{\gamma\}_\Gamma : \gamma \in \Gamma_{\mathcal{C}_0,\mathcal{C}_0}} \sum_{P \in \mathcal{P}(L_0(\gamma))} \frac{1}{[\Gamma_\gamma : \Gamma_\gamma \cap M]} \bullet \frac{1}{|\det(\mathrm{Ad}(\gamma)|\mathfrak{n} - 1)|}$$

$$\times p(\Gamma : P : \mathbf{H}(P))$$

$$\times \int_{M/A_0^\dagger(\gamma)\bullet(\Gamma_\gamma \cap M)} \alpha^P(m\gamma m^{-1}) v_{\mathfrak{a}_0^\dagger(\gamma)}(m) d_M(m).$$

[Note: For the sake of brevity, we have written $P = M \bullet A \bullet N$ in place of $P(\gamma) = M(\gamma) \bullet A(\gamma) \bullet N(\gamma)$.]

§6. Passage to Standard Form

The purpose of this § is to recast the expression obtained in Theorem 5.4 for

$$\mathbf{K}(\mathbf{H} : \alpha : \Gamma_{C_0,C_0})$$

so as to reflect the presence of the $P_{i\mu} = M_{i\mu} \bullet A_{i\mu} \bullet N_{i\mu}$ with $A_{i\mu}$ special (cf. [2-(a), pp. 65–70]), the point being that these are the parabolics of reference. The issue is therefore primarily one of bookkeeping, albeit a little on the involved side.

We can evidently write

$$\mathbf{K}(\mathbf{H} : \alpha : \Gamma_{C_0,C_0})$$

in the form

$$\sum_{\{\gamma\}_\Gamma : \gamma \in \Gamma_{C_0,C_0}} \sum_{P \in \mathcal{P}(L_0(\gamma))} \frac{1}{[\Gamma_\gamma : \Gamma_\gamma \cap M_0(\gamma)]}$$

$$\times p(\Gamma : P : \mathbf{H}(P))$$

$$\times \int_{G/A_0(\gamma)\bullet(\Gamma_\gamma \cap M_0(\gamma))} \alpha(x\gamma x^{-1}) v_{\mathfrak{a}_0^\dagger(\gamma)}(x) d_G(x),$$

the effect of which is to base part of the data at the $L_0(\gamma)$-level. Now introduce a parameter $\mathbf{H}_O \in \mathfrak{a}$—then still another way to write

$$\mathbf{K}(\mathbf{H} : \alpha : \Gamma_{C_0,C_0})$$

is

$$\sum_{\{\gamma\}_\Gamma : \gamma \in \Gamma_{C_0,C_0}} \sum_{P \in \mathcal{P}(L_0(\gamma))} \frac{1}{[\Gamma_\gamma : \Gamma_\gamma \cap M_0(\gamma)]}$$

$$\times p(\Gamma : P : \mathbf{H}(P) - \mathbf{H}_O(P))$$

$$\times \int_{G/A_0(\gamma)\bullet(\Gamma_\gamma \cap M_0(\gamma))} \alpha(x\gamma x^{-1}) v_{\mathfrak{a}_0^\dagger(\gamma)}(x : I_M(\mathbf{H}_O)) d_G(x).$$

As we shall see, the rationale here is that the introduction of $\mathbf{H}_O$ allows certain transitions to take place without the need for the introduction of compensating factors.

Consider an arbitrary $P \in \mathcal{P}(L_0(\gamma))$. Choose $\gamma(i : \mu) \in \Gamma$ such that

$$\gamma(i : \mu) P \gamma(i : \mu)^{-1} = P_{i\mu}$$

and let

$$\gamma_{i\mu} = \gamma(i : \mu)\gamma\gamma(i : \mu)^{-1}.$$

Obviously,

$$P_{i\mu} \in \mathcal{P}(L_0(\gamma_{i\mu}))$$

but there is no reason to expect that

$$L(\gamma_{i\mu}) = M(\gamma_{i\mu}) \bullet A(\gamma_{i\mu})$$

is the special $L_{i\mu} = M_{i\mu} \bullet A_{i\mu}$.

It is clear that

$$\begin{aligned} &p(\Gamma : P : \mathbf{H}(P) - \mathbf{H}_0(P)) \\ &= p(\Gamma : P_{i\mu} : \mathbf{H}(P_{i\mu}) - \mathbf{H}_0(P_{i\mu})). \end{aligned}$$

Additionally, one can verify that

$$\begin{aligned} &\int_{G/A_0(\gamma)\bullet(\Gamma_\gamma \cap M_0(\gamma))} \alpha(x\gamma x^{-1}) v_{\mathfrak{a}_0^\dagger(\gamma)}(x : I_M(\mathbf{H}_0)) d_G(x) \\ &= \int_{G/A_0(\gamma_{i\mu})\bullet(\Gamma_{\gamma_{i\mu}} \cap M_0(\gamma_{i\mu}))} \alpha(x\gamma_{i\mu} x^{-1}) v_{\mathfrak{a}_0^\dagger(\gamma_{i\mu})}(x : I_{M(\gamma_{i\mu})}(\mathbf{H}_0)) d_G(x). \end{aligned}$$

These are the two crucial relations which allow passage from γ to $\gamma_{i\mu}$. They would fail to hold if it were not for the presence of $\mathbf{H}_0$.

Accordingly, integrating out $N_{i\mu}$,

$$\mathbf{K}(\mathbf{H} : \alpha : \Gamma_{C_0, C_0})$$

becomes

$$\begin{aligned} &\sum_{\{\gamma\}_\Gamma : \gamma \in \Gamma_{C_0,C_0}} \sum_{P \in \mathcal{P}(L_0(\gamma))} \frac{1}{[\Gamma_{\gamma_{i\mu}} : \Gamma_{\gamma_{i\mu}} \cap M_0(\gamma_{i\mu})]} \\ &\times \frac{1}{|\det(\mathrm{Ad}(\gamma_{i\mu})|\mathfrak{n}_{i\mu} - 1)|} \\ &\times p(\Gamma : P_{i\mu} : \mathbf{H}(P_{i\mu}) - \mathbf{H}_0(P_{i\mu})) \\ &\times \int_{M(\gamma_{i\mu})/A_0^\dagger(\gamma_{i\mu})\bullet(\Gamma_{\gamma_{i\mu}} \cap M_0(\gamma_{i\mu}))} \alpha^{P_{i\mu}}(m\gamma_{i\mu} m^{-1}) v_{\mathfrak{a}_0^\dagger(\gamma_{i\mu})}(m : I_{M(\gamma_{i\mu})}(\mathbf{H}_0)) d_{M(\gamma_{i\mu})}(m). \end{aligned}$$

Next, determine $n_{i\mu} \in N_{i\mu}$ so that

$$n_{i\mu} A(\gamma_{i\mu}) n_{i\mu}^{-1} = A_{i\mu}.$$

Passing now from $\gamma_{i\mu}$ to

$$\delta_{i\mu} = n_{i\mu}\gamma_{i\mu} n_{i\mu}^{-1} \in \Gamma_{M_{i\mu}},$$

and agreeing to write

$$\begin{cases} M_0(\delta_{i\mu}) & \text{instead of } n_{i\mu}M_0(\gamma_{i\mu})n_{i\mu}^{-1} \\ A_0(\delta_{i\mu}) & \text{instead of } n_{i\mu}A_0(\gamma_{i\mu})n_{i\mu}^{-1}, \end{cases}$$

the foregoing then reads

$$\sum_{\{\gamma\}_\Gamma:\gamma\in\Gamma_{\mathcal{C}_0,\mathcal{C}_0}} \sum_{P\in\mathcal{P}(L_0(\gamma))} \frac{[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}} \cap M_0(\delta_{i\mu}) : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_0(\delta_{i\mu})]}{[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_0(\delta_{i\mu})]}$$

$$\times \frac{1}{|\det(\mathrm{Ad}(\gamma_{i\mu})|\mathfrak{n}_{i\mu} - 1)|}$$

$$\times p(\Gamma : P_{i\mu} : \mathbf{H}(P_{i\mu}) - \mathbf{H}_O(P_{i\mu}))$$

$$\times \int_{M_{i\mu}/A_0^\dagger(\delta_{i\mu})\bullet((\Gamma_{M_{i\mu}})_{\delta_{i\mu}}\cap M_0(\delta_{i\mu}))} \alpha^{P_{i\mu}}(m\delta_{i\mu}m^{-1})v_{\mathfrak{a}_0^\dagger(\delta_{i\mu})}(m : I_{M_{i\mu}}(\mathbf{H}_O))d_{M_{i\mu}}(m).$$

To get an inductive object out of this expression, it will be necessary to replace

$$\sum_{\{\gamma\}_\Gamma:\gamma\in\Gamma_{\mathcal{C}_0,\mathcal{C}_0}} \sum_{P\in\mathcal{P}(L_0(\gamma))}$$

by

$$\sum_{i,\mu} \sum_{\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}}} .$$

As a preliminary, we remark that our notation is slightly deceptive in that $\delta_{i\mu}$ really depends on γ. However, a choice of

$$\begin{cases} \gamma \in \Gamma_{\mathcal{C}_0,\mathcal{C}_0} \\ P \in \mathcal{P}(L_0(\gamma)) \end{cases}$$

singles out uniquely

$$P_{i\mu} \text{ and } \{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}}.$$

This said, recall that via the daggering procedure $\mathcal{C}_0$ will induce a disjoint union

$$\mathcal{C}_0^\dagger \equiv \coprod \mathcal{C}_0^\dagger(i : \mu).$$

Of course,

$$\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}} \in (\Gamma_{M_{i\mu}})_{\mathcal{C}_0^\dagger,\mathcal{C}_0^\dagger} \equiv \coprod (\Gamma_{M_{i\mu}})_{\mathcal{C}_0^\dagger(i:\mu),\mathcal{C}_0^\dagger(i:\mu)}.$$

And, the ambient $\Gamma_{M_{i\mu}}$-Levi subgroup of the ambient $\Gamma_{M_{i\mu}}$-cuspidal parabolic in $\mathcal{P}_0^\dagger(L_0(\delta_{i\mu}))$ is $A_0^\dagger(\delta_{i\mu}) \bullet M_0(\delta_{i\mu})$.

Consequently,

$$\mathbf{K}(\mathbf{H}:\alpha:\Gamma_{\mathcal{C}_0,\mathcal{C}_0})$$

reduces to

$$\sum_{i,\mu}\ \sum_{\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}}:\delta_{i\mu}\in(\Gamma_{M_{i\mu}})_{c_0^\dagger,c_0^\dagger}} C(P_{i\mu},\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}})$$

$$\times p(\Gamma:P_{i\mu}:\mathbf{H}(P_{i\mu})-\mathbf{H}_{\mathrm{O}}(P_{i\mu}))$$

$$\times\int_{M_{i\mu}/A_0^\dagger(\delta_{i\mu})\bullet((\Gamma_{M_{i\mu}})_{\delta_{i\mu}}\cap M_0(\delta_{i\mu}))}\alpha^{P_{i\mu}}(m\delta_{i\mu}m^{-1})v_{\mathfrak{a}_0^\dagger(\delta_{i\mu})}(m:I_{M_{i\mu}}(\mathbf{H}_{\mathrm{O}}))d_{M_{i\mu}}(m).$$

Here, the constant

$$C(P_{i\mu},\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}})$$

is, by definition, the sum

$$\sum\frac{[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}}\cap M_0(\delta_{i\mu}):n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1}\cap M_0(\delta_{i\mu})]}{[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1}:n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1}\cap M_0(\delta_{i\mu})]}$$

$$\times\frac{1}{|\det(\mathrm{Ad}(\gamma_{i\mu})|\mathfrak{n}_{i\mu}-1)|}$$

over all possible

$$\begin{cases}\gamma\in\Gamma_{\mathcal{C}_0,\mathcal{C}_0}\\ P\in\mathcal{P}(L_0(\gamma))\end{cases}$$

which give rise to a fixed

$$P_{i\mu}\text{ and }\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}}.$$

The main technical claim is then:

Lemma 6.1. *$C(P_{i\mu},\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}})$ is equal to*

$$\frac{1}{[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}}:(\Gamma_{M_{i\mu}})_{\delta_{i\mu}}\cap M_0(\delta_{i\mu})]}.$$

Proof. The way to keep track of what's coming and going is to look at the nonempty

$$\Delta_0(P)_{P_{i\mu}}\cap\delta_{i\mu}N_{i\mu}.$$

The cardinality of the Γ_γ-conjugacy class of P in $\mathcal{P}(L_0(\gamma))$ is

$$[\Gamma_\gamma:\Gamma_\gamma\cap P]$$

or still

$$[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_{i\mu}].$$

Since

$$\frac{[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_{i\mu}]}{[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_0(\delta_{i\mu})]}$$

$$= \frac{1}{[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_{i\mu} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_0(\delta_{i\mu})]},$$

$C(P_{i\mu}, \{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}})$ is equal to

$$\sum \frac{[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}} \cap M_0(\delta_{i\mu}) : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_0(\delta_{i\mu})]}{[n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_{i\mu} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_0(\delta_{i\mu})]}$$

$$\times \frac{1}{|\det(\mathrm{Ad}(\gamma_{i\mu})|\mathfrak{n}_{i\mu} - 1)|}$$

or still

$$\frac{1}{[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}} : (\Gamma_{M_{i\mu}})_{\delta_{i\mu}} \cap M_0(\delta_{i\mu})]}$$

$$\times \sum [(\Gamma_{M_{i\mu}})_{\delta_{i\mu}} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_{i\mu}]$$

$$\times \frac{1}{|\det(\mathrm{Ad}(\delta_{i\mu})|\mathfrak{n}_{i\mu} - 1)|},$$

the summation in either case extending over all nonempty

$$\Delta_0(P)_{P_{i\mu}} \cap \delta_{i\mu}N_{i\mu}.$$

It remains only to show that the second sum is one. For this, it will be enough to check that a given nonempty

$$\Delta_0(P)_{P_{i\mu}} \cap \delta_{i\mu}N_{i\mu}$$

includes

$$[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}} : n_{i\mu}\Gamma_{\gamma_{i\mu}}n_{i\mu}^{-1} \cap M_{i\mu}]$$

$\Gamma \cap N_{i\mu}$-conjugacy classes, there being precisely

$$|\det(\mathrm{Ad}(\delta_{i\mu})|\mathfrak{n}_{i\mu} - 1)|$$

of these. Since the problem is one of counting and therefore, in an obvious sense, is "conjugation invariant", we can assume without loss of generality that $\gamma_{i\mu} = \delta_{i\mu}$, then drop the i and the μ from the notation and finish by proving:

Lemma 6.2. *The $\Gamma \cap P$-conjugacy class of γ intersected with $\gamma N \cap \Gamma$ includes*

$$[(\Gamma_{M(\gamma)})_\gamma : \Gamma_\gamma \cap M(\gamma)]$$

$\Gamma \cap N$-conjugacy classes in $\gamma N \cap \Gamma$.

Proof. From the definitions,

$$\{\gamma\}_{\Gamma \cap P} \cap (\gamma N \cap \Gamma) = \{\eta\gamma\eta^{-1} : \eta \in (\Gamma_{M(\gamma)})_\gamma \bullet N \cap \Gamma\}.$$

And

$$(\Gamma_{M(\gamma)})_\gamma \bullet N \cap \Gamma \supset (\Gamma_\gamma \cap P) \bullet (\Gamma \cap N).$$

Conjugation by the latter gives a $\Gamma \cap N$-conjugacy class. Suppose that

$$\eta_1, \eta_2 \in (\Gamma_{M(\gamma)})_\gamma \bullet N \cap \Gamma$$

and suppose that $\eta_1\gamma\eta_1^{-1}$ and $\eta_2\gamma\eta_2^{-1}$ are $\Gamma \cap N$-conjugate, say

$$\eta_1\gamma\eta_1^{-1} = n_2\eta_2\gamma\eta_2^{-1}n_2^{-1}.$$

Then

$$\begin{aligned} &\gamma = \eta_1^{-1}n_2\eta_2\gamma\eta_2^{-1}n_2^{-1}\eta_1 \\ \Longrightarrow &\eta_1^{-1}n_2\eta_2 \in \Gamma_\gamma \\ \Longrightarrow &\eta_1(\Gamma_\gamma \cap P) = n_2\eta_2(\Gamma_\gamma \cap P) \\ \Longrightarrow &\eta_1(\Gamma_\gamma \cap P) \bullet (\Gamma \cap N) = \eta_2(\Gamma_\gamma \cap P) \bullet (\Gamma \cap N). \end{aligned}$$

Because each such coset fills out a full $\Gamma \cap N$-conjugacy class, the number of $\Gamma \cap N$-conjugacy classes in question is

$$[(\Gamma_{M(\gamma)})_\gamma \bullet N \cap \Gamma : (\Gamma_\gamma \cap P) \bullet (\Gamma \cap N)].$$

The map

$$\begin{cases} \varphi : \Gamma \cap P \to \Gamma_{M(\gamma)} \\ \quad \varphi(\eta) = \delta, \ \{\delta\} = M \cap \eta N \end{cases}$$

has kernel $\Gamma \cap N$, so our number is

$$[\varphi((\Gamma_{M(\gamma)})_\gamma \bullet N \cap \Gamma) : \varphi((\Gamma_\gamma \cap P) \bullet (\Gamma \cap N))]$$

or still

$$[(\Gamma_{M(\gamma)})_\gamma : \Gamma_\gamma \cap M(\gamma)],$$

as desired.

Hence:

Theorem 6.3. $\mathbf{K}(\mathbf{H} : \alpha : \Gamma_{\mathcal{C}_0,\mathcal{C}_0})$ *is equal to*

$$\sum_{i,\mu} \sum_{\{\delta_{i\mu}\}_{\Gamma_{M_{i\mu}}} : \delta_{i\mu} \in (\Gamma_{M_{i\mu}})_{\mathcal{C}_0^\dagger, \mathcal{C}_0^\dagger}} \frac{1}{[(\Gamma_{M_{i\mu}})_{\delta_{i\mu}} : (\Gamma_{M_{i\mu}})_{\delta_{i\mu}} \cap M_0(\delta_{i\mu})]}$$

$$\times p(\Gamma : P_{i\mu} : \mathbf{H}(P_{i\mu}) - \mathbf{H}_O(P_{i\mu}))$$

$$\times \int_{M_{i\mu}/A_0^\dagger(\delta_{i\mu}) \bullet ((\Gamma_{M_{i\mu}})_{\delta_{i\mu}} \cap M_0(\delta_{i\mu}))} \alpha^{P_{i\mu}}(m\delta_{i\mu}m^{-1}) v_{\mathfrak{a}_0^\dagger(\delta_{i\mu})}(m : I_{M_{i\mu}}(\mathbf{H}_O)) d_{M_{i\mu}}(m).$$

Denote by

$$\mathbf{Con} - \mathbf{Cl}(\mathbf{H} : \alpha : \Gamma_{\mathcal{C}_0,\mathcal{C}_0})$$

the result of setting $\mathbf{H}_O = \mathbf{H}$ in Theorem 6.3. Since

$$p(\Gamma : P_{i\mu} : 0) = 0$$

unless $P_{i\mu} = G$,

$$\mathbf{Con} - \mathbf{Cl}(\mathbf{H} : \alpha : \Gamma_{\mathcal{C}_0,\mathcal{C}_0})$$

can also be explicated by Theorem 5.4. It is then possible to go further but the discussion is combinatorially messy and unenlightening, thus will be omitted.

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