SETS AND CLASSES: OPERATIONAL THEORY

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ABSTRACT

The purpose of this book is to lay out certain aspects of descriptive set theory.

CONTENTS

Prerequisites

It is assumed that the reader is familiar with the language and notation employed in elementary algebra, analysis, set theory, and general topology.

As usual

$\mathbb{N}$ the set of positive integers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{P}$ the set of irrational numbers

$\mathbb{R}$ the set of real numbers

$\mathbb{C}$ the set of complex numbers.

The symbols $\mathbb{N}^n$, $\mathbb{Z}^n$, $\mathbb{Q}^n$, $\mathbb{P}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$ (n a positive integer) are then to be assigned their customary interpretations.

Tacitly, we shall always operate within the strictures of ZFC (Zermelo-Fraenkel Axioms + Axiom of Choice), unless the contrary is explicitly stated.

§1. Generalities

Throughout this book, whenever the word *set* is used, it is always understood to mean a subset of a given set which, generically, is denoted by X; we shall use the word *class* for a set of sets and the word *collection* for a set of classes. If S and T are subsets of X, then the union, intersection, difference, and symmetric difference of S and T are denoted by $S\cup T$, $S\cap T$, $S-T$, and $S\Delta T$, respectively. $\mathcal{P}(X)$ stands for the class of all subsets of X; $\emptyset$ stands for the empty set.

By card(X), we mean the cardinality of X. A set is said to be *countable* if its cardinality is $\aleph_0$, *finite* if its cardinality is $<\aleph_0$, *uncountable* if its cardinality is $>\aleph_0$, *infinite* if not finite, i.e., either countable or uncountable. If F is a finite set, then $\#(F)$ is the number of elements in F.

As is customary,

$$\aleph_0 < \aleph_1 < \dots < \aleph_\alpha < \dots$$

are the *infinite cardinals* and

$$\omega_0 < \omega_1 < \dots < \omega_\alpha < \dots$$

are the *infinite initial ordinals*. In this connection, bear in mind that α is an arbitrary ordinal and ω_α is the first ordinal such that

$$\operatorname{card}(\{\beta:\beta<\omega_\alpha\}) = \aleph_\alpha \quad .$$

Traditionally, ω_0 is denoted by ω, while ω_1 is denoted by Ω. By $\mathfrak{c}$, we understand the cardinality of the *continuum*, i.e., $\mathfrak{c} = 2^{\aleph_0}$. The *continuum hypothesis* is the statement that $2^{\aleph_0} = \aleph_1$; the *generalized continuum hypothesis* is the statement that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α. Both of these statements are independent of ZFC.

The *characteristic function* of a subset S of X is the function $\chi_S:X \to \underset{\sim}{R}$ defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x\in S \\ 0 & \text{if } x\in X-S \, . \end{cases}$$

There is a canonical identification between $\mathcal{P}(X)$ and the set $\operatorname{Fnc}(X,\{0,1\})$ of all functions from X to $\{0,1\}$, namely the rule $S \mapsto \chi_S$.

If S and T are subsets of X, then

$$\begin{cases} \chi_{S\cap T} = \chi_S\chi_T = \min(\chi_S,\chi_T) \\ \chi_{S\cup T} = \chi_S + \chi_T - \chi_{S\cap T} = \max(\chi_S,\chi_T) \end{cases}$$

with $\chi_S \leq \chi_T$ iff $S\subset T$. Furthermore,

$$\begin{cases} \chi_{S-T} = \chi_S(1 - \chi_T) \\ \chi_{S\Delta T} = |\chi_S - \chi_T|. \end{cases}$$

Let $\{S_i\}$ be a sequence of subsets of X -- then the set of all those points of X which belong to S_i for infinitely many values of i is called the *upper limit* or *limit superior* of the sequence and is denoted by $\overline{\lim}\, S_i$ or $\limsup S_i$, while the set of all those points of X which belong to S_i for all but a finite number of values of i is called the *lower limit* or *limit inferior* of the sequence and is denoted by $\underline{\lim}\, S_i$ or $\liminf S_i$. Evidently,

$$\begin{cases} \overline{\lim}\, S_i = \bigcap_{i=1}^{\infty}(\bigcup_{m=i}^{\infty} S_m) \\ \underline{\lim}\, S_i = \bigcup_{i=1}^{\infty}(\bigcap_{m=i}^{\infty} S_m). \end{cases}$$

In the event that

$$\left.\begin{matrix} \overline{\lim}\, S_i \\ \underline{\lim}\, S_i \end{matrix}\right\} = S, \text{ say,}$$

then S is said to be the *limit* of the sequence $S_1, S_2, \ldots$ and we write $S = \lim S_i$. For instance, if $\{S_i\}$ is an *increasing* (*decreasing*) sequence in the sense that $S_i \subset S_{i+1}$ ($S_i \supset S_{i+1}$) $\forall i$, then $\lim S_i = \cup S_i$ ($\cap S_i$). In general, it is always true that

$$\cap S_i \subset \underline{\lim}\, S_i \subset \overline{\lim}\, S_i \subset \cup S_i.$$

In terms of characteristic functions,

$$\begin{cases} \chi_{\overline{\lim}\, S_i} = \overline{\lim}\, \chi_{S_i} \\ \chi_{\underline{\lim}\, S_i} = \underline{\lim}\, \chi_{S_i} \end{cases} .$$

Example Suppose that $\{S_i\}$ is a sequence of pairwise disjoint subsets of X — then $\lim S_i = \emptyset$.

The preceding notions can be interpreted topologically. For this purpose, it will be convenient to consider first the elements of a useful abstract construction.

Thus let $(X, \mathfrak{T})$ be a topological space -- then by the *sequential modification* of $(X, \mathfrak{T})$ we mean the topological space whose underlying set is still X itself but whose topology $\mathfrak{T}_s$ consists, by definition, of the complements of those subsets S of X which are closed under pointwise convergence of sequences, i.e., a subset

S of X is $\mathfrak{T}_s$-closed iff for every sequence $\{x_i\} \subset S$, $x_i \to x \Longrightarrow x \in S$. It is easy to check that the class of closed subsets thereby singled out does in fact satisfy the usual axioms involved in defining a topology by closed sets. The canonical map $(X,\mathfrak{T}_s) \to (X,\mathfrak{T})$ is continuous, or, what amounts to the same, the $\mathfrak{T}_s$-topology on X is finer than the $\mathfrak{T}$-topology. In addition, it is clear that a sequence $\{x_i\}$ in X is $\mathfrak{T}$-convergent to a point x iff it is $\mathfrak{T}_s$-convergent to x. These remarks enable one to characterize the sequential modification of $(X,\mathfrak{T})$ in a simple way. Indeed, $\mathfrak{T}_s$ is the finest of all topologies $\mathfrak{T}_0 \supset \mathfrak{T}$ on X which have the following property: A sequence in X is $\mathfrak{T}$-convergent iff it is $\mathfrak{T}_0$-convergent.

The essential significance of the sequential modification is contained in:

Lemma 1 *Let* $f:X \to Y$ *be a map from* X *into a topological space* Y -- *then* f *is continuous per* $\mathfrak{T}_s$ *iff* f *is sequentially continuous per* $\mathfrak{T}$.

[We omit the elementary verification.]

In connection with the preceding developments, a modicum of caution must be exercised, viz.: The $\mathcal{T}_s$-closure of a subset S of X need not consist just of the sequential limits from S but, in general, will be much larger, as can be seen by simple examples (cf. Exer. 8). This can easily be made precise. Given S, let uS be the set of all $\mathcal{T}$-limits of sequences in S. Putting $u_0 S = S$, define by transfinite recursion

$$u_\alpha S = u\left(\bigcup_{\beta<\alpha} u_\beta S\right) \qquad (\alpha<\Omega).$$

Then the $\mathcal{T}_s$-closure of S is $\bigcup_{\alpha<\Omega} u_\beta S$. Another way to look at it is to let S_0 run through those subsets of S having cardinality $\leq\aleph_0$ -- then the union of the $\mathcal{T}_s$-closure of the S_0 is the $\mathcal{T}_s$-closure of S. In any event, the moral is that sequences do not ordinarily suffice; nets (or filters) will usually be needed.

[Note: Let $\mathbf{Top}$ be the category whose objects are topological spaces and whose morphisms are continuous maps; let $\mathbf{Top}_s$ be the category whose objects are the sequential topological spaces, i.e., those topological spaces in which every sequentially closed subset is closed, and whose morphisms are continuous maps -- then there is a canonically defined coreflective functor

$$\underset{\sim}{\text{Top}} \rightarrowtail \underset{\sim}{\text{Top}}_s ,$$

viz. the rule

$$(X,\mathfrak{T}) \rightarrow (X,\mathfrak{T}_s)$$

together with the obvious assignment of morphisms. $\underset{\sim}{\text{Top}}_s$ thus appears as a coreflective subcategory of $\underset{\sim}{\text{Top}}$ which, in fact, is monocoreflective , hence, on the basis of standard categorical generalities, is closed under the formation of quotients and coproducts in $\underset{\sim}{\text{Top}}$.]

Suppose now that X is again merely an abstract set but that Y is a topological space. Let Fnc(X,Y) be the set of all functions from X to Y equipped with the topology of pointwise convergence -- then by $\text{Fnc}(X,Y)_s$ we understand the sequential modification of Fnc(X,Y). The class of closed sets for the associated topology is thus comprised of those subsets of Fnc(X,Y) which are closed under pointwise convergence of sequences.

Example If X and Y are both topological spaces, then the closure in $\text{Fnc}(X,Y)_s$ of the subset of all continuous maps is known as the class of *Baire functions* (from X to Y).

The identification $\mathcal{P}(X) \approx \text{Fnc}(X,\{0,1\})$ enables one to topologize $\mathcal{P}(X)$ in a canonical way. Indeed, equipping $\{0,1\}$ with the discrete topology, place on $\text{Fnc}(X,\{0,1\})$ the topology of pointwise convergence -- then this topology may be pulled back to $\mathcal{P}(X)$, the upshot being that $\mathcal{P}(X)$ thus topologized is a compact Hausdorff space which, moreover, is totally disconnected. Write $\mathcal{P}(X)_s$ for the corresponding sequential modification -- then $\mathcal{P}(X)_s$ is still Hausdorff and totally disconnected but, in general, need not be compact (cf. Exer. 12). Given a sequence $\{S_i\} \subset \mathcal{P}(X)$, the relations

$$\begin{cases} \chi_{\overline{\lim}\, S_i} = \overline{\lim}\, \chi_{S_i} \\ \chi_{\underline{\lim}\, S_i} = \underline{\lim}\, \chi_{S_i} \end{cases}$$

then make it clear that $\lim S_i$ exists topologically, i.e., per $\mathcal{P}(X)_s$, iff $\lim S_i$ exists in the sense that $\overline{\lim}\, S_i = \underline{\lim}\, S_i$.

We shall terminate this § with some definitions and related notation.

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$. Write $\mathcal{S}_s$, $\mathcal{S}_\sigma$, $\mathcal{S}_d$, $\mathcal{S}_\delta$ for the class of subsets of X comprised of all nonempty finite unions, countable unions, nonempty finite intersections, countable intersections of sets in $\mathcal{S}$ (repetitions being permissible); write $\mathcal{S}_r$ for the class of subsets of X comprised of all sets in $\mathcal{S}$ and all differences of sets in $\mathcal{S}$; write $\mathcal{S}_c$ for the class of subsets of X comprised of all complements of sets in $\mathcal{S}$. Successive application of these operations is represented by juxtaposition of the symbols, e.g., $\mathcal{S}_{\sigma\delta} \equiv (\mathcal{S}_\sigma)_\delta$, the class of all countable intersections of countable unions of sets belonging to $\mathcal{S}$. Obviously,

$$\begin{cases} \mathcal{S} \subset \mathcal{S}_s = \mathcal{S}_{ss} \subset \mathcal{S}_\sigma = \mathcal{S}_{\sigma\sigma} \\ \mathcal{S} \subset \mathcal{S}_d = \mathcal{S}_{dd} \subset \mathcal{S}_\delta = \mathcal{S}_{\delta\delta} \end{cases}$$

$$\mathcal{S} = \mathcal{S}_{cc},\ \mathcal{S}_{c\sigma} = \mathcal{S}_{\delta c},\ \mathcal{S}_{c\delta} = \mathcal{S}_{\sigma c}\ .$$

The class $\mathcal{S}$ is termed *additive* (*σ-additive*) if it is nonempty and closed under the formation of nonempty finite (countable) unions, i.e., provided $\mathcal{S} = \mathcal{S}_s(\mathcal{S}_\sigma)$. The class $\mathcal{S}$ is termed multiplicative (δ-multiplicative) if it is nonempty and closed under the formation of nonempty finite (countable) intersections, i.e., provided $\mathcal{S} = \mathcal{S}_d(\mathcal{S}_\delta)$. If $\emptyset \in \mathcal{S}$ and if $\mathcal{S}$ is both additive and multiplicative (σ-additive or δ-multiplicative), then $\mathcal{S}$ is called a lattice (σ-lattice or δ-lattice). Every σ-lattice or δ-lattice is a lattice but, of course, not conversely. Naturally, a lattice of sets is an abstract lattice.

Example Let X be a topological space - then the class of all open (closed) subsets of X is a σ-lattice (δ-lattice).

If $\mathcal{S}$ is a nonempty subset of $\mathcal{P}(X)$ and if X_0 is an arbitrary subset of X, then the trace of $\mathcal{S}$ on X_0 is the class

$$tr_{X_0}(\mathcal{S}) = \{S \cap X_0 : S \in \mathcal{S}\}.$$

The trace operation will preserve certain structures, e.g., the trace of a lattice is again a lattice.

Notes and Remarks

The notion of characteristic function is due to Ch. de la Vallée Poussin, Trans. Amer. Math. Soc., 16(1915), pp. 435-501. Its use was, however, first anticipated by É. Borel, Lecons sur la Théorie des Fonctions, Gauthier-Villars, Paris, 1898 (see p. 109). É. Borel also introduced the upper limit and lower limit of a sequence of sets; cf. his Lecons sur les Fonctions de Variables Réelles, Gauthier-Villars, Paris, 1905 (see p. 18). Here

$$\begin{cases}\text{upper limit = limite complète}\\ \text{lower limit = limite restreinte.}\end{cases}$$

Strangely enough, the limit of a sequence of sets was formalized only later, viz. by Ch. de la Vallée Poussin (op. cit.), the term being limite unique, the notation being lim, and also, independently, by F. Hausdorff in his classic *Grundzüge der Mengenlehre*, Veit & Comp., Leipzig, 1914 (see p. 21), where also will be found the limit superior, limit inferior terminology. The notation $\overline{\lim}$ and $\underline{\lim}$ was codified by Ch. de la Vallée Poussin, *Intégrales de Lebesgue, Fonctions d'Ensemble, Classes de Baire*, Gauthier-Villars, Paris, 1916 (see p. 8). For an exhaustive study of closure operations and their modifications, consult E. Čech, *Topological Spaces*, Academia, Prague, 1966. The topologization of $\mathscr{P}(X)$ is the subject of a paper by R. Bagley, *Michigan Math. J.*, **3**(1955-56), pp. 105-108; see also L. Savel'ev [Л. Савельев], *Sibirsk. Mat. Ž.*, **6**(1965), pp. 1357-1364. An elementary but useful survey (with extensive references) on the various operations $\mathscr{S}_s$, $\mathscr{S}_\sigma$, $\mathscr{S}_d$, $\mathscr{S}_\delta$, $\mathscr{S}_r$, $\mathscr{S}_c$ (and much more) has been given by W. Sierpiński, *Proc. Benares Math. Soc.*, N.S. **9**(1947), pp. 1-24. The origin of the various subscripts used therein is this:

$$\begin{cases} s,\sigma: & \text{Summe}\\ d,\delta: & \text{Durchschnitt,}\end{cases}$$

r:relative (complement), c:complement. Sierpiński's *Hypothèse du Continu*, Chelsea, New York, 1956, is highly recommended as a source for additional information about the continuum hypothesis and its consequences. Many of the statements in this book have subsequently been approached from the point of view of Martin's axiom; cf. D. Martin and R. Solovay, *Ann. Math. Logic*, **2**(1970), pp. 143-178.

Exercises

(1) Let $S_i = [0,1]$ for odd values of i and $S_i = [-1,0]$ for even values of i — then $\overline{\lim}\, S_i = [-1,1]$ and $\underline{\lim}\, S_i = \{0\}$.

(2) Let $\{x_i\}$ be a sequence of real numbers; let $S_i =]-\infty, x_i[$ — then

$$\begin{cases}]-\infty, \overline{\lim}\, x_i [\ \subset \overline{\lim}\, S_i \subset\ ]-\infty, \overline{\lim}\, x_i] \\ ]-\infty, \underline{\lim}\, x_i [\ \subset \underline{\lim}\, S_i \subset\ ]-\infty, \underline{\lim}\, x_i]. \end{cases}$$

(3) Let $\{S_i\}$, $\{S_i'\}$, $\{S_i''\}$ be sequences of sets with $S_i' \subset S_i \subset S_i''$ for all i. Suppose that $\lim S_i' = \lim S_i'' = S$, say — then $\lim S_i$ exists and is equal to S.

(4) The union (intersection) of a sequence of sets $\{S_i\}$ can always be represented as the limit of an increasing (decreasing) sequence of sets.

[In fact

$$\begin{cases} \cup S_i = \lim(S_1 \cup \ldots \cup S_i) \\ \cap S_i = \lim(S_1 \cap \ldots \cap S_i).] \end{cases}$$

(5) Let $\{S_i\}$ be a sequence of sets — then $\lim(S_1 \Delta \ldots \Delta S_i)$ exists iff $\lim S_i = \emptyset$.

(6) If $\{S_i\}$ is a sequence of sets, then

$$X - \overline{\lim}\, S_i = \underline{\lim}(X - S_i),\ X - \underline{\lim}\, S_i = \overline{\lim}(X - S_i).$$

(7) If $\{S_i\}$, $\{T_i\}$ are sequences of sets, then

$$\begin{cases} \overline{\lim}(S_i \cup T_i) = \overline{\lim}\, S_i \cup \overline{\lim}\, T_i \\ \underline{\lim}(S_i \cup T_i) \supset \underline{\lim}\, S_i \cup \underline{\lim}\, T_i\ , \end{cases}$$

$$\begin{cases} \overline{\lim}(S_i \cap T_i) \subset \overline{\lim}\, S_i \cap \overline{\lim}\, T_i \\ \underline{\lim}(S_i \cap T_i) = \underline{\lim}\, S_i \cap \underline{\lim}\, T_i\ , \end{cases}$$

$$\begin{cases} \overline{\lim}(S_i - T_i) \subset \overline{\lim}\, S_i - \underline{\lim}\, T_i \\ \underline{\lim}(S_i - T_i) \supset \underline{\lim}\, S_i - \overline{\lim}\, T_i \end{cases} .$$

Consequently, if $\lim S_i$ and $\lim T_i$ exist, then so do $\lim(S_i \cup T_i)$, $\lim(S_i \cap T_i)$, and $\lim (S_i - T_i)$, with

$$\begin{cases} \lim(S_i \cup T_i) = \lim S_i \cup \lim T_i \\ \lim(S_i \cap T_i) = \lim S_i \cap \lim T_i \\ \lim(S_i - T_i) = \lim S_i - \lim T_i \end{cases} .$$

(8) Let $\chi_{\underset{\sim}{Q}}$ be the characteristic function of the rationals -- then $\chi_{\underset{\sim}{Q}}$ is the pointwise limit of no sequence of continuous real valued functions on $\underset{\sim}{R}$. However, $\chi_{\underset{\sim}{Q}}$ is a Baire function on $\underset{\sim}{R}$ since

$$\chi_{\underset{\sim}{Q}}(x) = \lim_{m\to\infty} [\lim_{n\to\infty} \{\cos(m!\pi x)\}^{2n}] \qquad (x \in \underset{\sim}{R}).$$

In addition,

$$1 - \chi_{\underset{\sim}{Q}}(x) = \lim_{m\to\infty} \operatorname{sgn}\{\sin^2(m!\pi x)\} \qquad (x \in \underset{\sim}{R}).$$

[Note: This example shows that sequences do not suffice to describe a closure in the sequential modification of a space.]

(9) Let $(X,\mathfrak{T})$ be a topological space, $(X,\mathfrak{T}_s)$ its sequential modification. Let Y be a subset of X; let $\mathfrak{T}(Y)$ and $\mathfrak{T}_s(Y)$ be the corresponding relative topologies -- then $\mathfrak{T}(Y)_s \supset \mathfrak{T}_s(Y)$, i.e., the sequential modification of the relative topology on Y is finer than the relativization to Y of the sequential modification of the topology on X, there being strict containment in general but equality if Y is in addition $\mathfrak{T}_s$-closed.

[To illustrate this phenomenon, take for X the following subset of the upper half-plane + the origin:

$$\{a_{mn} = (\tfrac{1}{m},\tfrac{1}{n}) : m,n = 1,2,\ldots\} \cup \{b_n = (0,\tfrac{1}{n}) : n = 1,2,\ldots\} \cup \{c = (0,0)\}.$$

Topologize X by specifying local open neighborhoods: The open neighborhoods of a_{mn} and b_n are to be the relativized usual open neighborhoods but the open neighborhoods of c are to be the relativization of the usual open neighborhoods of $\{0\} \times]0,\varepsilon[$ $(\varepsilon>0)$ with c added in. Consider $Y = \{a_{mn}\} \cup \{c\}$.]

(10) Let $(X',\mathfrak{T}')$, $(X'',\mathfrak{T}'')$ be topological spaces -- then

$$(X' \times X'', (\mathfrak{T}' \times \mathfrak{T}'')_s) = (X' \times X'', (\mathfrak{T}'_s \times \mathfrak{T}''_s)_s).$$

[To illustrate this phenomenon, take $X' = \underset{\sim}{Q}$ in the relative topology $\mathfrak{T}'$ and take $X'' = \underset{\sim}{Q}$ in the topology $\mathfrak{T}''$ obtained by specifying that the open neighborhoods at the nonzero points are to be the relativized usual open neighborhoods but the open neighborhoods at zero itself are to be the relativization of the usual open neighborhoods of the sequence $\{\frac{\sqrt{2}}{n} : n \in \underset{\sim}{N}\}$ with 0 added in. Consider the diagonal D, as well as $D - \{(0,0)\}$.]

(11) Suppose that X is finite or countable -- then the sequential modification $\mathcal{P}(X)_s$ of $\mathcal{P}(X)$ leaves $\mathcal{P}(X)$ unchanged.

[Observe that if X is finite or countable, then the topology of pointwise convergence on $\mathcal{P}(X)$ is metrizable.]

(12) Suppose that X is uncountable -- then the sequential modification $\mathcal{P}(X)_s$ of $\mathcal{P}(X)$ is never the same as $\mathcal{P}(X)$.

[In the topology of pointwise convergence, $\mathcal{P}(X)$ is, of course, compact. Show, therefore, that the uncountability of X necessarily forces $\mathcal{P}(X)_s$ to be noncompact.]

(13) Let $\mathcal{S}$ be a nonempty class of subsets of X -- then $\mathcal{S}_{sd} = \mathcal{S}_{ds}$ but, in general, $\mathcal{S}_{\sigma\delta} \neq \mathcal{S}_{\delta\sigma}$.

[The second point can be seen by taking for $\mathcal{S}$ the class of all bounded closed intervals of the line which have positive length -- then, by a category argument, $\mathbf{Q} \in \mathcal{S}_{\delta\sigma} - \mathcal{S}_{\sigma\delta}$.]

(14) There exist classes $\mathcal{S}$ such that

$$\mathcal{S} \neq \mathcal{S}_s = \mathcal{S}_d, \quad \begin{cases} \mathcal{S} \neq \mathcal{S}_s \neq \mathcal{S}_{sd} \\ \mathcal{S} \neq \mathcal{S}_d \neq \mathcal{S}_{ds} \end{cases} .$$

There exist classes $\mathcal{S}$ such that

$$\begin{cases} \mathcal{S} \neq \mathcal{S}_\sigma = \mathcal{S}_{\sigma\delta} \\ \mathcal{S} \neq \mathcal{S}_\sigma \neq \mathcal{S}_{\sigma\delta} = \mathcal{S}_{\sigma\delta\sigma}. \end{cases}$$

Admitting the continuum hypothesis, there exists a class $\mathcal{S}$ of subsets of the line such that

$$\mathcal{S} \neq \mathcal{S}_\sigma \neq \mathcal{S}_{\sigma\delta} \neq \mathcal{S}_{\sigma\delta\sigma} = \mathcal{S}_{\sigma\delta\sigma\delta} .$$

One can go much further (to any $\alpha < \Omega$!); cf. §6.

[Note: The last assertion is tied up with an old problem of A. Kolmogoroff; cf. Fund. Math., **25**(1935), p. 578. For the details on the line, see W. Sierpiński, Mat. Sb., N.S. **43**(1936), pp. 303-306.]

(15) There exist classes $\mathcal{S}$ for which $\mathcal{S}, \mathcal{S}_r, \mathcal{S}_{rr}, \mathcal{S}_{rrr}, \ldots$ are all distinct. If $\mathcal{S} = \mathcal{S}_r$, then $\mathcal{S} = \mathcal{S}_d$ but, in general, if $\mathcal{S} = \mathcal{S}_r$, then $\mathcal{S} \neq \mathcal{S}_s$.

(16) If $\mathcal{S}$ is a lattice, then $\mathcal{S}_{rr}$ is the class consisting of all unions of two sets from the class $\mathcal{S}_r$.

[Use the identities

$$\begin{cases} (S_1 - S_2) - (S_3 - S_4) = [S_1 - (S_2 \cup S_3)] \cup [(S_1 \cap S_4) - S_2] \\ (S_1 - S_2) \cup (S_3 - S_4) = [(S_1 \cup S_3) - (S_2 \cap S_4)] - [(S_2 \cup S_4) - ((S_1 \cap S_4) \cup (S_2 \cap S_3))].] \end{cases}$$

Problems

I. LIMITS OF LATTICES

Let $\mathscr{S}$ be a lattice in X; let $\overline{\lim}\,\mathscr{S}$ ($\underline{\lim}\,\mathscr{S}$) stand for the subsets of X which are the upper limit (lower limit) of a sequence of sets from $\mathscr{S}$ -- then

$$u\mathscr{S} = \overline{\lim}\,\mathscr{S} \cap \underline{\lim}\,\mathscr{S}\ .$$

[It suffices to prove that $u\mathscr{S} = \mathscr{S}_{\sigma\delta} \cap \mathscr{S}_{\delta\sigma}$. For this purpose, establish the following generality. Let $\{S'_{i,j}\}$, $\{S''_{i,j}\}$ be two double sequences of sets in X such that

$$S'_{i,j} \supset S'_{i,j+1}, \quad S''_{i,j} \subset S''_{i,j+1}$$

with

$$\bigcup_i \bigcap_j S'_{i,j} = \bigcap_i \bigcup_j S''_{i,j} = S .$$

Then

$$S = \lim((S'_{1,j} \cap S''_{1,j}) \cup (S'_{2,j} \cap S''_{1,j} \cap S''_{2,j}) \cup \ldots \cup (S'_{j,j} \cap S''_{1,j} \cap \ldots \cap S''_{j,j})).]$$

Ref W. Sierpiński, C.R. Acad. Sci. Paris, **192**(1931), pp. 1625-1627.

II. A THEOREM OF INSERTION

Let $\mathscr{S}$ be a lattice in X; let $S_\sigma \in \mathscr{S}_\sigma$, $S_\delta \in \mathscr{S}_\delta$ with $S_\sigma \supset S_\delta$ -- then there exists an $S \in \mathscr{S}_\sigma \cap \mathscr{S}_\delta$ such that $S_\sigma \supset S \supset S_\delta$.

[Use the following generality. Let $\{S_i'\}$, $\{S_i''\}$ be two sequences of sets in X such that

$$S_i' \subset S_{i+1}', \quad S_i'' \supset S_{i+1}''$$

with

$$\bigcup S_i' \supset \bigcap S_i'' .$$

Then

$$\cup(S_i' \cap S_i'') = S_1'' \cap (S_1' \cup S_2'') \cap (S_2' \cup S_3'') \cap \ldots .]$$

Ref W. Sierpiński, Fund. Math., 6(1924), pp. 1-5.

III. UPPER LIMIT OF A SEQUENCE OF SETS

Let $\mathcal{I}$ be the class of all infinite subsets of $\mathbb{N}$ -- then, given any sequence $\{S_i\}$ of subsets of X,

$$\overline{\lim}\, S_i = \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} S_i .$$

Supposing that $I \in \mathcal{I}$, say $I = \{i_j : j=1,2,\ldots\}$, let us agree to write $\overline{\lim}_I\, S_i$ for $\overline{\lim}\, S_{i_j}$.

It is easy to give examples where $\text{card}(\bigcap_{i \in I} S_i) \leq 1$ $\forall I \in \mathcal{I}$ and yet, e.g., $\text{card}(\overline{\lim}\, S_i) = \mathfrak{c}$. Accordingly, one asks instead: How does the cardinality of $\bigcap_{i \in I} S_i$ influence the cardinality of $\overline{\lim}_I\, S_i$?

(1) True or False?

(a) $\exists$ a sequence $\{S_i\}$ such that $\forall I \in \mathcal{I}$, $\bigcap_{i \in I} S_i$ is finite but $\overline{\lim}_I S_i$ is infinite.

(b) $\exists$ a sequence $\{S_i\}$ such that $\forall I \in \mathcal{I}$, $\text{card}(\bigcap_{i \in I} S_i) \leq \aleph_0$ but $\text{card}(\overline{\lim}_I S_i) \geq \aleph_1$.

(2) True or False?

(a) If $\forall I \in \mathcal{I}$, $\text{card}(\bigcap_{i \in I} S_i) \leq N$, then $\exists$ an $I_0 \in \mathcal{I}$ such that $\text{card}(\overline{\lim}_{I_0} S_i) \leq N$ $(N < \aleph_0)$.

(b) If $\forall I \in \mathcal{I}$, $\text{card}(\bigcap_{i \in I} S_i) < \aleph_0$, then there exists an $I_0 \in \mathcal{I}$ such that $\text{card}(\overline{\lim}_{I_0} S_i) \leq \aleph_0$.

In conclusion, let $\{S_i\}$ be a sequence of subsets of X such that $\text{card}(\bigcap_{i \in I} S_i) \leq \aleph_0$ $\forall I \in \mathcal{I}$ -- then $\{S_i\}$ admits a convergent subsequence iff $\exists$ an $I_0 \in \mathcal{I}$ such that $\text{card}(\overline{\lim}_{I_0} S_i) \leq \aleph_0$.

Ref M. Laczkovich, Anal. Math., 3(1977), pp. 199-206.

IV. THE CHARACTERISTIC FUNCTION OF A SEQUENCE OF SETS

Denote by $\text{Seq}(\mathcal{P}(X))$ the class of all sequences of subsets of X -- then by the characteristic function of an element $\mathcal{S} = \{S_i\}$ of $\text{Seq}(\mathcal{P}(X))$ we understand the function $\chi_{\mathcal{S}}: X \to \mathbf{R}$ defined by the series

$$\chi_{\mathcal{S}}(x) = 2 \cdot \sum_{i=1}^{\infty} \chi_{S_i}(x)/3^i \qquad (x \in X).$$

The range of $\chi_{\mathcal{S}}$ is evidently a subset of C, the classical Cantor set. In fact, the map $\mathcal{S} \mapsto \chi_{\mathcal{S}}$ implements an identification between $\text{Seq}(\mathcal{P}(X))$ and $\text{Fnc}(X,C)$.

Here are some elementary remarks.

(1) The sets in the sequence $\mathcal{S}$ are all one and the same iff $\chi_{\mathcal{S}}$ assumes only the values 0 and 1.

(2) The sets in the sequence $\mathcal{S}$ are pairwise disjoint iff $\chi_{\mathcal{S}}$ assumes only the value 0 and values of the form $2/3^n$.

(3) A sequence $\mathcal{S}$ of sets is increasing iff $\chi_{\mathcal{S}}$ assumes only the values 0, 1, and values of the form $1/3^n$.

(4) A sequence $\mathcal{S}$ of sets is decreasing iff $\chi_{\mathcal{S}}$ assumes only the values 0, 1, and values of the form $1-(1/3^n)$.

(5) A sequence $\mathcal{S}$ of sets is convergent iff $\chi_{\mathcal{S}}$ assumes only the values 0, 1, and values of the form $m/3^n$.

Characterize those $\mathcal{S}$ for which:

(a) $\overline{\chi_{\mathcal{S}}(X)} = C$; (b) $\chi_{\mathcal{S}}(X) = C$.

Show that if X is in addition a topological space, then $\chi_{\mathcal{S}}$ is continuous iff all the sets in $\mathcal{S}$ are open and closed.

[Note: Suppose that X is a metric space with weight $\aleph_0$ -- then, upon consideration of the characteristic function of a base of cardinality $\aleph_0$, one can readily establish the following well-known results:

X is the continuous image of a subset of C. Furthermore, if

$$\begin{cases} X \text{ is compact} \\ X \text{ is compact and totally disconnected} \\ X \text{ is compact, totally disconnected, and perfect,} \end{cases}$$

then

$$\begin{cases} X \text{ is a continuous image of } C \\ X \text{ is a homeomorphic image of a closed subset of } C \\ X \text{ is a homeomorphic image of } C.] \end{cases}$$

Ref E. Szpilrajn, Fund. Math., 31(1938), pp. 207-223.

[A transfinite generalization can be found in M. Stone, Fund. Math., 33(1945), pp. 27-33.]

V. THE EQUALITY (INEQUALITY) OF $\mathcal{S}_{\sigma\delta}$ AND $\mathcal{S}_{\delta\sigma}$

Let X be a set of cardinality $\aleph_0$; let $\mathcal{S} \subset \mathcal{P}(X)$ be nonempty -- then, of necessity, $\mathcal{S}_{\sigma\delta} = \mathcal{S}_{\delta\sigma}$.

[This is easy, the point being that the complement of a countable subset of X is either countable or finite.]

Let X be a set of cardinality $>\aleph_0$ -- then there necessarily exists an $\mathcal{S} \subset \mathcal{P}(X)$ for which $\mathcal{S}_{\sigma\delta} \neq \mathcal{S}_{\delta\sigma}$.

[There is no loss of generality in supposing that X is a subset of $\mathbf{R}$ of cardinality $\aleph_1$. Let $\mathcal{S}$ be the class of all sets of the form $X \cap I_{k,n}$, $I_{k,n} =]\frac{k}{2^n}, \frac{k+1}{2^n}[$ a generic dyadic open interval. We claim that $\mathcal{S}_{\sigma\delta} \neq \mathcal{S}_{\delta\sigma}$. To prove this, select in each nonempty $X \cap I_{k,n}$ some point $x_{k,n}$, say. Denote by X_0 the totality of all such -- then $X - X_0 \in \mathcal{S}_{\sigma\delta}$ but $X - X_0 \notin \mathcal{S}_{\delta\sigma}$.

Ref W. Sierpiński, Spis. Bulgar. Akad. Nauk, 53(1936), pp. 181-195.

[Note: Let X be any set; let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$. Write $\mathcal{S}_\Sigma$, $\mathcal{S}_\Delta$ for the class of subsets of X comprised of all nonempty unions, nonempty intersections of sets in $\mathcal{S}$ (repetitions being permissible) -- then always $\mathcal{S}_{\Sigma\Delta} = \mathcal{S}_{\Delta\Sigma}$.]

VI. DIFFERENCES, UNIONS, INTERSECTIONS

Let X be a set of cardinality $\aleph_0$; Let $\mathcal{S} \subset \mathcal{P}(X)$ be nonempty -- then

$$\mathcal{S}_{r\sigma r\sigma r} = \mathcal{S}_{r\sigma r\sigma}, \ \mathcal{S}_{r\delta r\delta r} = \mathcal{S}_{r\delta r\delta}$$

but, in general,

$$\mathcal{S}_{r\sigma r\sigma} \neq \mathcal{S}_{r\sigma r}, \ \mathcal{S}_{r\delta r\delta} \neq \mathcal{S}_{r\delta r}.$$

Discuss the effect of permuting the roles of r and σ or of r and δ.

What happens if X is a set of cardinality $>\aleph_0$?

Ref S. Picard, Fund. Math., 26 (1936), pp. 262-266.

[See also the paper of Sierpiński's referred to in Prob. V.]

VII. FILTERS AND ULTRAFILTERS

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$ - then $\mathcal{S}$ is said to be a *filter* on X if:

(i) $\emptyset \notin \mathcal{S}$;

(ii) $\mathcal{S} = \mathcal{S}_d$;

(iii) $S \in \mathcal{S}$, $S \subset T \Longrightarrow T \in \mathcal{S}$.

The collection $\mathcal{F}il(X)$ of all filters on X is ordered by the inclusion relation (induced from that on $\mathcal{P}(\mathcal{P}(X))$).

[Note: Occasionally, condition (i) is dropped, $\mathcal{P}(X)$ itself being regarded as a filter (cf., e.g., Exer. 9(§4)).]

An ultrafilter on X is a filter which is properly contained in no other filter on X. I.e.: The ultrafilters on X are the maximal elements in the ordered set $\mathcal{Fil}(X)$.

A filter $\mathcal{S}$ on X is an ultrafilter iff for each $S \subset X$, either $S \in \mathcal{S}$ or $X - S \in \mathcal{S}$. If $S_1, \ldots, S_m$ are subsets of X whose union $\bigcup_{i=1}^{m} S_i$ is an element of an ultrafilter $\mathcal{S}$ on X, then at least one of the S_i belongs to $\mathcal{S}$.

Every filter is contained in an ultrafilter; moreover, every filter is the intersection of the ultrafilters containing it.

A filterbase on X is a class $\mathcal{S}$ of nonempty subsets of X with the property that

$$\forall S_1, S_2 \in \mathcal{S}, \exists S_3 \in \mathcal{S} \text{ st } S_3 \subset S_1 \cap S_2.$$

A class $\mathcal{S}$ is contained in a filter iff it is a filterbase. If $\mathcal{S}$ is a filterbase, then

$$\mathcal{Fil}(\mathcal{S}) = \{T \subset X: \exists S \in \mathcal{S} \text{ st } S \subset T\}$$

is the smallest filter containing $\mathcal{S}$ or still, the filter generated by $\mathcal{S}$.

A class $\mathcal{S}$ is said to have the finite intersection property if the intersection of the members of any finite subclass of $\mathcal{S}$ is nonempty. Suppose that $\mathcal{S}$ has the finite intersection property - then $\mathcal{S}_d$ is a filterbase, thus $\mathcal{S}$ is contained in $\mathcal{Fil}(\mathcal{S}_d)$, thence in an ultrafilter. Every filter has the finite intersection property.

Ref H. Cartan, C.R. Acad. Sci. Paris, 205(1937), pp. 595-598 and pp. 777-779.

[Note: The purpose of this problem is merely to fix the terminology and recall some basic facts.]

VIII. COMPACT AND COUNTABLY COMPACT CLASSES

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$ - then $\mathcal{S}$ is said to be compact (countably compact) if every subclass (countable subclass) of $\mathcal{S}$ with the finite intersection property has a nonempty intersection.

There is no a priori connection between the compactness (countable compactness) of a class and the topology of pointwise convergence on $\mathcal{P}(X)$ (or of its sequential modification).

Example Let X be a compact (countably compact) Hausdorff space - then the class of all closed subsets of X is compact (countably compact).

There are countably compact classes which are not compact.

The main stabilization result is this: Suppose that

$$\mathcal{S} \text{ is } \begin{cases} \text{compact} \\ \\ \text{countably compact.} \end{cases}$$

Then

$$\begin{cases} \mathcal{S}_{s\Delta} & \text{is compact} \\ \\ \mathcal{S}_{s\delta} & \text{is countably compact.} \end{cases}$$

[Since compactness (countable compactness) is evidently preserved by operation $\Delta(\delta)$, it suffices in either case to deal just with $\mathcal{S}_s$. Consider, therefore, a class (countable class) $\{S_i : i \in I\}$ of elements of $\mathcal{S}_s$ with the finite intersection property. Fix an ultrafilter $\mathcal{S}_0$ on X such that $S_i \in \mathcal{S}_0$ $\forall i$. Write $S_i = \bigcup_j S_{ij}$ $(j \in J_i)$, J_i a finite set $(S_{ij} \in \mathcal{S}\ \forall j)$. Choose, as is possible (cf. Prob. VII), an index $j_i \in J_i$ for which $S_{ij_i} \in \mathcal{S}_0$ - then the class consisting of the S_{ij_i} $(i \in I)$ has the finite intersection property, so $\cap S_{ij_i} \neq \emptyset \Longrightarrow \cap S_i \neq \emptyset$.]

Ref E. Marczewski, Fund. Math., 40(1953), pp. 113-124.

[The notion of a countably compact class of sets is due to W. Sierpiński, *Fund. Math.*, **21**(1933), pp. 250-275.]

§2. Partitions

Let X be a nonempty set -- then by a *partition* of X we understand a class $P(X) = \{X_i : i \in I\}$ of nonempty, pairwise disjoint subsets X_i of X such that $X = \cup X_i$, i.e., such that $\chi_X = \sum \chi_{X_i}$. The X_i are called the *components* of $P(X)$. Associated with the partition $P(X)$ is a surjective map $f: X \to I$, viz. the map assigning to $x \in X_i \hookrightarrow X$ the index i; conversely, associated with a surjective map $f: X \to I$ is a partition $P(X)$, viz. the partition whose i^{th}-component X_i is the fiber $f^{-1}(i)$.

Example The equivalence classes determined by an equivalence relation on X form a partition of X.

A partition $Q(X)$ is said to be a *refinement* of the partition $P(X)$, written

$$Q(X) \geq P(X) \quad \text{or} \quad P(X) \leq Q(X),$$

if every component of $Q(X)$ is contained in a component of $P(X)$. Evidently, $Q(X)$ refines $P(X)$ iff every component of $P(X)$ is a union of components of $Q(X)$. The partition whose components are the elements of X refines every partition of X; every partition of X refines the partition whose sole component is X itself.

Let

$$\begin{cases} P'(X) = \{X_{i'}: i' \in I'\} \\ P''(X) = \{X_{i''}: i'' \in I''\} \end{cases}$$

be two partitions of X -- then by the *meet* of $P'(X)$ and $P''(X)$ we mean that partition $P'(X) \wedge P''(X)$ of X whose components are the nonempty $X_{i'} \cap X_{i''}$. It is clear that $P'(X) \wedge P''(X)$ is a simultaneous refinement of both $P'(X)$ and $P''(X)$; moreover, $P'(X) \wedge P''(X)$ is refined by every partition with this property. Since the relation of refinement is reflexive and transitive, it follows that the collection of all partitions of X is in fact a directed set.

[Note: The collection of all partitions of X carries the structure of a lattice possessing certain supplementary characteristics (cf. Exer. 3).]

Example Suppose that $f:X \to X$ is a map without fixed points -- then there exists a disjoint decomposition

$$X = X_1 \cup X_2 \cup X_3$$

of X such that $X_i \cap f(X_i) = \emptyset$ $(i=1,2,3)$.

[Note: Strictly speaking, this decomposition need not be a partition of X since a given X_i may be empty. For the easy details, see M. Katětov, Comment. Math. Univ. Carolin., **8**(1967), pp 431-433.]

In many of the applications, the emphasis is not so much on partitioning X by certain of its subsets as it is on partitioning the elements of a given class of subsets of X by elements from that class.

Let, then, $\mathcal{S}$ be a nonempty class of subsets of X; it is not required but it is not excluded that X itself belongs to $\mathcal{S}$. Let $S \in \mathcal{S}$ -- then by an $\mathcal{S}$-*partition* of S, we understand a class $P(S) = \{S_i : i \in I\}$ of nonempty, pairwise disjoint subsets $S_i \in \mathcal{S}$ such that $S = \cup S_i$, i.e., such that $\chi_S = \sum \chi_{S_i}$. The S_i are called the *components* of P(S).

The collection of all $\mathcal{S}$-partitions of S need not be directed by the relation of refinement, the point being that there is no reason to expect that the meet of two $\mathcal{S}$-partitions be again an $\mathcal{S}$-partition. However, there is a simple condition on $\mathcal{S}$ which will guarantee this, namely that $\mathcal{S}$ be a multiplicative class. The multiplicativity of $\mathcal{S}$, an essentially minimal requirement, also ensures that it is permissible to take the trace of an $\mathcal{S}$-partition.

Thus let $S,T \in \mathcal{S}$ with $S \supset T \neq \emptyset$. Suppose that $P(S) = \{S_i : i \in I\}$ is an $\mathcal{S}$-partition of S -- then by the *trace* of $P(S)$ on T we mean that $\mathcal{S}$-partition $tr_T(P(S))$ of T whose components are the non-empty $S_i \cap T$. To within the empty set, this notation agrees with that introduced in §1.

Partitions of restricted cardinality (viz. $\leq \aleph_0$) figure prominently in the theories of the integral and derivative. To stress this, let us agree that an $\mathcal{S}$-partition of $S \in \mathcal{S}$ is *finite* (*countable*) if this is so of the corresponding index set. The class of all components arising from all possible finite (countable) $\mathcal{S}$-partitions of S will be denoted by $Com_{\mathcal{S}}(S)$ (σ-$Com_{\mathcal{S}}(S)$) while the collection of all possible finite (countable) $\mathcal{S}$-partitions of S will be denoted by $Par_{\mathcal{S}}(S)$ (σ-$Par_{\mathcal{S}}(S)$). If $\mathcal{S}$ is multiplicative, then, per the relation of refinement, both $Par_{\mathcal{S}}(S)$ and σ-$Par_{\mathcal{S}}(S)$ are directed sets. Conventionally, S admits *infinite* $\mathcal{S}$-*partitions* if σ-$Par_{\mathcal{S}}(S)$ is nonempty; of course, for this to be the case, $\mathcal{S}$ itself must be at least countable.

Example Take $X = [0,1]$ -- then the traditional notion of a partition of X consists in the specification of points $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$. Observe, however, that the intervals $[x_0,x_1],\ldots,[x_{n-1},x_n]$ do not partition X. The way out is to use instead the intervals $[x_0,x_1[,[x_1,x_2[,\ldots,[x_{n-1},x_n]$ or the intervals $[x_0,x_1],]x_1,x_2],\ldots,]x_{n-1},x_n]$. Note too that while the intervals $]x_0,x_1[,\ldots,]x_{n-1},x_n[$ do not partition X, they do constitute a topological partition of X; cf. infra. In passing, we remark that it is easy to exhibit countable partitions of X, e.g., $\{0\}$, and the $]\frac{1}{n+1}, \frac{1}{n}]$ $(n=1,2,\ldots)$. Consider now the class $\mathcal{S}$ of all closed subintervals of X; $\mathcal{S}$ is multiplicative, singletons (as well as the empty set) belonging to $\mathcal{S}$. Given $[a,b]$ in $\mathcal{S}$, it is clear that

$$\mathrm{Par}_{\mathcal{S}}([a,b]) = \{[a,b]\},\ \sigma\text{-}\mathrm{Par}_{\mathcal{S}}([a,b]) = \emptyset .$$

Therefore, in so far as it is a question of finite or countable partitions, $\mathcal{S}$ is inutile. Trivially, of course, $[a,b] = \bigcup_{a \leq x \leq b} \{x\}$, an uncountable union (if $b > a$).

The preceding example, its essential simplicity notwithstanding, already contains a degree of unpleasantness. Our strictly set-theoretic definition of partition allows for no overlap in the components. In certain situations, however, this turns out to be an unduly restrictive condition, particularly in the presence of other structures, for instance, a topology. Though this will not be a point of concern at present, nevertheless an illustration may prove helpful.

Let X be a topological space -- then by a *topological partition* of X we understand a class $P(X) = \{X_i : i \in I\}$ of nonempty, pairwise disjoint, open and connected subsets X_i of X such that $\bigcup X_i$ is dense in X. The X_i are called the *components* of $P(X)$. A topological partition $Q(X)$ is said to be a *refinement* of the topological partition $P(X)$, written

$$Q(X) \underset{T}{\leq} P(X) \quad \text{or} \quad P(X) \underset{T}{\geq} Q(X).$$

if every component of $Q(X)$ is contained in a component of $P(X)$.

Specialize now and suppose that X is actually a metric space with metric d. Let $\varepsilon > 0$ -- then an *ε-partition* of X is a topological partition with the property that each of its components has diameter $<\varepsilon$. X is called *d-partitionable* if for every $\varepsilon > 0$, there exists an ε-partition of X.

Example The metric space (X,d) is *strongly d-partitionable* if for every $\varepsilon > 0$, there exists a finite ε-partition of X. We then ask: What metric spaces are strongly d-partitionable? It turns out that there is a very simple answer. To give it, recall that X has *property S* if for every $\varepsilon > 0$, X can be written as the union of a finite number of connected subsets each of diameter less than ε. In terms of this notion, the sought for characterization then reads: X is strongly d-partitionable iff X has property S. Consequently, if X is strongly d-partitionable, then for every $\varepsilon > 0$, it is possible to find a finite ε-partition of X all components of which have property S, hence there is a sequence $P_1(X)$, $P_2(X)$, ... such that $P_i(X)$ is a finite $1/i$-partition of X and $P_{i+1}(X)$ is a refinement of $P_i(X)$. Assume in addition that X is compact and connected, i.e., that X is a continuum -- then, as is well-known, X is locally connected iff X has property S. By definition, a **continuous curve** is a locally connected continuum. In view of what has been said, therefore, every continuous curve is strongly d-partitionable, a theorem of R. Bing.

[Note: For a complete discussion of these and other related results, see R. Bing, Bull. Amer. Math. Soc., **55**(1949), pp. 1101-1110, and **58**(1952), pp. 536-556.]

Notes and Remarks

Partitions, in one guise or another, have been around from the beginning. They will play a central role in the sequel. Incidentally, it should be noted that partitions and equivalence relations are coextensive notions, both being descriptions of the same mathematical reality. Observe too that the axiom of choice is entirely equivalent to the statement that every partition of every set has a **set of representatives**, i.e., if $P(X) = \{X_i : i \in I\}$ is a partition of X, then there exists a subset $C_{P(X)}$ of X such that $C_{P(X)} \cap X_i = \{x_i\}$ $\forall X_i$. The discovery that continuous curves could be topologically partitioned was one of the most important combinatorial developments of the 1950's. The term continuous curve arises, of course, from the famous theorem of Hahn-Mazurkiewicz which states that a metric space is a continuous curve iff it is the continuous image of [0,1]. For this reason, continuous curves are sometimes referred to as **Peano spaces**. A systematic treatment of these matters can be found in G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, New York, 1942, and T. Radó, *Length and Area*, Amer. Math. Soc. Colloquium Publications, vol. 30, New York, 1948. Finally, for much additional information on the general theory of partitions, the reader can consult with profit O. Ore, *Duke Math. J.*, **9**(1942), pp. 573-627.

Exercises

(1) For $n = 1,2,\ldots$, let p_n be the number of partitions of a set of n elements -- then the p_n satisfy the recursion relation

$$p_{n+1} = 1 + \sum_{k=1}^{n} \binom{n}{k} p_k \quad .$$

What is the relationship between the p_n and $\exp(\exp x - 1)$?

(2) Let $\mathfrak{X} = \{X_i : i \in I\}$ be a class of nonempty subsets of a set X -- then $\mathfrak{X}$ determines a partition $P_{\mathfrak{X}}$ of X which partitions each of the X_i and is refined by any partition of X with this property.

[Given a subset E of I, put

$$X_E = \bigcap_{i \in E} X_i \cap \bigcap_{i \in I - E} (X - X_i).$$

Consider the nonempty X_E.]

(3) Let

$$\begin{cases} P'(X) = \{X_{i'} : i' \in I'\} \\ \\ P''(X) = \{X_{i''} : i'' \in I''\} \end{cases}$$

be two partitions of X -- then by the *join* of $P'(X)$ and $P''(X)$ we mean that partition $P'(X) \vee P''(X)$ of X whose components are the minimal nonempty $\cup X_{i'} = \cup X_{i''}$. It is clear that $P'(X) \vee P''(X)$ is refined simultaneously by both $P'(X)$ and $P''(X)$; moreover, $P'(X) \vee P''(X)$ refines every partition with this property.

[Note: In the technical language of the trade, the collection of all partitions of X is a relatively complemented, semimodular, complete lattice with largest and smallest elements. It is called the *partition lattice* attached to X. Up to isomorphism, every abstract lattice appears as a sublattice of some such partition lattice; cf. P. Whitman, Bull. Amer. Math. Soc., **52**(1946), pp. 507-522.]

(4) Suppose that $X = \bigcup_{i=1}^{m} X_i$ is the union of $m = 2^n$ nonempty, distinct subsets X_i -- then there exist n+1 nonempty, pairwise disjoint subsets Y_j of X such that $X = \bigcup_{j=1}^{n+1} Y_j$.

[There are two ways to look at this. The first method consists in remarking that X must have at least $n+1$ distinct elements, say $x_1, \ldots, x_{n+1}$, so

$$X = \{x_1\} \cup \ldots \cup \{x_n\} \cup (X - \{x_1, \ldots, x_n\})$$

which is certainly a partition of X with the desired property. However, while the axiom of choice has not been used, the construction can hardly be considered effective. The second (effective) method consists in considering $M = \{1, \ldots, m\}$, the $2^m - 1$ nonempty subsets of which, $\{i_1, \ldots, i_s\}$, can be arranged into a finite sequence according to the size of the number $2^{i_1} + \ldots + 2^{i_s}$. Denoting by $\{M_k\}$ the sequence thereby obtained, put

$$Z_k = \bigcap_{i \in M_k} X_i - \bigcup_{i \in M - M_k} X_i .$$

The Z_k may be used to determine the Y_j.]

(5) Suppose that $X = \bigcup_{i=1}^{\infty} X_i$ is the union of countably many nonempty, distinct subsets X_i -- then there exist countably many nonempty, pairwise disjoint subsets Y_j of X such that $X = \bigcup_{j=1}^{\infty} Y_j$.

[The axiom of choice is not needed here (Kuratowski); cf. A. Tarski, Fund. Math., **6**(1924), pp. 45-95 (see pp. 94-95).]

(6) Let X be a set; let $f:X \to X$ be a map. Suppose that f is injective -- then X can be uniquely decomposed as a countable union of pairwise disjoint sets $X_0, X_1, \ldots$ (possibly $\emptyset$) such that

$$f(X_0) = X_0, \quad f(X_i) = f(X_{i+1}) \quad (i \geq 1).$$

[Take

$$X_0 = \bigcap_{i=1}^{\infty} f^i(X), \quad X_i = f^{i-1}(X) - f^i(X) \qquad (i \geq 1),$$

where $f^0(X) = X$.]

(7) Let X and Y be sets; let $f:\mathcal{P}(X) \to \mathcal{P}(Y)$ and $g:\mathcal{P}(Y) \to \mathcal{P}(X)$ be maps. Suppose that

$$\begin{cases} \forall S,T \in \mathcal{P}(X), \ S \subset T \Longrightarrow f(S) \subset f(T) \\ \forall S,T \in \mathcal{P}(Y), \ S \subset T \Longrightarrow g(S) \subset g(T). \end{cases}$$

Then there exist disjoint decompositions $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$ such that $f(X_1) = Y_1$, $g(Y_2) = X_2$. Must these decompositions be partitions of X or Y?

[First prove that if M is a set, $\phi:\mathcal{P}(M) \to \mathcal{P}(M)$ a map such that

$$\forall A,B \in \mathcal{P}(M), \ A \subset B \Longrightarrow \phi(A) \subset \phi(B),$$

then for some subset M_0 of M, $\phi(M_0) = M_0$. This done, specialize and for $S \subset X$, put

$$\phi(S) = X - g(Y-f(S)).$$

The preceding remark implies that ϕ has a fixed point X_1, say. Take, then, $X_2 = X - X_1$, $Y_1 = f(X_1)$, $Y_2 = Y - Y_1$.]

(8) There exists a nonempty set X and a nonempty class $\mathcal{S}$ of subsets of X with the following property: Every nonempty $S \in \mathcal{S}$ admits a partition by three elements of $\mathcal{S}$ but no nonempty $S \in \mathcal{S}$ admits a partition by two elements of $\mathcal{S}$. Can $\mathcal{S}$ be taken multiplicative?

(9) Let $\mathcal{S}$ be a nonempty class of subsets of X with the property that every nonempty element of $\mathcal{S}$ can be written as the union of three distinct elements of $\mathcal{S}$ -- then every nonempty element of $\mathcal{S}$ can be written as the union of two distinct elements of $\mathcal{S}$.

(10) There exist a nonempty set X and a nonempty class $\mathcal{S}$ of subsets of X with the following property: Every nonempty $S \in \mathcal{S}$ admits a partition by two elements of $\mathcal{S}$ but no nonempty $S \in \mathcal{S}$ admits a partition by countably many elements of $\mathcal{S}$. Can $\mathcal{S}$ be taken multiplicative?

[Note: Suppose that $X = \mathbb{N}$ -- then in this case, if every nonempty $S \in \mathcal{S}$ can be partitioned by two elements of $\mathcal{S}$, it must actually be the case that every nonempty $S \in \mathcal{S}$ can be partitioned by countably many elements of $\mathcal{S}$.]

(11) Exhibit an explicit countable partition of $\mathbb{N}$, each component of which is countable.

(12) Exhibit an explicit partition of $\mathbb{R}$, each component of which consists of two elements.

(13) Exhibit an explicit partition of $[0,1]$, each component of which consists of two elements.

(14) Take $X = \mathbf{R}$ -- then there exists a subset S of X and a countable set of real numbers $\{s_i\}$ such that

$$X = \bigcup_{i=1}^{\infty} (s_i + S) ,$$

where

$$i \neq j \Longrightarrow (s_i + S) \cap (s_j + S) = \emptyset$$

[This is easy: Put $S = [0,1[$ and choose the s_i in the obvious way.]

(15) Take $X = [0,1]$ -- then there exists a subset S of X and a countable set of real numbers $\{s_i\}$ such that

$$X = \bigcup_{i=1}^{\infty} (s_i + S) ,$$

where

$$i \neq j \Longrightarrow (s_i + S) \cap (s_j + S) = \emptyset.$$

[This is difficult; cf. J. v. Neumann, Fund. Math., **11**(1928), pp. 230-238. We remark that the axiom of choice is needed here; naturally, neither S, nor any of its translates is Lebesgue measurable.]

(16) The continuum hypothesis is equivalent to the statement that the real line - the origin can be partitioned into countably many rationally independent sets.

[This result is due to P. Erdös and S. Kakutani, Bull. Amer. Math. Soc., **49**(1943), pp. 457-461. In brief, the argument runs as follows.

Admit the continuum hypothesis. Let $\{x_\beta : \beta < \omega_1\}$ be a Hamel basis for $\underset{\sim}{R}$. Given nonzero rational numbers $r_1,\ldots,r_n$, write $\underset{\sim}{R}(r_1,\ldots,r_n)$ for the set of all $x \in \underset{\sim}{R}$ such that $x = r_1 x_{\beta_1} + \ldots + r_n x_{\beta_n}$ $(\beta_1 < \ldots < \beta_n)$ -- then, in an obvious notation,

$$\underset{\sim}{R} = \{0\} \cup \bigcup_{(r_1,\ldots,r_n)} \underset{\sim}{R}(r_1,\ldots,r_n) \quad \text{(disjoint union)}.$$

Decompose each $\underset{\sim}{R}(r_1,\ldots,r_n)$ by considering for every $\beta < \omega_1$ the subset comprised of those x for which $\beta_n = \beta$.

Deny the continuum hypothesis. Let $\{x_\beta : \beta < \omega_\alpha\}$ be a Hamel basis for $\underset{\sim}{R}$ -- then $\alpha \geqq 2$. Let $\{X_i\}$ be any countable partition of $\underset{\sim}{R} - \{0\}$ -- then there exists an index i for which

$$\text{card}(\{\omega_1 \leq \beta < \omega_\alpha : i(\beta) = i\}) \geqq \aleph_2,$$

where $i(\beta)$ is defined by requiring that there be ordinals $\beta'_\beta, \beta''_\beta$ with

$$\begin{cases} \beta'_\beta < \omega_1, \ \beta''_\beta < \omega_1 \\ \beta'_\beta < \beta''_\beta \end{cases} \qquad \begin{cases} x_{\beta'_\beta} + x_\beta \in X_{i(\beta)} \\ x_{\beta''_\beta} + x_\beta \in X_{i(\beta)} \end{cases}.$$

Conclude from this that there exist ordinals

$$\begin{cases} \beta' < \omega_1, \ \beta'' < \omega_1 \\ \beta' < \beta'' \end{cases} \qquad \begin{cases} \omega_1 \leq \beta^*, \ \beta^{**} < \omega_\alpha \\ \beta^* < \beta^{**} \end{cases} \quad i = i(\beta^*) = i(\beta^{**})$$

such that

$$x_{\beta'} + x_{\beta^*}, \ x_{\beta''} + x_{\beta^*}, \ x_{\beta'} + x_{\beta^{**}}, \ x_{\beta''} + x_{\beta^{**}}$$

all belong to X_i.]

Problems

1. MESH FUNCTIONS

Let $\mathcal{S}$ be a nonempty class of subsets of X; let $S \in \mathcal{S}$ -- then by a *mesh function* δ on $\text{Par}_{\mathcal{S}}(S)$ we understand a rule which assigns to each $P(S)$ in $\text{Par}_{\mathcal{S}}(S)$ a positive real number $\delta(P(S))$ subject to the following rule: $\forall \varepsilon > 0\ \exists P_{\varepsilon}(S) \in \text{Par}_{\mathcal{S}}(S)$ such that $\delta(P_{\varepsilon}(S)) < \varepsilon$. If $\text{Par}_{\mathcal{S}}(S)$ admits a mesh function δ, then δ can be used to direct $\text{Par}_{\mathcal{S}}(S)$: $Q(S) \underset{\delta}{\leq} P(S)$ iff $\delta(Q(S)) \leq \delta(P(S))$. It is to be stressed that if $Q(S)$ is a refinement of $P(S)$, then there may be no relation between $\delta(Q(S))$ and $\delta(P(S))$; in fact, δ need not decrease upon refinement.

[Take $X = [0,1[$ and let $\mathcal{S}$ be the class of all left closed and right open subintervals $[a,b[$ of X. Fix $S = [a,b[$ in $\mathcal{S}$ -- then an element $P(S)$ in $\text{Par}_{\mathcal{S}}(S)$ has the form $\{[a_i,b_i[: i=1,\ldots,n\}$, where, say, $a_1 = a$, $b_n = b$ and $a_1 < b_1 = a_2 < b_2 \ldots$. Put $\delta(P(S)) = \max(b_i - a_i)$ -- then δ is a mesh function on $\text{Par}_{\mathcal{S}}(S)$ which, moreover, does decrease upon refinement. Define now a function σ on X via the following stipulation: $\sigma(x) = 0$ if x is irrational, $\sigma(x) = 1/q$ if $x = p/q$ is rational $(0 \leq p \leq q,\ q \text{ min.})$. Put $\delta(P(S)) = \sum(\sigma(a_i) + \sigma(b_i)) + \max(b_i - a_i) - (\sigma(a) + \sigma(b))$ -- then δ is a mesh function on $\text{Par}_{\mathcal{S}}(S)$ which, this time, need not decrease upon refinement.]

Ref L. Cesari, Trans. Amer. Math. Soc., **102**(1962), pp. 94-113.

[Note: Suppose that X is a continuous curve. Let Top-Par(X) be the collection of all finite ε-partitions of X -- then the rule which assigns to each $P(X)$ in Top-Par(X) the maximum diameter of its components can be viewed, in the obvious way, as a mesh function on Top-Par(X) which, moreover, decreases upon refinement.]

II. THEOREMS OF RAMSEY AND SIERPIŃSKI

Given a set X and a natural number n, let us agree to write $\langle X \rangle_n$ for the class of all subsets of X of cardinality n.

Theorem (Ramsey) *Let* X *be a set of cardinality* $\aleph_0$; *let* $\{\mathcal{X}_1, \ldots, \mathcal{X}_m\}$ *be a finite partition of* $\langle X \rangle_n$ -- *then there exists an infinite subset* S *of* X *and an index* i *such that* $\langle S \rangle_n \subset \mathcal{X}_i$.

[There is no loss of generality in taking $X = \mathbf{N}$. This being so, it will then be enough to prove that for any map $f : \langle \mathbf{N} \rangle_n \to \{1, \ldots, m\}$, there exists an infinite subset S of $\mathbf{N}$ such that f is constant on $\langle S \rangle_n$. Proceed by induction on n. If $n = 1$, the result is clear so assume that it holds for $n \geq 1$. Let $f : \langle \mathbf{N} \rangle_{n+1} \to \{1, \ldots, m\}$ be a map. Given $x \in \mathbf{N}$, write f_x for the function on $\langle \mathbf{N} - \{x\} \rangle_n$ defined by the rule

$$f_x(?) = f(\{x\} \cup ?).$$

Apply the induction hypothesis in an appropriate way to f_x.]

Ref F. Ramsey, *Proc. London Math. Soc.* (2), **30**(1930), pp. 264-286.

One possible generalization of Ramsey's theorem might read: Let X be a set of cardinality $\aleph_1$; let $\{\mathcal{X}_1, \ldots, \mathcal{X}_m\}$ be a finite partition of $\langle X \rangle_n$ -- then there exists a subset S of X of cardinality $\aleph_1$ and an index i such that $\langle S \rangle_n \subset \mathcal{X}_i$. This statement is, however, false. In fact, even more can be said:

Theorem (Sierpiński) *Let* X *be a set of cardinality* $2^{\aleph_0}$ -- *then there exists a finite partition* $\{\mathfrak{X}_1,\ldots,\mathfrak{X}_m\}$ *of* $\langle X\rangle_n$ *with the following property: For every subset* S *of* X *of cardinality* $\aleph_1$, $\langle S\rangle_n \not\subset \mathfrak{X}_i$ $(i=1,\ldots,m)$.

[There is no loss of generality in taking $X = \mathbf{R}$. Furthermore, it can be supposed that $m=2$, $n=2$, the general case being a consequence of this one. Let $<$ be the usual ordering of $\mathbf{R}$; let $<_w$ be some well-ordering of $\mathbf{R}$ -- then we define a map $f:\langle \mathbf{R}\rangle_2 \to \{0,1\}$ by requiring that $f(\{x,y\}) = 0$ if $<$ and $<_w$ order the pair $\{x,y\}$ in the same way and $f(\{x,y\}) = 1$ if $<$ and $<_w$ order the pair $\{x,y\}$ in the opposite way. If now S were a subset of $\mathbf{R}$ of cardinality $\aleph_1$ such that either $f(\langle S\rangle_2) = 0$ or $f(\langle S\rangle_2) = 1$, then of necessity either the natural order or its inverse would well-order S, an impossibility. The partition of $\langle \mathbf{R}\rangle_2$ canonically associated with f thus has the desired properties.]

Ref W. Sierpiński, *Ann. Scoula Norm. Sup. Pisa Cl. Sci.* (2), **2**(1933), pp. 285-287.

[A useful survey on this interesting subject was given by P. Erdös and R. Rado, *Bull. Amer. Math. Soc.*, **62**(1956), pp. 427-489. See also P. Erdös, A. Hajnal, and R. Rado, *Acta Math. Acad. Sci. Hungar.*, **16**(1965), pp. 93-196; P. Erdös and A. Hajnal, *Proc. Symp. Pure Math.*, **13**(1971), pp. 17-48. For an account of recent developments (and additional references), cf. R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, Wiley, New York, 1980.]

III. DISJOINT AND NONDISJOINT CLASSES

Let $\mathcal{S}$ be an infinite class of sets -- then there necessarily exists an infinite subclass $\mathcal{S}_0$ of $\mathcal{S}$ such that either

$$\forall S', S'' \in \mathcal{S}_0 : S' \neq S'' \Longrightarrow S' \cap S'' = \emptyset$$

or

$$\forall S', S'' \in \mathcal{S}_0 : S' \neq S'' \Longrightarrow S' \cap S'' \neq \emptyset.$$

On the other hand, there exists an uncountable class $\mathcal{S}$ of sets such that $\mathcal{S}$ contains no uncountable subclass having one or the other of the preceding properties.

Ref W. Sierpiński, Fund. Math., 35(1948), pp. 165-174.

IV. PARTITIONS OF THE PLANE

The continuum hypothesis is equivalent to the statement that the plane can be partitioned into two sets X and Y, where $X(Y)$ intersects every line parallel to the $x(y)$-axis in a finite or countable set.

Ref W. Sierpiński, Bull. Acad. Sci. Cracovie, A(1919), pp. 1-3; W. Sierpiński, Fund. Math., 5(1924), p. 177-187.

The plane cannot be partitioned into two sets X and Y, where X intersects every line parallel to the x-axis in a finite set and Y intersects every line parallel to the y-axis in a finite or countable set.

Ref H. Tietze, Math. Ann., 88(1923), pp. 290-312.

[Note: By comparison, the continuum hypothesis is equivalent to the statement that space can be partitioned into three sets X, Y, and Z, where X(Y,Z) intersects every line parallel to the x(y,z)-axis in a finite set; see W. Sierpiński, Rend. Circ. Mat. Palermo (2), **1**(1952), pp. 7-10.]

The axiom of choice implies that the plane can be partitioned into two sets X and Y, where X(Y) intersects every line parallel to the x(y)-axis in a set of cardinality $<2^{\aleph_0}$.

Ref W. Sierpiński, Soc. Sci. Lett. Varsovie C.R. Cl. III Sci. Math. Phys., **25**(1932), pp. 9-12.

The continuum hypothesis is equivalent to the statement that there exist in the plane three straight lines L_1, L_2, L_3 with the property that the plane is the union of three sets S_1, S_2, S_3 such that S_i intersects every line parallel to L_i (i=1,2,3) in a finite set.

Ref F. Bagemihl, Z. Math. Logic Grundlag. Math., **7**(1961), pp. 77-79; R. Davies, Z. Math. Logic Grundlag. Math., **8**(1962), pp. 109-111.

The axiom of choice implies that the plane can be partitioned into countably many sets, none of which contains the vertices of an equilateral triangle.

Ref J. Ceder, Rev. Roumaine Math. Pures Appl., **14**(1969), pp. 1247-1251.

The continuum hypothesis implies that the plane can be partitioned into countably many sets, none of which containes the vertices of an isosceles triangle.

Ref R. Davies, Proc. Cambridge Philos. Soc., **72**(1972), pp. 179-183.

[Note: There is an extensive literature on these and related themes. For additional results, together with a variety of conjectures, see P. Erdös, Real Anal. Exchange, **4**(1978-79), pp. 113-138.]

§3. Semirings

Let X be a nonempty set; let $\mathcal{S}$ be a subset of $\mathcal{P}(X)$ containing the empty set -- then $\mathcal{S}$ is said to be a *semiring* (*σ-semiring*) if $\mathcal{S}$ is multiplicative and if for all nonempty $S, T \in \mathcal{S}$ with $S \supset T$, there exists a finite (finite or countable) $\mathcal{S}$-partition of S having T as a component. A *semialgebra* (*σ-semialgebra*) is a semiring (σ-semiring) containing X. It is clear that every semiring is a σ-semiring but the converse is not true. Conventionally, $\{\emptyset\}$ is both a semiring and a σ-semiring.

Examples (1) Take $X = \mathbf{R}$. Let $\mathcal{S}$ be the class consisting of all bounded, open intervals, and all singletons -- then $\mathcal{S}$ is a semiring.

(2) Take $X = \mathbf{R}$. Let $\mathcal{S}$ be the class consisting of all bounded, left closed and right open intervals, and all singletons -- then $\mathcal{S}$ is a σ-semiring but not a semiring.

Partition theory leads at once to the consideration of semirings (σ-semirings). Indeed, let $\mathcal{S}$ be a multiplicative class; let $S \in \mathcal{S}$ -- then the class consisting of the empty set and the elements of $\mathrm{Com}_{\mathcal{S}}(S)$ $(\sigma\text{-}\mathrm{Com}_{\mathcal{S}}(S))$ is a semiring (σ-semiring).

[Note: Tacitly, of course, $S \neq \emptyset$. Accordingly, $S \in \mathrm{Com}_{\mathcal{S}}(S)$, hence $\mathrm{Com}_{\mathcal{S}}(S)$ is not empty. On the other hand, $\sigma\text{-}\mathrm{Com}_{\mathcal{S}}(S)$ may very well be empty (cf. §2).]

Semirings (or σ-semirings) also arise naturally in the presence of certain chain conditions. Thus let $\mathcal{S}$ be a multiplicative class containing the empty set -- then $\mathcal{S}$ is said to satisfy the *finite (countable) chain condition* if for all $S, T \in \mathcal{S}$ with $S \supset T$, there exists a finite (countable) class $\{S_i\} \subset \mathcal{S}$ such that

$$T = S_1 \subset S_2 \subset \ldots \subset \bigcup_i S_i = S,$$

where $S_i - S_{i-1} \in \mathcal{S}$ for each $i > 1$. Here, of course, repetitions are allowed. Any multiplicative class containing the empty set for which the finite (countable) chain condition holds is evidently a semiring (σ-semiring).

Example Let $\mathcal{S} \subset \mathcal{P}(X)$ be a lattice -- then the class of all sets of the form $S - T$, where $S, T \in \mathcal{S}$ and $S \supset T$, is a semiring. Indeed, the condition as regards the empty set is trivial (take $S = T$). Let now $S_1 - T_1$ and $S_2 - T_2$ be in our class. Multiplicativity is then a consequence of the identity

$$(S_1 - T_1) \cap (S_2 - T_2) = (S_1 \cap S_2) - (S_1 \cap S_2) \cap (T_1 \cup T_2).$$

If in addition, $S_1 - T_1$ is contained in $S_2 - T_2$, then

$$S_1 - T_1 \subset (S_1 \cap S_2) - (T_1 \cap T_2) \subset S_2 - T_2,$$

from which it follows that the finite chain condition is in force, as can be seen by a direct set-theoretic calculation.

Lemma 1   *Let $\mathcal{S}$ be a semiring; let $S_1,\ldots,S_m$ be nonempty, pairwise disjoint elements of $\mathcal{S}$, contained in some fixed element S of $\mathcal{S}$ -- then there exists a finite $\mathcal{S}$-partition $P(S)$ of S of the form*

$$\{S_1,\ldots,S_m, S_{m+1},\ldots,S_n\}.$$

Proof   The proof is by induction on the integer m. If $m = 1$, then the assertion is true by the very definition of semiring. Assuming now its validity for $m \geq 1$, suppose that $\emptyset \neq T \subset S$ and intersects none of the $S_1,\ldots,S_m$ -- then

$$T = T\cap S_{m+1} \cup \ldots \cup T\cap S_n \qquad \text{(disjoint union)}.$$

In turn, making the obvious conventions, write

$$\left.\begin{array}{rl} S_{m+1} & = T\cap S_{m+1} \cup S_{m+1}(1) \cup \ldots \cup S_{m+1}(r_{m+1}) \\ & \vdots \\ S_n & = T\cap S_n \cup S_n(1) \cup \ldots \cup S_n(r_n). \end{array}\right. \qquad \text{(disjoint union)}$$

Then

$$\{S_1,\ldots,S_m,\ T,\ S_{m+i}(j)\}$$

is an $\mathcal{S}$-partition of S, thereby completing the proof. //

Lemma 2 Let $\mathcal{S}$ be a semiring; let $S_1,\ldots,S_m$ be nonempty, distinct elements of $\mathcal{S}$ -- then the union $S_1 \cup \ldots \cup S_m$ can be represented in the form

$$S_1(1) \cup \ldots \cup S_1(r_1) \cup \ldots \cup S_m(1) \cup \ldots \cup S_m(r_m),$$

where the $S_i(j)$ are nonempty, pairwise disjoint, belong to $\mathcal{S}$, and

$$S_i \supset S_i(1),\ldots,S_i(r_i) \qquad (i=1,\ldots,m).$$

Proof The proof is by induction on the integer m. As the assertion is trivially true when $m=1$, let us assume that it is valid for $m \geq 1$. Given S_{m+1}, consider the $S_{m+1} \cap S_i(j)$. If each of these intersections is empty, then our contention is evident. Suppose, therefore, that $S_{m+1} \cap S_i(j) \neq \emptyset$ for certain i and j -- then there are two possibilities:

$$\begin{cases} S_{m+1} \cap S_i(j) = S_{m+1} \\ S_{m+1} \cap S_i(j) \neq S_{m+1} \end{cases}.$$

If the first possibility obtains, then i and j are unique. Accordingly, in view of the definition of semiring, the difference $S_i(j) - S_{m+1}$, if not empty, can be written as a finite union of nonempty, pairwise disjoint elements of $\mathcal{S}$, leading, thereby, to the desired decomposition. If the second possibility obtains, then the $S_{m+1} \cap S_i(j)$ are proper, pairwise disjoint subsets of S_{m+1}. The proof can then be completed by an appeal to Lemma 1. //

We shall leave it up to the reader to decide if Lemmas 1 and 2 admit meaningful formulations in terms of σ-semirings, the issue being, of course, countable versus finite (cf. Exer. 5).

In passing, it should be noted that the trace of a semiring (σ-semiring) is again a semiring (σ-semiring).

Example Take for X a bounded, closed interval in $\mathbf{R}^n$, say:

$$X = \{x: a_1 \le x_1 \le b_1, \ldots, a_n \le x_n \le b_n\}.$$

Let $\mathcal{S}$ be the class consisting of the empty set and all intervals

$$\{x: \alpha_1 \le x_1 < \beta_1, \ldots, \alpha_n \le x_n < \beta_n\} \qquad (a_i \le \alpha_i < \beta_i \le b_i)$$

if $\beta_i < b_i$ for every i, but if $\beta_i = b_i$ for some i, then the inequality $x_i < \beta_i$ is to be replaced by $x_i \le \beta_i$. With this agreement, $\mathcal{S}$ is a semialgebra.

By comparison, note that the class of all closed subintervals of X, while multiplicative, is not a semiring, although the class of all finite unions of such is a lattice.

[Note: There are, of course, numerous simple variants on this theme.]

Notes and Remarks

The notion of a semiring is frequently attributed to J. v. Neumann, Functional Operators, Annals of Mathematics Studies, vol. 21, Princeton, 1950. This, however, is inaccurate, the priority belonging to A. Kolmogoroff, Math. Ann., **103**(1930), pp. 654-696. There one will find the term *zerlegbarer Bereich* employed in context for what we have called semiring or σ-semiring. Actually, v. Neumann (op. cit.) did not work with semirings per se but rather with multiplicative classes satisfying the finite chain condition; they were called by him *halfrings* (see p. 85 of that work). The term semiring appears in Halmos, Measure Theory, D. Van Nostrand, New York, 1950 (see p. 22), but still only in reference to the finite chain condition. Semirings were used early on by V. Glivenko [В. Гливенко] in his book The Stieltjes Integral [Интеграл Стилтьеса], ОНТИ, Москва-Ленинград, 1936 (see pp. 175-207). That semirings and σ-semirings might be made the basis for measure theory was suggested by N. d. Bruijn and A. Zaanen, Indag. Math., **16**(1954), pp. 456-466; their perspective is quite different from that of Kolmogoroff's (op. cit.), being didactic rather than innovative.

Exercises

(1) Give an example of a semiring of finite cardinality which does not satisfy the finite chain condition.

(2) Give an example of a semiring of infinite cardinality which satisfies neither the finite chain condition nor the countable chain condition.

(3) Give an example of a semiring of infinite cardinality which does not satisfy the finite chain condition but does satisfy the countable chain condition.

(4) Give an example of a σ-semiring which is not a semiring and which does not satisfy the countable chain condition.

(5) Take $X = [0,1[$ and consider the semiring $\mathcal{S}$ consisting of all left closed and right open subintervals of X -- then every $\mathcal{S}$-partition of X is finite or countable. Does there exist an $\mathcal{S}$-partition $P(X)$ of X such that each $[a,b[\subset X$ $(a<b)$ with rational endpoints is partitioned by the components of $P(X)$ lying therein?

[What is the relevance of this exercise to Lemmas 1 and 2?]

(6) Let X be a set of finite cardinality n, say. In terms of n, how many semirings does $\mathcal{P}(X)$ contain?

(7) By definition, a nonempty, bounded subset of $\mathbb{R}^n$ is called a *convex polyhedron* provided that it can be written as a finite intersection of open or closed halfspaces. Show that the class consisting of the empty set and all convex polyhedra is a semiring satisfying the finite chain condition.

(8) Take for X the Banach space (c_0) of all real sequences $x = \{x_i\}$ which converge to zero, the norm being given by $\|x\| = \sup|x_i|$. Let $\{r_i(+)\}$ be a sequence in $\bar{\mathbb{R}}$ such that $0 < r_i(+) \leq +\infty$, $\underline{\lim}\, r_i(+) > 0$; let $\{r_i(-)\}$ be a sequence in $\bar{\mathbb{R}}$ such that $0 > r_i(-) \geq -\infty$, $\overline{\lim}\, r_i(-) < 0$ -- then by $S(\{r_i(-)\}, \{r_i(+)\})$ we understand the set of all $x \in X$ such that $r_i(-) \leq x_i < r_i(+)$ $\forall i$. Explain why the class consisting of the empty set and all possible $S(\{r_i(-)\}, \{r_i(+)\})$ is neither a semiring nor a σ-semiring.

[Note: It was claimed to the contrary by P. Maserick in Pacific J. Math., **17**(1966), pp. 137-148, that the class in question was a σ-semiring satisfying the countable chain condition.]

Problem

NORMAL CLASSES

Let $\mathcal{S}$ be a multiplicative class -- then $\mathcal{S}$ is said to be *normal* if for any $S \in \mathcal{S}$ admitting infinite $\mathcal{S}$-partitions, each element $P(S) = \{S_1, \ldots, S_m, \ldots\}$ in $\sigma\text{-Par}_{\mathcal{S}}(S)$ has the property that $\forall m$, there exists a finite $\mathcal{S}$-partition

$$\{S_1, \ldots, S_m, T_1, \ldots, T_{r_m}\}$$

of S. Every semiring is a normal class (cf. Lemma 1).

(1) There exist multiplicative classes which are not normal.

[Take for X a countable set $\{x_1, x_2, \ldots\}$. Put $\mathcal{S} = \{\emptyset, X, \{x_1\}, \{x_2\}, \ldots\}$ -- then $\mathcal{S}$ is multiplicative but not normal.]

(2) There exist σ-semirings which are not normal classes.

[Let X be an infinite set. Let $P(X) = \{X_1, \ldots, X_m\}$ be a finite partition of X by subsets X_i, each of which we suppose in turn can be countably partitioned by subsets of X_{ij} -- then the class $\mathcal{S}$ consisting of $\emptyset$, X, the X_i and the X_{ij} is a σ-semiring but is not normal.]

(3) There exist normal classes which are not σ-semirings.

[Take for X a countable set $\{x_1, x_2, \ldots\}$. Put $\mathcal{S} = \{\emptyset, X, \{x_2\}, \{x_3\}, \ldots, \{x_i, x_{i+1}, \ldots\} (i=2,3,\ldots)\}$ -- then $\mathcal{S}$ is normal but is not a σ-semiring.]

Let $\mathcal{S}$ be a multiplicative class -- then $\mathcal{S}$ is normal iff for any $S \in \mathcal{S}$ admitting infinite $\mathcal{S}$-partitions and for any $P(S) \in \text{Par}_{\mathcal{S}}(S)$, each element $\{S_1, \ldots, S_m, \ldots\}$ in $\sigma\text{-Par}_{\mathcal{S}}(S)$ which refines $P(S)$ has the property that $\forall m$, there exists a finite $\mathcal{S}$-partition

$$\{S_1, \ldots, S_m, T_1, \ldots, T_{r_m}\}$$

refining $P(S)$.

Ref D. Procenko [Д. Проценко], Soobšč. Akad. Nauk Gruzin. SSR, 40(1965), pp. 271-278.

§4. Rings, σ-Rings, δ-Rings

Let X be a nonempty set; let $\mathcal{S}$ be a subset of $\mathcal{P}(X)$ containing the empty set -- then $\mathcal{S}$ is said to be a *ring* if

$$S,T\in\mathcal{S} \Longrightarrow S\Delta T\in\mathcal{S} \text{ and } S\cap T\in\mathcal{S}.$$

Since

$$\begin{cases} S\cup T = (S\Delta T)\,\Delta\,(S\cap T) \\ S-T = S\,\Delta\,(S\cap T), \end{cases}$$

a ring is closed under the formation of finite unions and differences and, in fact, is characterized by these requirements. An *algebra* is a ring containing X. Trivially, $\{\emptyset\}$ is a ring while $\{\emptyset,X\}$ and $\mathcal{P}(X)$ are algebras.

Example (Kolmogoroff) Any ring is a semiring. We have seen in §3 that every lattice gives rise in a natural manner to a semiring; in turn, every semiring gives rise in a natural manner to a ring. Thus let $\mathcal{S}$ be a semiring and consider the class $\text{Kol}(\mathcal{S})$ of all sets of the form $\coprod_{i=1}^{m} S_i$, the S_i being elements of $\mathcal{S}$ which, without loss of generality, can be taken pairwise disjoint (cf. Lemma 2 (§3)) -- then we claim that $\text{Kol}(\mathcal{S})$ is a ring. Indeed, if $S = \coprod_{i=1}^{m} S_i$, $T = \coprod_{j=1}^{n} T_j$ are disjoint unions of elements $S_i,T_j\in\mathcal{S}$, then so is

$$S\cap T = \coprod_{i=1}^{m}\ \coprod_{j=1}^{n} (S_i\cap T_j).$$

As for $S\Delta T$, use Lemma 1 (§3) to write

$$\begin{cases} S_i = \bigsqcup_{j=1}^{n} (S_i \cap T_j) \cup \bigsqcup_{k=1}^{r_i} S_{ik} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{(disjoint union)} \\ T_j = \bigsqcup_{i=1}^{m} (S_i \cap T_j) \cup \bigsqcup_{k=1}^{r_j} T_{jk} . \end{cases}$$

Then we have

$$S\Delta T = \bigsqcup_{i=1}^{m} (\bigsqcup_{k=1}^{r_i} S_{ik}) \cup \bigsqcup_{j=1}^{n} (\bigsqcup_{k=1}^{r_j} T_{jk}),$$

which again is a disjoint union of elements in $\mathcal{S}$. Accordingly, the class $\mathrm{Kn1}(\mathcal{S})$ is a ring.

[Note: Suppose that $P(X) = \{X_i : i \in I\}$ is a partition of X -- then the class consisting of $\emptyset$ and the X_i is a semiring. Therefore the class formed by the empty set and all nonempty finite unions of the components of $P(X)$ is a ring.]

The justification of the term "ring of sets" lies in the following remarks. In $\mathcal{P}(X)$ itself, introduce operations of addition and multiplication via the stipulations

$$\begin{cases} S + T \equiv S \Delta T \\ S \cdot T \equiv S \cap T. \end{cases}$$

Then by an elementary if slightly tedious verification, one checks that $\mathcal{P}(X)$ thus equipped is a commutative ring with zero element

$\emptyset$ and multiplicative identity X. It is a point of some importance that these operations, when viewed as maps

$$\mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X),$$

are jointly continuous, i.e., $\mathcal{P}(X)$ is a topological ring; on the other hand, these operations, when viewed as maps

$$\mathcal{P}(X)_s \times \mathcal{P}(X)_s \to \mathcal{P}(X)_s,$$

are separately continuous.

Utilizing now the customary algebraic terminology, a subring of $\mathcal{P}(X)$ is a subset containing the zero element, i.e., $\emptyset$, and closed under addition and multiplication or still, under symmetric differences and intersections; in other words, subring of $\mathcal{P}(X)$ = ring of subsets of X. In addition, a subalgebra of $\mathcal{P}(X)$ is a subring containing the multiplicative identity, i.e., X; in other words: subalgebra of $\mathcal{P}(X)$ = algebra of subsets of X.

[Note: A ring (algebra) of sets is evidently a Boolean ring (algebra). It must be stressed, however, that a ring $\mathcal{S}$ may well admit a multiplicative identity, thus is a Boolean algebra, but is not an algebra, the point being that generally $X \notin \mathcal{S}$. Consider, e.g., $\mathcal{S} = \mathcal{P}(S)$, S a nonempty proper subset of X. Accordingly, we shall use the term *ring with unit* to refer to a ring $\mathcal{S}$ possessing a multiplicative identity; in particular, therefore, every algebra is a ring with unit. It is easy to check that a ring $\mathcal{S}$ is a ring with unit iff $\cup\mathcal{S} \in \mathcal{S}$. If $\cup\mathcal{S} \notin \mathcal{S}$, then the class $\hat{\mathcal{S}}$ consisting of all S, $\cup\mathcal{S} - S$ $(S \in \mathcal{S})$ is a ring with unit containing $\mathcal{S}$. Finally, it should be recalled that every Boolean ring is of characteristic 2, hence may be regarded as an algebra over the field $\mathbb{Z}_2$.]

The usual algebraic notions then admit easy descriptive interpretations. Consider, e.g., the notion of an ideal $\mathcal{I}$ in the ring $\mathcal{S}$ -- then, descriptively, $\mathcal{I}$ can be characterized as a nonempty subclass of $\mathcal{S}$ which is closed under the formation of finite unions and is hereditary in the sense that $I \in \mathcal{I}$, $S \in \mathcal{S}$, $S \subset I \implies S \in \mathcal{I}$. The corresponding quotient $\mathcal{S}/\mathcal{I}$ is a Boolean ring, elements $S, T \in \mathcal{S}$ being equivalent mod $\mathcal{I}$ iff $S \Delta T \in \mathcal{I}$ or still, iff $S = (T - I) \cup J$ $(I, J \in \mathcal{I})$.

Lemma 1 *Let $\mathcal{S}$ be a ring; let $\mathcal{I} \neq \mathcal{S}$ be an ideal -- then:*

(1) *$\mathcal{I}$ is contained in a maximal ideal;*

(2) *$\mathcal{I}$ is maximal iff $\mathcal{I}$ is prime;*

(3) *$\mathcal{I}$ is the intersection $\bigcap_{\mathfrak{p} \supset \mathcal{I}} \mathfrak{p}$, $\mathfrak{p}$ prime.*

[There is nothing to be gained by giving the proof in extenso. The point is this. $\mathcal{S}$ need not have a multiplicative identity and, as is well-known, if a ring does not have a multiplicative identity, then, e.g., generic ideals need not be contained in maximal ideals, maximal ideals need not be prime, prime ideals need not be maximal, etc. But $\mathcal{S}$ is a Boolean ring, hence carries compensating structure. To illustrate, consider (1). Since $\mathcal{I} \neq \mathcal{S}$, $\exists S_0 \in \mathcal{S}$, $S_0 \notin \mathcal{I}$. Let $\mathfrak{m}$ be any ideal in $\mathcal{S}$ maximal with respect to the property that $\mathfrak{m} \supset \mathcal{I}$, $S_0 \notin \mathfrak{m}$ (Zorn's lemma ensures the existence of $\mathfrak{m}$) -- then $\mathfrak{m}$ is in fact a maximal ideal, as can be checked without difficulty ($S_0^2 = S_0$!). Statement (2) is also easy, as is (3).]

A *σ-ring* is a ring $\mathcal{S}$ which is closed under the formation of countable unions, i.e.,

$$\{S_i : i=1,2,\ldots\} \subset \mathcal{S} \Longrightarrow \cup S_i \in \mathcal{S},$$

or still, $\mathcal{S} = \mathcal{S}_\sigma$. A *σ-algebra* is a σ-ring containing X. A *δ-ring* is a ring $\mathcal{S}$ which is closed under the formation of countable intersections, i.e.,

$$\{S_i : i=1,2,\ldots\} \subset \mathcal{S} \Longrightarrow \cap S_i \in \mathcal{S},$$

or still, $\mathcal{S} = \mathcal{S}_\delta$. A *δ-algebra* is a δ-ring containing X.

A *σ-ideal* (*δ-ideal*) is an ideal in a ring which is closed under the formation of countable unions (intersections).

Example Let X be a topological space-then the class $\mathcal{S}$ of all subsets of X having the Baire property is a σ-algebra containing the σ-ideal of all first category subsets of X.

[Note: Recall that a set $S \subset X$ is said to have the *Baire property* if there exists an open set G such that $S - G$ and $G - S$ are of the first category.]

A σ-ring is a δ-ring. To see this, put $S = \cup S_i \, (S_i \in \mathcal{S})$ -- then

$$\cap S_i = S - \cup(S - S_i).$$

Consequently, if $\{S_i\}$ is a sequence of sets in a σ-ring $\mathcal{S}$, then

$$\overline{\lim}\, S_i \in \mathcal{S},\ \underline{\lim}\, S_i \in \mathcal{S}.$$

In particular: A σ-ring is necessarily closed in $\mathcal{P}(X)_s$. Furthermore, due to the separate continuity of the operations

$$\mathcal{P}(X)_s \times \mathcal{P}(X)_s \rightarrow \mathcal{P}(X)_s,$$

the closure in $\mathcal{P}(X)_s$ of a ring is again a ring, thus is actually a σ-ring.

Example There are δ-rings which are not σ-rings. For instance, take $X = \mathbf{R}^n$ and consider the class $\mathcal{S}$ of all relatively compact subsets.

Lemma 2 *Let $\mathcal{S}$ be a ring -- then $\mathcal{S}$ is a δ-ring iff for every $S_0 \in \mathcal{S}$, the set $\{S \in \mathcal{S} : S \subset S_0\}$ is a σ-algebra in S_0.*

[We omit the elementary verification.]

It follows from Lemma 2 that every δ-ring which admits a multiplicative identity is necessarily a σ-ring.

A ring $\mathcal{S}$ is said to be *complete* if $\mathcal{S}$ is closed under the formation of arbitrary nonempty unions. A complete ring is evidently also closed under the formation of arbitrary nonempty intersections. If $\mathcal{S}$ is complete, then $\mathcal{S}$ is a ring with unit $\cup\mathcal{S}$; of course, $\cup\mathcal{S} \neq X$ in general, hence $\mathcal{S}$ need not be an algebra.

Example Let X be a set of cardinality $\aleph_0$; let $\mathcal{S}$ be a σ-ring in X -- then $\mathcal{S}$ is complete.

Consider $\mathcal{P}(X)$, equipped with the topology of pointwise convergence -- then a net $\{S_i\}$ in $\mathcal{P}(X)$ is convergent with limit S, say, iff it is order convergent, i.e.,

$$\bigcap_i \bigcup_{j \geq i} S_j = \bigcup_i \bigcap_{j \geq i} S_j ,$$

the order limit being exactly S.

This being so, suppose that $\mathcal{S}$ is a complete ring in X -- then $\mathcal{S}$ is closed in $\mathcal{P}(X)$. If $\mathcal{S}$ is a ring but is not complete, then the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$ in $\mathcal{P}(X)$ is a complete ring in X, the completion of $\mathcal{S}$. Every complete subring of $\mathcal{P}(X)$ containing $\mathcal{S}$ must contain $\bar{\mathcal{S}}$, therefore the completion of $\mathcal{S}$ is the minimal complete ring in X containing $\mathcal{S}$ or still, the complete ring generated by $\mathcal{S}$ (cf. §6).

Example Let $\mathcal{S}$ be a ring in X. Suppose that $\forall x \in X$, $\{x\} \in \mathcal{S}$ -- then the completion of $\mathcal{S}$ is $\mathcal{P}(X)$.

Let $\mathcal{S}$ be a ring -- then a nonempty set $A \in \mathcal{S}$ is said to be an *atom* if, apart from the empty set, A properly contains no other element of $\mathcal{S}$. We write $At(\mathcal{S})$ for the class of all atoms in $\mathcal{S}$.

If every nonempty $S \in \mathcal{S}$ contains an atom, then $\mathcal{S}$ is said to be *atomic*; on the other hand, if no nonempty $S \in \mathcal{S}$ contains an atom, then $\mathcal{S}$ is said to be *antiatomic*.

Example Let X be a Hausdorff topological space, X_{isol} its set of isolated points -- then X can be written as a disjoint union $X = X_{perf} \cup X_{scat}$, where X_{perf} is the perfect kernel of X, i.e., the union of all subsets of X which are dense in themselves, and $X_{scat} \supset X_{isol}$ is the corresponding complement. X_{perf} is closed while X_{scat} is open; one of them may, of course, be empty. Assume now that X is in addition, locally compact and totally disconnected. Consider the ring $\mathcal{S}$ of all open and compact subsets of X -- then $At(\mathcal{S}) = \{\{x\}: x \in X_{isol}\}$, so

$$\begin{cases} \mathcal{S} \text{ is atomic iff } X = \bar{X}_{isol} \\ \mathcal{S} \text{ is antiatomic iff } X = X_{perf}. \end{cases}$$

In this connection, note that $X = X_{perf}$ iff $X_{scat} = \emptyset$ but $X = \bar{X}_{isol}$ does not imply that $X_{perf} = \emptyset$, as can be seen by example. It is also easy to envision intermediate situations, a particularly transparent case being when X is extremally disconnected.

Any complete ring $\mathcal{S}$ is atomic, there being an easy characterization of the atoms. Thus define an equivalence relation in $\cup\mathcal{S}$ by requiring that x be equivalent to y iff every set in $\mathcal{S}$ which contains x also contains y. The equivalence class $[x]$ $(x \in \cup\mathcal{S})$ belongs to $\mathcal{S}$, as can be seen by noting that

$$[x] = \bigcap_{x \in S} S \qquad (S \in \mathcal{S}).$$

The atoms of $\mathcal{S}$ are just the $[x]$ $(x \in \cup\mathcal{S})$. Every nonempty $S \in \mathcal{S}$ is partitioned by the atoms which it contains.

Let now $\mathcal{S}$ be an arbitrary ring in X -- then there is a canonical map

$$\phi : \mathcal{S} \to \mathcal{P}(\mathrm{At}(\mathcal{S})),$$

namely the rule which assigns to each $S \in \mathcal{S}$ the class $\phi(S)$ of all atoms $A \subset S$. It is clear that ϕ is a homomorphism of rings. Furthermore:

(1) If $\mathcal{S}$ is atomic, then ϕ is injective. Indeed, if $S, T \in \mathcal{S}$, $S \neq T$, then $S - T \neq \emptyset$, say, thus $\exists A \in \mathrm{At}(\mathcal{S})$, $A \subset S - T$, and so $A \in \phi(S)$, $A \notin \phi(T)$.

(2) If $\mathcal{S}$ is complete, then ϕ is surjective. Indeed, if $\{A_i\}$ is any class of atoms, then $\cup A_i \in \mathcal{S}$ and $\phi(\cup A_i) = \{A_i\}$.

We have seen above that every complete ring is atomic. Therefore, in this case, ϕ is an isomorphism of rings. We remark that ϕ is then even a complete isomorphism in that it preserves arbitrary unions and intersections.

In passing, it should be noted that the trace of a ring (σ-ring, δ-ring) is again a ring (σ-ring, δ-ring), the same also being true of complete rings.

Notes and Remarks

The theory presented in this § can be approached more generally, viz. from the point of view of abstract Boolean rings and Boolean algebras; cf. R. Sikorski, Boolean Algebras, Springer-Verlag, Berlin, 1969, as well as D. Ponasse and J-C. Carrega, Algèbre et Topologie Booléennes, Masson, Paris, 1979. The terminology, particularly in the older literature, is tangled. Specifically, what we have termed a lattice is frequently called a ring while what we have termed a ring is frequently called a field; cf. F. Hausdorff, Grundzüge der Mengenlehre, Veit & Comp., Leipzig, 1914 (see pp. 14-16), the German being **Ring** and **Körper**, respectively. To compound the confusion, M. Fréchet, Bull. Soc. Math. France, **43**(1915), pp. 248-265, refers to a σ-ring as a **famille additive d'ensembles**, whereas O. Nikodym, Fund. Math., **15**(1930), pp. 131-179, understands by *corps d'ensembles* a σ-algebra. There are other permutations and combinations too; e.g., R. de Possel, J. Math. Pures Appl. (9), **15**(1936), pp. 391-409, has suggested the term *tribe* (*tribu* in French) for σ-ring, a *clan* then being a ring. In the sense employed in the text, the term ring appears in J. v. Neumann, Functional Operators, Annals of Mathematics Studies, vol. 21, Princeton, 1950 (see p. 84). That semirings lead naturally to rings was pointed out by A. Kolmogoroff, Math. Ann., **103**(1930), pp. 654-696. Ideals in rings have been investigated systematically by A. Tarski, Fund. Math., **32**(1939), pp. 45-63, Fund. Math., **33**(1945), pp. 51-65, and Soc. Sci. Lett. Varsovie C.R. Cl. III Sci. Math. Phys., **30**(1937), pp. 151-181. The notion of atom is generally attributed to M. Fréchet, Fund. Math., **5**(1924), pp. 206-251, although it can be traced back to E. Schröder, Vorlesungen über die Algebra der Logik, II (Bd. 1), B.G. Teubner, Leipzig, 1891 (see §47). The fact that every complete ring is isomorphic to the power set of its atoms is due to Lindenbaum and Tarski; cf. A. Tarski, Fund. Math., **24**(1935), pp. 177-198.

Exercises

(1) Take $X = \underset{\sim}{R}$. For $n=0,1,\ldots$, let $\mathcal{D}_n$ be the class consisting of the empty set and all nonempty finite disjoint unions of dyadic left closed and right open intervals of order n, i.e., the $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[$. Verify that $\mathcal{D}_n$ is a ring. Noting that $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots$, put $\mathcal{D} = \cup \mathcal{D}_n$, the class of all finite unions of dyadic left closed and right open intervals of any order. Verify that $\mathcal{D}$ is a ring. Formulate and prove a multidimensional generalization.

[Observe that

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[= \left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right[\cup \left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right[.]$$

(2) Let X be a topological space -- then the class $\mathcal{S}$ comprised of all sets $S \subset X$ whose boundary is nowhere dense is an algebra of subsets of X.

(3) Let X be a nonempty set -- then the class $\mathcal{S}$ comprised of all sets $S \subset X$ such that either $\text{card}(S) < \aleph_\alpha$ or $\text{card}(X-S) < \aleph_\alpha$ is an algebra of subsets of X.

(4) Given a ring $\mathcal{S}$, consider the following conditions:

(C_1) Every subset of $\mathcal{S}$ consisting of nonempty, pairwise disjoint elements is finite or countable.

(C_2) Every subset of $\mathcal{S}$ consisting of nonempty, pairwise comparable elements is finite or countable.

Show by example that there exist infinite rings which satisfy (C_1) but not (C_2) and vice-versa.

(5) Let $\mathcal{S}$ be a subset of $\mathbf{P}(X)$ containing the empty set. Suppose that $\mathcal{S}$ is multiplicative -- then the following are equivalent:

(i) The class $\mathcal{S}$ is a semiring;

(ii) The class consisting of all sets expressible as a finite union of pairwise disjoint sets from $\mathcal{S}$ is a ring;

(iii) Given elements $S_1,\ldots,S_m$ of $\mathcal{S}$, there exist pairwise disjoint elements $T_1,\ldots,T_n$ of $\mathcal{S}$ such that each S_i is a union of certain of the T_j.

(6) Let $\mathcal{S}$ be a semiring. Consider the class of all sets of the form $\bigcup_{i=1}^{\infty} S_i$, the S_i being elements of $\mathcal{S}$ which, without loss of generality, can be taken pairwise disjoint (cf. Lemma 2(§3)). Show by example that this class need not be a ring.

(7) True or False? $\mathbf{P}(X)_s$ is a topological ring, i.e., the operations of addition and multiplication

$$\mathbf{P}(X)_s \times \mathbf{P}(X)_s \to \mathbf{P}(X)_s$$

are jointly continuous.

[Is Exer. 10(§1) revelant here?]

(8) Let $\mathcal{S}$ be a ring -- then the following are equivalent:

(i) $\mathcal{S}$ admits a nonprincipal prime ideal;

(ii) $\mathcal{S}$ admits a nonprincipal ideal;

(iii) $\mathcal{S}$ is infinite.

[If (iii) is in force, then $\mathcal{S}$ must possess countably many nonempty, pairwise disjoint elements (cf. Exer. 5(§2)).]

(9) In a ring with unit, there is a natural one-to-one correspondence between ideals and filters, the two concepts being dual to one another; under this correspondence, prime ideals are matched with ultrafilters.

[Let $\mathcal{S}$ be a ring with unit $U\mathcal{S}$ -- then the correspondence in question is simply complementation relative to $U\mathcal{S}$.]

(10) Let X be a locally compact, totally disconnected, Hausdorff space; let $\mathcal{S}$ be the ring of open and compact subsets of X. Is $\mathcal{S}$ a σ-ring?

(11) Let X and Y be nonempty sets; let $f:X \to Y$ be a map -- then there is an induced map $f^{-1}:\mathcal{P}(Y) \to \mathcal{P}(X)$. Show that if $\mathcal{T}$ is a ring (σ-ring) in Y, then $\{f^{-1}(T):T\in\mathcal{T}\}$ is a ring (σ-ring) in X, and if $\mathcal{S}$ is a ring (σ-ring) in X, then $\{T \subset Y:f^{-1}(T)\in\mathcal{S}\}$ is a ring (σ-ring) in Y. Are these assertions true if ring (σ-ring) is replaced by algebra (σ-algebra)?

(12) Let $\mathcal{S}$ be a σ-ring in X not containing X -- then the classes

$$\begin{cases}\{A \subset X:A\in\mathcal{S} \text{ or } X - A \in \mathcal{S}\}\\ \{A \subset X:S\in\mathcal{S} \Longrightarrow A \cap S\in \mathcal{S}\}\end{cases}$$

are σ-algebras in X containing $\mathcal{S}$, the latter containing the former.

(13) Prove that there does not exist an infinite σ-algebra $\mathcal{S}$ with countably many members. Can σ-algebra be replaced by σ-ring in this assertion?

[Bear in mind Exer. 5(§2).]

(14) Let $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots$ be a strictly increasing chain of subsets of $\mathcal{P}(X)$. Show that if the $\mathcal{S}_i$ are algebras in X, then the union $\cup\mathcal{S}_i$ is again an algebra in X but if the $\mathcal{S}_i$ are σ-algebras in X, then the union $\cup\mathcal{S}_i$ is never a σ-algebra in X. What happens if, instead, the $\mathcal{S}_i$ are rings?

[To discuss the second assertion, first show that there exists a sequence $\{S_i\}$ of nonempty, pairwise disjoint sets $S_i : S_i \in \mathcal{S}_{i+1} - \mathcal{S}_i \ \forall i$ (change the indexing if necessary). This done, proceed by contradiction and assume that $\cup\mathcal{S}_i$ is a σ-algebra -- then eventually the

$$\mathbf{N}_i = \{S \subset \mathbf{N} : \bigcup_{j \in S} S_j \in \mathcal{S}_i\}$$

are σ-algebras in $\mathbf{N}$.]

(15) True or False? Let $\aleph$ be an infinite cardinal. Let X be a set of cardinality $\aleph$; let $\mathcal{S}$ be a ring in X which is closed under the formation of unions of cardinality $\leq\aleph$ -- then $\mathcal{S}$ is complete.

(16) Let $P(X) = \{X_i : i \in I\}$ be a partition of X -- then the class consisting of the empty set and all possible nonempty unions of the X_i is a complete algebra. Conversely, let $\mathcal{S}$ be a complete algebra -- then there exists a partition $P(X) = \{X_i : i \in I\}$ of X such that the class consisting of the empty set and all possible nonempty unions of the X_i is $\mathcal{S}$.

[Note: The correspondence between partitions and complete algebras is evidently one-to-one.]

(17) A ring $\mathcal{S}$ such that it and all its subrings are atomic is called *superatomic*.

True or False? There exist infinite superatomic rings.

(18) Let $\mathcal{S}$ be a ring -- then the following are equivalent:

(i) There exist a prime ideal containing $At(\mathcal{S})$;

(ii) There exists a proper ideal containing $At(\mathcal{S})$;

(iii) There exists an infinite class $\{S_i\} \subset \mathcal{S}$ of nonempty, pairwise disjoint sets S_i and a set $S \in \mathcal{S}$ such that $\cup S_i \subset S$.

[What, if any, is the connection between the three conditions figuring here and the three which appear in Exer. 8?]

(19) Let $\mathcal{S}$ be a ring; let $\aleph$ be an infinite cardinal -- then the following are equivalent:

(i) $\mathcal{S}$ is complete and the cardinality of $At(\mathcal{S})$ is $\aleph$;

(ii) $\mathcal{S}$ is closed under the formation of unions of cardinality $\leq\aleph$ and $\aleph$ is the largest cardinal for which there exists a class $\mathcal{X} \subset \mathcal{S}$ of cardinality $\aleph$ comprised of nonempty, pairwise disjoint sets.

(20) Let $\mathcal{S}$ be a ring -- then the following are equivalent:

(i) $\mathcal{S}$ is complete and $At(\mathcal{S})$ is countable;

(ii) $\mathcal{S}$ is an infinite σ-ring with the property that every class $\mathcal{X} \subset \mathcal{S}$ of nonempty, pairwise disjoint sets is finite or countable.

[What additional fact must be cited in order to make this exercise a corollary to the preceding exercise?]

Taking into account Exer. 16, explicate the significance of this result for the collection of countable partitions of X.

(21) Construct an example of an atomic ring $\mathcal{S}$ possessing elements which cannot be written as a union of atoms.

(22) True or False?

(a) If $\mathcal{S}$ is an antiatomic ring, then every nonempty $S \in \underline{\mathcal{S}}$ contains $\aleph_0$ nonempty, pairwise disjoint sets $S_i \in \underline{\mathcal{S}}$.

(b) If $\mathcal{S}$ is an antiatomic σ-ring, then every nonempty $S \in \underline{\mathcal{S}}$ contains $\aleph_1$ nonempty, pairwise disjoint sets $S_i \in \underline{\mathcal{S}}$.

(23) Let L_X stand for the collection of all σ-algebras on X. Given $\mathcal{S}', \mathcal{S}'' \in L_X$, write $\mathcal{S}' \leq \mathcal{S}''$ if $\mathcal{S}' \subset \mathcal{S}''$ - then, with this definition of order, L_X is a complete lattice with largest and smallest elements. However, in general, L_X is neither distributive nor modular. If $\text{card}(X) \leq \aleph_0$, then L_X is isomorphic to the partition lattice on X (cf. Exer. 3 (§2)), thus is complemented but, as can be shown, this fails if $\text{card}(X) > \aleph_0$.

[Note: It is necessary to admit here the notion of generated σ-algebra (see §6). For the details (and additional information), see K. Bhaskara Rao and B. Rao, Dissertationes Math., **190** (1981), pp. 1-68.]

(24) Every abstract lattice is isomorphic to a sublattice of L_X for some X.

[Combine the theorem of Whitman (Exer. 3 (§2)) with Exer. 16.]

Problem

TOPOLOGICAL REPRESENTATION OF BOOLEAN RINGS

Rings of sets and their quotients are the Boolean rings of primary importance in analysis. To deal with both simultaneously, it is most economical to consider an arbitrary Boolean ring. Such rings were studied intensively by Stone in the 1930's. The foundational results of this theory, a sketch of which will be given below, can be regarded as but simple exercises in the modern theory of schemes. Accordingly, the reader who is familiar with the language of contemporary algebraic geometry should have no difficulty in filling in the omitted details.

By a *Boolean space*, we shall understand a topological space X whose topology $\mathfrak{T}$ is locally compact, totally disconnected, and Hausdorff. Open subsets of a Boolean space are Boolean spaces, as are the closed subsets. Associated with every Boolean space X is a ring $\mathbb{A}(X)$, viz. the ring of open and compact subsets of X. The prime ideals in $\mathbb{A}(X)$ are parameterized by the points $x \in X : \mathfrak{p}_x$ = elements of $\mathbb{A}(X)$ not containing x.

[Note: Owing to the Urysohn metrization theorem, a compact Boolean space is metrizable iff the cardinality of $\mathbb{A}(X)$ is $\leq \aleph_0$.]

Let $\mathbb{A}$ be a Boolean ring -- then attached to $\mathbb{A}$ is the set Spec($\mathbb{A}$) of all prime (= maximal) ideals of $\mathbb{A}$. Given $f \in \mathbb{A}$, put

$$\mathbb{A}_f = \{\mathfrak{p} \in \text{Spec}(\mathbb{A}) : f \notin \mathfrak{p}\}.$$

Then the map

$$\mathbb{A} \to \mathbb{P}(\text{Spec}(\mathbb{A}))$$

which assigns to each f in $\mathbf{A}$ the set $\mathbf{A}_f$ in $\text{Spec}(\mathbf{A})$ is an injective homomorphism of rings. The range of this map is a multiplicative class, hence is a base for a topology on $\text{Spec}(\mathbf{A}) = \cup \mathbf{A}_f$, the so-called *spectral topology*. In the spectral topology, $\text{Spec}(\mathbf{A})$ is a locally compact, totally disconnected, Hausdorff space, i.e., is a Boolean space, the $\mathbf{A}_f$ then being the ring of open and compact subsets of $\text{Spec}(\mathbf{A})$. Because $\text{Spec}(\mathbf{A})$ is compact iff $\mathbf{A}$ admits a multiplicative identity, in the noncompact case, compactifying $\text{Spec}(\mathbf{A})$ by the Alexandroff procedure is equivalent to formally passing from $\mathbf{A}$ to the Boolean ring $\hat{\mathbf{A}}$ obtained by adjunction of a unit. If $\mathbf{A}$ is infinite, then the weight of $\text{Spec}(\mathbf{A})$ is the cardinality of $\mathbf{A}$; if $\mathbf{A}$ is finite, then $\mathbf{A}$ has 2^n elements and therefore $\text{Spec}(\mathbf{A})$ is a discrete space with n elements.

The set $\text{Spec}(\mathbf{A})$, equipped with the spectral topology, is called the *Stone space* of $\mathbf{A}$. We shall denote it by the symbol $\text{ST}(\mathbf{A})$. Evidently, the Stone spaces of isomorphic Boolean rings are homeomorphic and conversely.

[Note: In reality, $\text{ST}(\mathbf{A})$ comes supplied with a sheaf of rings. However, this additional structure, while fundamental from the scheme-theoretic point of view, plays no explicit role in the present considerations, the ring $\mathcal{SC}(\mathbf{A})$ of open and compact subsets of $\text{ST}(\mathbf{A})$ being its replacement.]

If X is a Boolean space, then the Stone space of $\mathbf{A}(X)$ can be identified with X.

Examples (1) Let X be an infinite set, equipped with the discrete topology. Let $\mathcal{S}$ be the algebra consisting of the finite and cofinite subsets of X. Fix a point ∞ which is not in X -- then the map $\phi : \mathcal{S} \rightarrow \mathbf{P}(X \cup \{\infty\})$ defined by the rule

$$\begin{cases} \phi(S) = S & \text{if } S \text{ is finite} \\ \phi(S) = S \cup \{\infty\} & \text{if } S \text{ is infinite} \end{cases}$$

sets up an isomorphism between $\mathcal{S}$ and an algebra $\mathcal{S}_\infty$ of subsets of $X \cup \{\infty\}$. Topologize $X \cup \{\infty\}$ by taking the class $\mathcal{S}$ as a basis -- then $X \cup \{\infty\}$ can be viewed as the Stone space of $\mathcal{S}$ or still, the Stone space of $\mathcal{S}$ is the Alexandroff compactification of X.

(2) Let X be a set, equipped with the discrete topology -- then the Stone space of $\mathcal{P}(X)$ has cardinality

$$\begin{cases} 2^{2^{\text{card}(X)}} & \text{if } X \text{ is infinite} \\ \text{card}(X) & \text{if } X \text{ is finite} \end{cases}$$

and can be identified with the Stone-Čech compactification of X.

(3) Let A be a commutative ring with unit; let $\mathcal{I}(A)$ be the set of idempotents of A -- then $\mathcal{I}(A)$ is a Boolean algebra, the operations being

$$\begin{cases} f + g \equiv f + g - 2fg \\ f \cdot g \equiv fg. \end{cases}$$

Suppose now that A is regular in the sense of von Neumann, i.e., that every principal ideal is idempotent. Consider $X = \text{Spec}(A)$ -- then, topologized in the usual way, X is a compact Boolean space and $\mathcal{A}(X)$ is isomorphic to $\mathcal{I}(A)$, implying, therefore, that X can be regarded as the Stone space of $\mathcal{I}(A)$.

The fact that A is isomorphic to $\mathcal{S}\mathcal{C}(A)$ means that purely algebraic notions per A can be reinterpreted vis-à-vis topological properties of the corresponding open and compact sets in $ST(A)$. On the other hand, it is to be emphasized that this correspondence may break down when it becomes a question of infinite operations. For example, $\mathcal{ST}(A)$ need not be a σ-ring even if A is.

We shall write BR for the category whose objects are Boolean rings $A, B, \ldots,$ and whose morphisms are the ring homomorphisms $\phi : A \to B$ such that $\phi(A) \not\subset \mathfrak{q}\ \forall \mathfrak{q} \in \text{Spec}(B)$. Any morphism $\phi : A \to B$ of Boolean rings induces a continuous map $\psi : ST(B) \to ST(A)$ of the corresponding Stone spaces This map is, moreover, proper.

[Note: We remark that if A and B are both Boolean algebras, then the condition that $\phi : A \to B$ be a morphism of Boolean rings is equivalent to the requirement that $\phi : A \to B$ be a homomorphism of rings taking the multiplicative identity of A to the multiplicative identity of B.]

We shall write BS for the category whose objects are Boolean spaces $X, Y, \ldots,$ and whose morphisms are the proper continuous maps $\psi : X \to Y$. Any morphism $\psi : X \to Y$ of Boolean spaces induces a morphism $\phi : A(Y) \to A(X)$ of Boolean rings.

Example Let A be a Boolean ring without a multiplicative identity, $\hat{A}$ the Boolean ring obtained by adjunction of a unit -- then the canonical injection $A \hookrightarrow \hat{A}$ is not a morphism in BR. Put $X = \text{Spec}(A)$, $\hat{X} = \text{Spec}(\hat{A})$ -- then the canonical injection $X \hookrightarrow \hat{X}$ is not a morphism in BS.

These considerations can best be interpreted functorially.

(1) $\exists$ a contravariant functor

$$\mathbf{F}_{\mathbf{RS}} : \mathbf{BR} \rightsquigarrow \mathbf{BS}.$$

Here

$$\begin{cases} A \to ST(A) \\ \phi \in \mathrm{Hom}(A,B) \to \psi \in \mathrm{Hom}(ST(B),ST(A)). \end{cases}$$

(2) $\exists$ a contravariant functor

$$\mathbf{F}_{\mathbf{SR}} : \mathbf{BS} \rightsquigarrow \mathbf{BR}.$$

Here

$$\begin{cases} X \to A(X) \\ \psi \in \mathrm{Hom}(X,Y) \to \phi \in \mathrm{Hom}(A(Y),A(X)). \end{cases}$$

Call $\mathbf{1}_{\mathbf{BR}}$, $\mathbf{1}_{\mathbf{BS}}$ the identity functors in $\mathbf{BR}$, $\mathbf{BS}$ -- then it is easy to check that $\mathbf{F}_{\mathbf{SR}} \circ \mathbf{F}_{\mathbf{RS}}$ is isomorphic to $\mathbf{1}_{\mathbf{BR}}$ and $\mathbf{F}_{\mathbf{RS}} \circ \mathbf{F}_{\mathbf{SR}}$ is isomorphic to $\mathbf{1}_{\mathbf{BS}}$. The categories $\mathbf{BR}$ and $\mathbf{BS}$ are therefore dual.

Ref The results discussed above are surveyed in M. Stone, _Bull. Amer. Math. Soc._, **44**(1938), pp. 807-816, the complete account being given in M. Stone, _Trans. Amer. Math. Soc._, **40**(1936), pp. 37-111, and _Trans. Amer. Math. Soc._, **41**(1937), pp. 375-481.

§5. Products and Sums

Let X and Y be nonempty sets - then by

$$\begin{cases} \pi_X: X \times Y \to X \\ \pi_Y: X \times Y \to Y, \end{cases}$$

we shall understand the projections of $X \times Y$ onto X and Y, respectively. Given a subset E of $X \times Y$ and points $x \in X$, $y \in Y$, put

$$\begin{cases} E_x = \pi_Y[\pi_X^{-1}(x) \cap E] \\ E^y = \pi_X[\pi_Y^{-1}(y) \cap E], \end{cases}$$

the

$$\begin{cases} \text{vertical} \\ \text{horizontal} \end{cases}$$

sections of E over $\begin{cases} x \\ y \end{cases}$. It is easy to check that

$$\begin{cases} (\cup E_i)_x = \cup(E_i)_x \\ (\cup E_i)^y = \cup(E_i)^y \end{cases} \qquad \begin{cases} (\cap E_i)_x = \cap(E_i)_x \\ (\cap E_i)^y = \cap(E_i)^y \end{cases}$$

$$\begin{cases} (X \times Y - E)_x = Y - E_x \\ (X \times Y - E)^y = X - E^y . \end{cases}$$

Let $S \subset X$, $T \subset Y$ — then the rectangle R determined by S and T is the Cartesian product $S \times T \hookrightarrow X \times Y$, S and T being its sides.

One has $\chi_R = \chi_S \cdot \chi_T$. It is clear that a rectangle is empty iff one of its sides is empty. Furthermore, if $R_1 = S_1 \times T_1$ and $R_2 = S_2 \times T_2$ are nonempty rectangles, then $R_1 \subset R_2$ iff $S_1 \subset S_2$ and $T_1 \subset T_2$. Consequently, two nonempty rectangles are equal iff both of their sides are equal.

There are some simple identities governing the manipulation of rectangles which we had best record explicitly as they will be used tacitly in what follows.

$\bigcup$:

$$\left(\bigcup_{i\in I} S_i\right) \times \left(\bigcup_{j\in J} T_j\right) = \bigcup_{(i,j)\in I\times J} S_i \times T_j$$

[In particular:

$$(S_1 \cup S_2) \times (T_1 \cup T_2)$$

$$= (S_1 \times T_1) \cup (S_1 \times T_2) \cup (S_2 \times T_1) \cup (S_2 \times T_2).]$$

$\bigcap$:

$$\left(\bigcap_{i\in I} S_i\right) \times \left(\bigcap_{j\in J} T_j\right) = \bigcap_{(i,j)\in I\times J} S_i \times T_j$$

[In particular:

$$(S_1 \cap S_2) \times (T_1 \cap T_2) = (S_1 \times T_1) \cap (S_2 \times T_2).]$$

—:

$$\begin{cases} (S_1 - S_2) \times T = (S_1 \times T) - (S_2 \times T) \\ S \times (T_1 - T_2) = (S \times T_1) - (S \times T_2) \end{cases}$$

$$(S_1 \times T_1) - (S_2 \times T_2) = \begin{cases} [(S_1 - S_2) \times (T_1 \cap T_2)] \cup [S_1 \times (T_1 - T_2)] \\ [(S_1 - S_2) \times T_1] \cup [(S_1 \cap S_2) \times (T_1 - T_2)] \end{cases}$$

[In particular: The difference of two rectangles can be written as the disjoint union of two other rectangles.]

Consider now the natural map

$$\mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y),$$

namely the rule assigning to each pair (S, T) the rectangle $R = S \times T$. As this map is evidently bilinear, it must factor canonically

$$\begin{array}{ccc} \mathcal{P}(X) \times \mathcal{P}(Y) & \to & \mathcal{P}(X) \otimes \mathcal{P}(Y) \\ & \searrow & \downarrow \\ & & \mathcal{P}(X \times Y) \end{array}$$

Here, the tensor product is taken over $\mathbb{Z}$ or still, since it amounts to the same, over $\mathbb{Z}_2$. After a moments reflection, the reader will agree that the vertical arrow is actually an injection, its range being the class consisting of those sets in $X \times Y$ which can be written as a finite union of rectangles. Because the image of the ring $\mathfrak{P}(X) \otimes \mathfrak{P}(Y)$ contains all singletons, the associated completion is $\mathfrak{P}(X \times Y)$.

To illustrate these remarks, suppose that $\mathfrak{S}$ is a subring of $\mathfrak{P}(X)$ and that $\mathfrak{T}$ is a subring of $\mathfrak{P}(Y)$ — then, since everything in sight is flat,

$$\mathfrak{S} \otimes \mathfrak{T} \hookrightarrow \mathfrak{P}(X) \otimes \mathfrak{P}(Y).$$

Accordingly, $\mathfrak{S} \otimes \mathfrak{T}$ may be regarded as the class of all subsets of $X \times Y$ of the form

$$\bigcup_{i=1}^{m} (S_i \times T_i) \qquad (S_i \in \mathfrak{S},\ T_i \in \mathfrak{T}),$$

it not being restrictive to suppose that any such union is even disjoint.

Generally, if $\mathfrak{S}$ is a nonempty class of subsets of X and if $\mathfrak{T}$ is a nonempty class of subsets of Y, then we shall write $\mathfrak{S} \boxtimes \mathfrak{T}$ for the class of all rectangles $R = S \times T (S \in \mathfrak{S}, T \in \mathfrak{T})$. In other words, $\mathfrak{S} \boxtimes \mathfrak{T}$ is simply the image of $\mathfrak{S} \times \mathfrak{T}$ under the natural map

$$\mathfrak{P}(X) \times \mathfrak{P}(Y) \rightarrow \mathfrak{P}(X \times Y).$$

Observe that:

(1) If $\mathfrak{S}$ and $\mathfrak{T}$ are multiplicative classes, then $\mathfrak{S} \boxtimes \mathfrak{T}$ is a multiplicative class.

(2) If $\mathfrak{S}$ and $\mathfrak{T}$ are additive classes, then $\mathfrak{S} \boxtimes \mathfrak{T}$ need not be an additive class.

Lemma 1 Let $\mathfrak{S}$ and $\mathfrak{T}$ be semirings — then $\mathfrak{S} \boxtimes \mathfrak{T}$ is a semiring.

[We omit the elementary verification.]

Suppose that $\mathfrak{S}$ and $\mathfrak{T}$ are rings — then $\mathfrak{S} \boxtimes \mathfrak{T}$ is a semiring but rarely a ring. However, if we apply the Kolmogoroff procedure to $\mathfrak{S} \boxtimes \mathfrak{T}$ (cf. §4), the result will be a ring, viz. $\mathfrak{S} \otimes \mathfrak{T}$.

Suppose that $\mathfrak{S}$ and $\mathfrak{T}$ are σ-rings — then $\mathfrak{S}$ is necessarily closed in $\mathfrak{P}(X)_s$ and $\mathfrak{T}$ is necessarily closed in $\mathfrak{P}(Y)_s$. Nevertheless, $\mathfrak{S} \otimes \mathfrak{T}$ is not necessarily closed in $\mathfrak{P}(X \times Y)_s$, hence ordinarily fails to be a σ-ring.

Example Take $X = Y$ of cardinality $\aleph_0$ and let $\mathfrak{S} = \mathfrak{T}$ be the class of all subsets of cardinality $\leq \aleph_0$ — then the diagonal D belongs to the closure of $\mathfrak{S} \otimes \mathfrak{S}$ in $\mathfrak{P}(X \times X)_s$ but is certainly not in $\mathfrak{S} \otimes \mathfrak{S}$ itself.

If $\mathfrak{S}$ and $\mathfrak{T}$ are rings, then in what follows we shall write $\mathfrak{S} \bar{\otimes} \mathfrak{T}$ for the closure of $\mathfrak{S} \otimes \mathfrak{T}$ in $\mathfrak{P}(X \times Y)_s$. Needless to say, $\mathfrak{S} \bar{\otimes} \mathfrak{T}$ is a σ-ring; of course, $\mathfrak{S} \otimes \mathfrak{T} \neq \mathfrak{S} \bar{\otimes} \mathfrak{T}$ in general, even if both $\mathfrak{S}$ and $\mathfrak{T}$ are σ-rings (cf. supra).

Lemma 2 Let $\mathfrak{S}$ and $\mathfrak{T}$ be σ-rings; let $E \in \mathfrak{S} \bar{\otimes} \mathfrak{T}$ — then

$$\begin{cases} E_x \in \mathfrak{T} & \forall\, x \in X \\ E^y \in \mathfrak{S} & \forall\, y \in Y . \end{cases}$$

[One need only note that the class of all subsets of $X \times Y$ with the stated property contains $\mathfrak{S} \otimes \mathfrak{T}$ and is closed in $\mathfrak{P}(X \times Y)_s$.]

Here is a corollary. Let $R = S \times T$ be a nonempty rectangle in $X \times Y$ — then $R \in \mathfrak{S} \bar{\otimes} \mathfrak{T}$ iff $S \in \mathfrak{S}$ and $T \in \mathfrak{T}$.

[Note: The converse to Lemma 2 is false as can be seen by a slight alteration of the preceding example, namely this time take $X = Y$ of cardinality $> \aleph_0$ and, with $\mathfrak{S} = \mathfrak{T}$ as there, consider again the diagonal D .]

Example Take $X = Y$. Consider the following question: Is $\mathfrak{P}(X) \otimes \mathfrak{P}(X)$ dense in $\mathfrak{P}(X \times X)_s$? The answer depends on the cardinality of X .

(1) Suppose that card $(X) > \underline{c}$ — then $\mathfrak{P}(X) \otimes \mathfrak{P}(X)$ is not dense in $\mathfrak{P}(X \times X)_s$.

Re (1) Proceed by contradiction — then, of necessity, the diagonal D would belong to $\mathcal{P}(X) \bar{\otimes} \mathcal{P}(X)$. Therefore, in view of a simple property of the sequential modification (cf. §1), one could find a ring $\mathcal{S}$ in X of cardinality $\leq \aleph_0$ such that D actually belongs to $\mathcal{S} \bar{\otimes} \mathcal{S}$. Denote by $\sigma\text{-}\mathbf{Rin}(\mathcal{S})$ the closure of $\mathcal{S}$ in $\mathcal{P}(X)_s$ — then, thanks to Lemma 2,

$$\forall x \in X : \{x\} \in \sigma\text{-}\mathbf{Rin}(\mathcal{S}) .$$

Let $S_1, S_2, \ldots$ be an enumeration of the elements of $\mathcal{S}$ — then we claim that the characteristic function $f : X \to C$ of the S_i (cf. Prob. IV (§1)),

$$f(x) = 2 \cdot \sum_{i=1}^{\infty} \chi_{S_i}(x)/3^i \quad (x \in X),$$

is one-to-one, hence that $\text{card}(X) \leq \mathfrak{c}$. Indeed, if $f(x) = f(y)$, then $\forall i, x \in S_i$ iff $y \in S_i$. But the class of all subsets $S \subset X$ such that either $\{x,y\} \subset S$ or $\{x,y\} \cap S = \emptyset$ is a σ-ring containing $\mathcal{S}$, thus contains the singletons and so $x = y$, as claimed.

[Note: For a somewhat different approach to this result, see Exer. 21(§6).]

Re (2) There is no loss of generality in taking X to be a subset of $\underline{R}$. If $\text{card}(X) \leq \aleph_0$, then the assertion is clear. We shall therefore suppose that $\text{card}(X) = \aleph_1$. For the purposes at hand, let us agree that a curve in $X \times X$ is simply any set of the form

$$\{(x, f(x)) : x \in \text{dom}(f)\}, \ \{(g(x), x) : x \in \text{dom}(g)\},$$

where

$$\text{dom}(f) \subset X , \quad \text{dom}(g) \subset X$$

and $f : \text{dom}(f) \to X$, $g : \text{dom}(g) \to X$ are functions. Every curve is in the closure of $\mathcal{P}(X) \otimes \mathcal{P}(X)$ in $\mathcal{P}(X \times X)_s$. To see this, note that

$$\{(x, f(x)) : x \in \text{dom}(f)\} = \bigcap_{m=1}^{\infty} E_m$$

where

$$E_m = \bigcup_{i=-\infty}^{\infty} E_{im}$$

with

$$E_{im} = \{x \in \mathrm{dom}(f): \frac{i}{m} \leq f(x) < \frac{i+1}{m}\} \times X \cap [\frac{i}{m}, \frac{i+1}{m}[$$

and similarly for g. To prove (2), therefore, it need only be shown that $X \times X$ can be written as a countable union of curves. To this end, well-order X: $\{x_\alpha: \alpha < \Omega\}$. Divide $X \times X$ into complementary sets E and F by the definitions

$$\begin{cases} E = \{(x_\alpha, x_\beta): \beta < \alpha\} \\ F = \{(x_\alpha, x_\beta): \alpha \leq \beta\}. \end{cases}$$

It is clear that the vertical sections of E are finite or countable, as are the horizontal sections of F. For each $x \in X$, arrange E_x into a sequence $\{x_n\}$, it being understood that the sequence is to be completed in an arbitrary way if it is finite to begin with. Define now functions $f_n: X \to X$ by the prescription $f_n(x) = x_n$. Analogous considerations apply to the horizontal sections F^x of F leading to functions $g_n: X \to X$. Taken together, the curves

$$\{(x, f_n(x)): x \in X\}, \{(g_n(x), x): x \in X\}$$

cover $X \times X$.

[Note: The last part of the preceeding argument is virtually the same as that needed in the first part of Prob. IV(§2).]

Re (3) On the basis of (2), this is immediate.

[Note: Actually, one can get away with less here in that Martin's axiom alone suffices to force the conclusion if $\aleph_1 < \mathrm{card}(X) \leq \mathfrak{c}$; cf. K. Kunen, Inaccessibility Properties of Cardinals, Ph.D. Thesis, Stanford University, 1968.]

Partitions in X and Y are closely related to partitions in $X \times Y$ and vice versa.

Lemma 3 Let $R = S \times T$ be a nonempty rectangle; let $\{R_k = S_k \times T_k\}$ be a class of nonempty rectangles — then the R_k partition R iff

(i) $R = \cup R_k$; (ii) $S = \cup S_k$, $T = \cup T_k$;

(iii) $\forall\, k \neq \ell$

$$\begin{cases} S_k \cap S_\ell \neq \emptyset \Longrightarrow T_k \cap T_\ell = \emptyset \\ \text{or} \\ T_k \cap T_\ell \neq \emptyset \Longrightarrow S_k \cap S_\ell = \emptyset . \end{cases}$$

[We omit the elementary verification.]

Let R be a nonempty rectangle — then a partition $P(R) = \{R_k : k \in K\}$ of R by rectangles is said to be a *network* on R if

$$\begin{cases} \text{the } \pi_X(R_k) \text{ partition } \pi_X(R) \\ \text{and} \\ \text{the } \pi_Y(R_k) \text{ partition } \pi_Y(R) . \end{cases}$$

[Note: Here we are admitting a small solecism in that repetitions may, of course, be present in the classes $\pi_X(R_k)$, $\pi_Y(R_k)$.]

Lemma 4 *Let* $\mathfrak{S} \subset \mathfrak{P}(X)$, $\mathfrak{T} \subset \mathfrak{P}(Y)$ *be multiplicative classes; let* $R = S \times T \in \mathfrak{S} \boxtimes \mathfrak{T}$ *be a nonempty rectangle. Suppose that* $P(R)$ *is a finite* $\mathfrak{S} \boxtimes \mathfrak{T}$*-partition of* R — *then there exists a partition in* $\text{Par}_{\mathfrak{S} \boxtimes \mathfrak{T}}(R)$ *which refines* $P(R)$ *and is a network on* R.

Proof It can be assumed that $P(R)$ is not a network on R. Denoting the components of $P(R)$ by R_k, let $S_k = \pi_X(R_k)$, $T_k = \pi_Y(R_k)$ — then $S = \cup S_k$, $T = \cup T_k$. Consider the S_k. Define an equivalence relation on X by stipulating that x_1 be equivalent to x_2 iff

$$\forall\, y \in Y\,,\ (x_1, y) \sim (x_2, y),$$

the latter equivalence being that corresponding to $P(R)$. Given $x \in X$, the equivalence class $[x]$ determined by x is simply the intersection of the S_k containing x. All told, therefore, this procedure produces a finite $\mathfrak{S}$-partition $P(S) = \{S_i: i \in I\}$ of S. Work with the T_k in an analogous fashion to produce a finite $\mathfrak{T}$-partition $P(T) = \{T_j: j \in J\}$ of T. The $S_i \times T_j$ then constitute a finite $\mathfrak{S} \boxtimes \mathfrak{T}$ partition of R , refining P(R) and forming a network on R . //

Retaining the notation of Lemma 4, suppose that P(R) is a countable $\mathfrak{S} \boxtimes \mathfrak{T}$-partition of R . We then ask: Does there exist a partition in σ - $\text{Par}_{\mathfrak{S} \boxtimes \mathfrak{T}}(R)$ which refines P(R) and is a network on R? Unfortunately, even after imposing about as much additional structure on $\mathfrak{S}$ and $\mathfrak{T}$ as can be reasonably expected, the answer will in general be negative.

Examples (1) Take $X = [-1, 1[$, $Y = [0, +\infty[$. Let $\mathfrak{S}$ be the class consisting of all left closed and right open subintervals of X ; let $\mathfrak{T} = \mathfrak{P}(Y)$ - then $\mathfrak{S}$ is a semiring and $\mathfrak{T}$ is a complete ring. Consider the countable $\mathfrak{S} \boxtimes \mathfrak{T}$-partition of $X \times Y$ by the rectangles

$$[-1, 1[\times [0, 1[$$

$$\begin{cases} [-1, -\frac{1}{n}[\times [n-1, n[\\ [-\frac{1}{n}, \frac{1}{n}[\times [n-1, n[\qquad (n > 1) . \\ [\frac{1}{n}, 1[\times [n-1, n[\end{cases}$$

Because $0 \in [-\frac{1}{n}, \frac{1}{n}[\ \forall n$, it is impossible to find a countable $\mathfrak{S} \boxtimes \mathfrak{T}$ - network on $X \times Y$ which refines this partition.

(2) Take $X =]0, 1[$, $Y =]0, 1[\cap \mathbb{Q}$. Let $\mathfrak{S} = \mathfrak{P}(X)$, $\mathfrak{T} = \mathfrak{P}(Y)$ - then both $\mathfrak{S}$ and $\mathfrak{T}$ are complete rings. Consider the countable $\mathfrak{S} \boxtimes \mathfrak{T}$-partition of $X \times Y$ by the rectangles

$$\begin{cases}]0, q[\times \{q\} \\ \qquad\qquad\qquad (0 < q < 1,\ q \in \underset{\sim}{Q}). \\ [q, 1[\times \{q\} \end{cases}$$

Suppose that the $S_i \times T_j$ $(i \in I,\ j \in J)$ refine this partition and form a network on $X \times Y$ — then, of necessity,

$$\begin{cases} \text{card}(I) > \aleph_0 \\ \text{card}(J) = \aleph_0 , \end{cases}$$

so $I \times J$ must be uncountable.

Up until this point, the discussion has dealt exclusively with products involving two factors. The extension of the theory to $n > 2$ $(n \in \underset{\sim}{N})$ factors is purely formal, hence need not be considered in detail. We remark only that tacitly one makes throughout the usual conventions as regards the associativity of the relevant operations.

The situation for products involving an arbitrary number of factors is only slightly more complicated, it being a matter of setting up the definitions in a succinct fashion. Let, then, $\{X_i : i \in I\}$ be a class of nonempty sets X_i indexed by an infinite set I — then we shall agree that a rectangle in $\prod X_i$ is any set of the form $\prod S_i$, where $S_i \subset X_i$ $\forall_i$ and $S_i = X_i$ for all but a finite set of i. If $S = \prod S_i$ and if

$$\begin{cases} S' = \prod S_i' \\ S'' = \prod S_i'' \end{cases}$$

are nonempty rectangles, then $S = S' \cup S''$ with $S' \cap S'' = \emptyset$ iff there exists a unique index i_0 such that

$$\begin{cases} i \neq i_0 \Longrightarrow S_i = S_i' = S_i'' \\ i = i_0 \Longrightarrow S_i = S_i' \cup S_i'' , \ S_i' \cap S_i'' = \emptyset . \end{cases}$$

Consider now the tensor product $\otimes \mathcal{P}(X_i)$ - then, $\forall i$, $\exists$ a canonical homomorphism

$$\iota_i : \mathcal{P}(X_i) \longrightarrow \otimes \mathcal{P}(X_i),$$

namely the rule which assigns to each $S_i \subset X_i$ the tensor whose i^{th} entry is S_i and whose j^{th} entry is $X_j (j \neq i)$. The subalgebra of $\otimes \mathcal{P}(X_i)$ generated by the $\iota_i(\mathcal{P}(X_i))$ is composed of all finite sums of elements of the form $\otimes S_i$, where $S_i = X_i$ except for a finite number of indices. Algebraists customarily refer to this subalgebra of $\otimes \mathcal{P}(X_i)$ as the *tensor product of the algebras* $\mathcal{P}(X_i)$. We shall denote it by $\otimes^* \mathcal{P}(X_i)$. Since the index set I is infinite, it differs in general from $\otimes \mathcal{P}(X_i)$.

[Note: Consideration of $\otimes^* \mathcal{P}(X_i)$ is, of course, necessary from the categorical point of view.]

Denote by $\prod^{*} \mathscr{P}(X_i)$ that subset of $\prod \mathscr{P}(X_i)$ consisting of the (S_i) such that $S_i = X_i$ for all but a finite set of i. There is a commutative triangle

$$\begin{array}{ccc} \prod^{*} \mathscr{P}(X_i) & \longrightarrow & \otimes^{*} \mathscr{P}(X_i) \\ & \searrow & \downarrow \\ & & \mathscr{P}(\prod X_i) \end{array}$$

The vertical arrow is an injection, its range being the class of those sets in $\prod X_i$ which can be written as a finite union of rectangles.

Finally, we come to the one big difference between infinite as opposed to finite products, namely this: It is necessary to consider algebras $\mathscr{S}_i \subset \mathscr{P}(X_i)$ rather than just rings. The reason is easy enough to see. Indeed, if we proceed as above to form $\otimes^{*} \mathscr{S}_i$, then each of the $\mathscr{S}_i$'s must at least be rings with unit and to ensure compatibility, it is best to assume that they are actually algebras. Under these circumstances,

$$\otimes^{*} \mathscr{S}_i \hookrightarrow \otimes^{*} \mathscr{P}(X_i)$$

meaning, therefore, that $\otimes^{*} \mathscr{S}_i$ can be thought of as sitting inside $\mathcal{P}(\prod X_i)$, the characterization reading as in the finite case, i.e., the class of all finite disjoint unions of rectangles $\prod S_i$, where $S_i \in \mathcal{S}_i \ \forall i$. This being so, we shall then write $\bar{\otimes}^*\mathcal{S}_i$ for the closure of $\otimes^*\mathcal{S}_i$ in $\mathcal{P}(\prod X_i)_s$. Evidently, $\bar{\otimes}^*\mathcal{S}_i$ is a σ-algebra.

Keeping to the preceding notation, put $X = \prod X_i$, $\mathcal{S} = \bar{\otimes}^*\mathcal{S}_i$. Let $I = I_1 \cup I_2$ be a partition of I. Let $X_1 = \prod_{I_1} X_i$, $X_2 = \prod_{I_2} X_i$; let $\mathcal{S}_1 = \bar{\otimes}^*_{I_1}\mathcal{S}_i$, $\mathcal{S}_2 = \bar{\otimes}^*_{I_2}\mathcal{S}_i$ — then X may be identified with $X_1 \times X_2$ and, when this is done, we have $\mathcal{S} = \mathcal{S}_1 \bar{\otimes} \mathcal{S}_2$. Therefore, in a certain sense, we are right back at the beginning.

Example Let $\{X_i : i \in I\}$ be a class of compact Hausdorff spaces X_i indexed by an infinite set I. Take for $\mathcal{S}_i$ the algebra of open and compact subsets of X_i — then $\otimes^*\mathcal{S}_i$ is the algebra of open and compact subsets of $\prod X_i$.

[Let us consider an important special case. Equip $\{0,1\}$ with the discrete topology. Given any $i = 1,2,\ldots,$ put $X_i = \{0,1\}$ — then, in the product topology, $2^{\mathbb{N}} = \prod X_i$ is a compact, totally disconnected, Hausdorff space of weight $\aleph_0$, the so-called Cantor space. Of course, the terminology arises from the fact that $2^{\mathbb{N}}$ is homeomorphic to C, viz. (cf. Prob. IV(§1)):

$$(f \in 2^{\mathbb{N}}) \longmapsto (2 \cdot \sum_{i=1}^{\infty} f(i)/3^i \in C).$$

Let $\mathcal{S}_i$ be the algebra of all subsets of X_i — then $\otimes^*\mathcal{S}_i$ is the algebra of open and compact subsets of $2^{\underset{\sim}{N}}$ and $\bar{\otimes}^* \mathcal{S}_i$ is the σ-algebra of Borel subsets of $2^{\underset{\sim}{N}}$ (cf. §6).]

Let $\{X_i: i \in I\}$ be a class of nonempty sets X_i indexed by a nonempty set I (finite of infinite), the X_i being, in addition, pairwise disjoint. Write $\oplus\ \mathcal{P}(X_i)$ for the direct sum of the $\mathcal{P}(X_i)$. Suppose that $\forall i$, $\mathcal{S}_i$ is a ring in X_i — then the direct sum $\oplus\ \mathcal{S}_i$ of the $\mathcal{S}_i$ is a subring of $\oplus\ \mathcal{P}(X_i)$. The elements in $\oplus\ \mathcal{S}_i$ may be viewed as those subsets S of $\bigcup X_i$ with the property that $S \cap X_i \in \mathcal{S}_i$ for all i, or still, as the class of all unions $\cap S_i$, where $S_i \in \mathcal{S}_i$ $(\forall i)$. If each of the $\mathcal{S}_i$ is a σ-ring, then so is $\oplus\ \mathcal{S}_i$.

[Note: If the X_i are not initially pairwise disjoint, then this may always be arranged by looking instead at the $X_i \times \{i\}$.]

Example Let $\mathcal{S}$ be a σ-ring in X. Fix a countable partition $P(X) = \{X_i: i \in I\}$ of X, where $X_i \in \mathcal{S}$ $\forall i$. Put $\mathcal{S}_i = tr_{X_i}(\mathcal{S})$ — then $\mathcal{S} = \oplus\ \mathcal{S}_i$.

Notes and Remarks

Just who was the first to consider products in abstracto is not completely clear. The following papers are relevant: H. Hahn, Ann. Scuola Norm. Sup. Pisa, **2**(1933), pp. 429-452; F. Maeda, Tôhoku Math. J., **37**(1933), pp. 446-453;

Z. Lomnicki and S. Ulam, Fund. Math., **23**(1934), pp. 237-278 (see too Ulam's paper in the proceedings of the 1932 International Congress); J. Ridder, Fund. Math., **24**(1934), pp. 72-117; W. Feller, Bull. Int. Acad. Youg., **28**(1934), pp. 30-45; B. Jessen, Acta Math., **63**(1934), pp. 249-323.
The question of the density of $\mathfrak{P}(X) \otimes \mathfrak{P}(X)$ in $\mathfrak{P}(X \times X)_s$ is an old problem of Ulam and has been considered by a number of authors; cf. B. Rao, Acta Math. Acad. Sci. Hungar., **22**(1971), pp. 197-198. Lemma 4 is a variation on a well known theme; it is explicitly stated and proved in D. Goguadze [Д. Гогуадзе], Kolmogoroff Integrals and Some of their Applications [Об Интегралах Колмогорова И Их Некоторых Приложениях], Мецниереба, Тбилиси, 1979 (see pp. 152-153). This author goes on to claim (statement 13.8, p. 154) that if $\mathfrak{S}$ and $\mathfrak{T}$ are semirings, then Lemma 4 is true when "finite" is replaced by "countable". As we have seen in the text, this is false. It may have occurred to the reader that the language of category theory might be helpful at certain points in this §; some comments in this direction may be found in L. Auslander and C. Moore, Mem. Amer. Math. Soc., **62**(1966), pp.1-199.

Exercises

(1) True or False?

(a) There exists a nonempty set E such that $E \times E \subset E$;

(b) There exists a nonempty set E such that $E \subset E \times E$.

(2) Discuss the continuity of the natural maps

$$\begin{cases} \mathfrak{P}(X) \times \mathfrak{P}(Y) \to \mathfrak{P}(X \times Y) \\ \mathfrak{P}(X)_s \times \mathfrak{P}(Y)_s \to \mathfrak{P}(X \times Y)_s \end{cases} .$$

(3) Let $\mathfrak{Kol}(?)$ be the ring obtained from the semiring ? via the Kolmogoroff procedure (cf. §4).

True or False? If $\mathfrak{S}$ is a semiring in X and if $\mathfrak{T}$ is a semiring in Y (so that $\mathfrak{S} \boxtimes \mathfrak{T}$ is a semiring in $X \times Y$), then

$$\mathfrak{Kol}(\mathfrak{S}) \otimes \mathfrak{Kol}(\mathfrak{T}) = \mathfrak{Kol}(\mathfrak{S} \boxtimes \mathfrak{T}) .$$

(4) Let $\mathfrak{S}$ and $\mathfrak{T}$ be σ-rings; let $E \in \mathfrak{S} \bar{\otimes} \mathfrak{T}$ — then there exist $S \in \mathfrak{S}$, $T \in \mathfrak{T}$ such that $E \subset S \times T$.

(5) Let X and Y be nonempty sets — then

$$\mathfrak{P}(X \times Y) = \mathfrak{P}(X) \bar{\otimes} \mathfrak{P}(Y)$$

if

$$\begin{cases} \text{card}(X) \leq \aleph_1 \\ \qquad\qquad\qquad \text{(or even } \leq \mathfrak{c} \text{ under Martin's axiom),} \\ \text{card}(Y) \leq \aleph_1 \end{cases}$$

but

$$\mathfrak{P}(X \times Y) \neq \mathfrak{P}(X) \bar{\otimes} \mathfrak{P}(Y)$$

if both X and Y are uncountable and at least one of them has cardinality $> \mathfrak{c}$.

(6) Suppose that $P(X) = \{X_i : i \in I\}$ is a partition of X ; suppose that $P(Y) = \{Y_j : j \in J\}$ is a partition of Y — then the *product* of $P(X)$ and $P(Y)$ is that partition $P(X) \times P(Y)$ of $X \times Y$ whose components are the $X_i \times Y_j$. Check that a product is a network and that, conversely, a network is a product.

(7) Suppose that there is attached to each i in an uncountable set I a nonempty set X_i and a nontrivial σ-algebra $\mathcal{S}_i \subset \mathcal{P}(X_i)$ — then $\bar{\otimes}^* \mathcal{S}_i$ is antiatomic.

[Note: This need not be true, of course, if I is countable.]

(8) Given a class of nonempty, pairwise disjoint sets X_i , let $\mathcal{S}_i$ be an atomic ring in X_i — then $\oplus \mathcal{S}_i$ is an atomic ring in $\cup X_i$.

Problem

PROJECTIONS

Let X and Y be nonempty sets — then by *projection onto* X we understand the map from $\mathcal{P}(X \times Y)$ onto $\mathcal{P}(X)$ defined by the rule

$$\text{Pro}_X(E) = \{x \in X: E_x \neq \emptyset\} .$$

Verify that

$$\begin{cases} \text{Pro}_X(\cup E_i) = \cup \text{Pro}_X(E_i) \\ \text{Pro}_X(\cap E_i) \subset \cap \text{Pro}_X(E_i) , \end{cases}$$

the second containment being strict in general, even for a decreasing sequence, although for rectangles it is true that

$$\text{Pro}_X((S_1 \times T_1) \cap (S_2 \times T_2) \cap \ldots) = S_1 \cap S_2 \cap \ldots$$

if $T_1 \cap T_2 \cap \ldots \neq \emptyset$.

Let $\mathcal{S}$ be a nonempty class of subsets of X ; let $\mathcal{T}$ be a nonempty class of subsets of Y — then, for any nonempty E ,

$$E \in (\mathcal{S} \boxtimes \mathcal{T})_s \Longrightarrow \text{Pro}_X(E) \in \mathcal{S}_s$$
$$E \in (\mathcal{S} \boxtimes \mathcal{T})_d \Longrightarrow \text{Pro}_X(E) \in \mathcal{S}_d$$
$$E \in (\mathcal{S} \boxtimes \mathcal{T})_\sigma \Longrightarrow \text{Pro}_X(E) \in \mathcal{S}_\sigma$$
$$E \in (\mathcal{S} \boxtimes \mathcal{T})_\delta \Longrightarrow \text{Pro}_X(E) \in \mathcal{S}_\delta$$
$$E \in (\mathcal{S} \boxtimes \mathcal{T})_{\delta s} \Longrightarrow \text{Pro}_X(E) \in \mathcal{S}_{\delta s}$$
$$E \in (\mathcal{S} \boxtimes \mathcal{T})_{d\sigma} \Longrightarrow \text{Pro}_X(E) \in \mathcal{S}_{d\sigma}$$

What can be said about the other operations, e.g., $s\delta$, $\sigma\delta$ etc.?

Example Take $X = Y = [0, 1]$. Let $\mathcal{S}$ be the class comprised of all closed subintervals of X ; let $\mathcal{T} = \mathcal{P}(Y)$ — then

$$\mathrm{Pro}_X((\mathcal{S} \boxtimes \mathcal{T})_{s\delta}) = \mathcal{P}(X) \ .$$

So, the moral is that some assumptions will have to be imposed if a positive result is to be obtained.

This said, prove that if $\mathcal{T}$ is countably compact (Prob.VIII (§1)), then for any nonempty E,

$$E \in (\mathcal{S} \boxtimes \mathcal{T})_{s\delta} \Longrightarrow \mathrm{Pro}_X(E) \in \mathcal{S}_{s\delta} \ .$$

[Recall that the countable compactness of $\mathcal{T}$ implies the countable compactness of $\mathcal{T}_{sd} = \mathcal{T}_{ds}$ (cf. op. cit.). With this in mind, establish the following lemma. If $E_1 \supset E_2 \supset \ldots$ $(E_i \subset X \times Y \ \ \forall_i)$, and if $\forall \ x \in X$, the class $\{(E_i)_x : i = 1, 2, \ldots\}$ is countably compact, then

$$\mathrm{Pro}_X(\cap E_i) = \cap \mathrm{Pro}_X(E_i) . \ ]$$

Maintaining the above hypothesis on $\mathcal{T}$, it can also be shown that

$$E \in (\mathcal{S} \boxtimes \mathcal{T})_{\sigma\delta} \Longrightarrow \mathrm{Pro}_X(E) \in \mathcal{S}_A \ .$$

Here, the sub-A refers to operation A (cf. §8).

[Note: $\mathcal{S}_A \supset \mathcal{S}_{\sigma\delta}$ but the result cannot be improved to read $\mathrm{Pro}_X(E) \in \mathcal{S}_{\sigma\delta}$, as may be seen by example.]

Ref E. Marczewski and C. Ryll-Nardzewski, Fund. Math., 40(1953), pp.160-164.

§6. Extension and Generation

Let X be a nonempty set. Let $\bigstar$ be a property of certain nonempty classes of subsets of X -- then $\bigstar$ is said to be *extensionally attainable* if for every subset $\mathcal{S}$ of $\mathcal{P}(X)$, there exists a subset $\bigstar(\mathcal{S})$ of $\mathcal{P}(X)$ which

$$\begin{cases} \text{(a)} \quad \text{contains } \mathcal{S} \\ \text{(b)} \quad \text{possesses } \bigstar \end{cases}$$

and, in addition, in minimal with respect to (a) and (b). $\bigstar(\mathcal{S})$, if it exists, is said to be the $\bigstar$-class *generated* by $\mathcal{S}$.

Lemma 1 *Property $\bigstar$ is extensionally attainable iff $\mathcal{P}(X)$ has property $\bigstar$ and the intersection of any nonempty collection of classes having property $\bigstar$ also has property $\bigstar$.*

[We omit the elementary verification.]

Suppose that $\bigstar$ is extensionally attainable -- then, for any $\mathcal{S}$,

$$\bigstar(\mathcal{S}) = \cap \mathcal{S}_i ,$$

the $\mathcal{S}_i$ running over all those classes which contain $\mathcal{S}$ and which possess ★.

Here are some typical examples of extensionally attainable properties:

$$\left\{\begin{array}{l}
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a lattice} \\
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a ring (algebra)} \\
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a } \sigma\text{-ring } (\sigma\text{-algebra}) \\
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a } \delta\text{-ring } (\delta\text{-algebra}) \\
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a complete ring.}
\end{array}\right.$$

On the other hand, the stipulations that

$$\left\{\begin{array}{l}
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a ring with unit} \\
\mathcal{S} \text{ has property } \bigstar \text{ iff } \mathcal{S} \text{ is a semiring}
\end{array}\right.$$

are not extensionally attainable.

Examples (1) The intersection of two rings with unit need not be a ring with unit.

[Take $X = [0,3]$. If $\left\{\begin{matrix}\mathcal{S}\\ \mathfrak{T}\end{matrix}\right.$ is the class of all subsets of $\left\{\begin{matrix}[0,2]\\ [1,3]\end{matrix}\right.$ which are either finite or have a finite complement per $\left\{\begin{matrix}[0,2]\\ [1,3]\end{matrix}\right.$, then both $\mathcal{S}$ and $\mathfrak{T}$ are rings with unit, but their intersection $\mathcal{S} \cap \mathfrak{T}$ consists of all finite subsets of [1,2], hence is not a ring with unit.]

(2) The intersection of two semirings need not be a semiring.

[Take X = {1,2,3} -- then

$$\begin{cases} \mathcal{S} = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\} \\ \mathcal{T} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2,3\}\} \end{cases}$$

are both semirings, but their intersection

$$\mathcal{S} \cap \mathcal{T} = \{\emptyset, \{1\}, \{1,2,3\}\}$$

is not.]

Suppose that $\star$ is extensionally attainable -- then $\star$ determines a map

$$M_\star : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X)),$$

namely the rule which assigns to each $\mathcal{S}$ its $\star$-class $\star(\mathcal{S})$. The fixed points for this map are exactly those classes $\mathcal{S}$ having property $\star$. The central question to be considered now is this: Given $\mathcal{S}$, describe $\star(\mathcal{S})$. Naturally, the description itself will depend on $\star$. In terms of $M_\star$, there is a variant in that typically a generic nonempty fiber $M_\star^{-1}(\mathcal{S}_0)$ is fixed in advance, the point being that each $\mathcal{S}$ in this fiber generates the $\star$-class

$\mathfrak{S}_0$, i.e., $\bigstar(\mathfrak{S}) = \mathfrak{S}_0$, implying, therefore, that $\mathfrak{S}_0$ can be studied in a variety of ways.

[Note: In what follows, we shall leave it up to the reader to struggle with the empty class.]

Let us begin with a simple illustration. Take $\bigstar$ to be the property: ? is a lattice. Given a nonempty $\mathfrak{S}$, we then call $\bigstar(\mathfrak{S})$ the lattice generated by $\mathfrak{S}$ and denote it by $\mathfrak{Lat}(\mathfrak{S})$. In terms of $\mathfrak{S}$, $\mathfrak{Lat}(\mathfrak{S})$ is the class $\mathfrak{S}_{sd}(= \mathfrak{S}_{ds})$ with, if necessary, the empty set adjoined.

A slightly more complicated situation arises when we take $\bigstar$ to be the property: ? is a ring. Given any nonempty $\mathfrak{S}$, we then call $\bigstar(\mathfrak{S})$ the ring generated by $\mathfrak{S}$ and denote it by $\mathfrak{Rin}(\mathfrak{S})$. Viewed abstractly, $\mathfrak{Rin}(\mathfrak{S})$ is simply the intersection of all rings in X containing $\mathfrak{S}$. Thus, an algebraic grounds, $\mathfrak{Rin}(\mathfrak{S})$ can be described as the class of all finite symmetric differences $S_1 \Delta \ldots \Delta S_m$, each S_i being in turn a finite intersection of sets belonging to $\mathfrak{S}$. Consequently, if $\mathfrak{S}$ is finite (countable), then so is $\mathfrak{Rin}(\mathfrak{S})$.

[Note: Other characterizations of $\mathfrak{Rin}(\mathcal{S})$ may be found in Exer. 3. Trivially, every element of $\mathfrak{Rin}(\mathcal{S})$ is contained in some element of $\mathcal{S}_s$ (cf. Exer. 8).]

Example Let $\mathcal{S}$ be a semiring -- then

$$\mathfrak{Rin}(\mathcal{S}) = \mathfrak{Kol}(\mathcal{S}).$$

Take now for $\bigstar$ the property :? is a σ-ring (δ-ring). Given any nonempty $\mathcal{S}$, we then call $\bigstar(\mathcal{S})$ the σ-ring (δ-ring) generated by $\mathcal{S}$ and denote it by $\sigma\text{-}\mathfrak{Rin}(\mathcal{S})$ ($\delta\text{-}\mathfrak{Rin}(\mathcal{S})$). Observe that the notation is unambiguous in that the σ-ring (δ-ring) generated by $\mathcal{S}$ is in fact the same as the σ-ring (δ-ring) generated by $\mathfrak{Rin}(\mathcal{S})$. Obviously,

$$\delta\text{-}\mathfrak{Rin}(\mathcal{S}) \subset \sigma\text{-}\mathfrak{Rin}(\mathcal{S}),$$

$\sigma\text{-}\mathfrak{Rin}(\mathcal{S})$ being in fact the class of all countable unions of elements from $\delta\text{-}\mathfrak{Rin}(\mathcal{S})$, i.e.,

$$\sigma\text{-}\mathfrak{Rin}(\mathcal{S}) = [\delta\text{-}\mathfrak{Rin}(\mathcal{S})]_\sigma.$$

Examples (1) Let X be a topological space -- then the σ-ring generated by the open (or, equivalently, closed) subsets of X is called the σ-ring of Borel sets in X and is denoted by $\mathfrak{Bo}(X)$.

(2) Let X be a Hausdorff topological space -- then the δ-ring generated by the compact subsets of X is called the δ-ring of *bounded Borel sets* in X and is denoted by $\mathfrak{Ba}_b(X)$.

[Note: X is taken to be Hausdorff here in order to ensure that every compact subset of X is a Borel set (all compacta then being closed, of course). By comparison, observe that if X is equipped with the indiscrete topology, then the Borel sets are $\emptyset$ and X, but every subset of X is compact.]

One cannot, in general describe the σ-ring generated by a class of sets in purely algebraic terms. There are, however, useful alternative procedures, essentially transfinite in nature.

We have already encountered one such. Indeed, given $\mathcal{S}$, σ-$\mathbf{Rin}(\mathcal{S})$ is simply the closure of $\mathbf{Rin}(\mathcal{S})$ in $\mathfrak{P}(X)_s$ (cf. §4) or still (cf. §1),

$$\sigma\text{-}\mathbf{Rin}(\mathcal{S}) = \bigcup_{\alpha<\Omega} u_\alpha(\mathbf{Rin}(\mathcal{S})).$$

In this connection, let us recall that $u_\alpha(\mathbf{Rin}(\mathcal{S}))$ is the class comprised of those sets $S \subset X$ for which there exists a sequence $\{S_i\} \subset \bigcup_{\beta<\alpha} u_\beta(\mathbf{Rin}(\mathcal{S}))$ such that $\lim S_i = S$. The $u_\alpha(\mathbf{Rin}(\mathcal{S}))$ are rings which increase with α. Consequently, inside σ-$\mathbf{Rin}(\mathcal{S})$ is a transfinite sequence of rings

$$\mathbf{Rin}(\mathcal{S}) \subset \ldots \subset u_\alpha(\mathbf{Rin}(\mathcal{S})) \subset \ldots \quad (\alpha<\Omega),$$

whose union is precisely σ-$\mathbf{Rin}(\mathcal{S})$ itself.

Example Let $\mathcal{S}$ be a ring in X; let $\mathcal{T}$ be a ring in Y -- then

$$\sigma\text{-}\mathbf{Rin}(\mathcal{S} \otimes \mathcal{T}) = \mathcal{S} \bar{\otimes} \mathcal{T}.$$

More generally, let $\mathcal{S}_i$ be an algebra in X_i ($i \in I$, I infinite) -- then

$$\sigma\text{-}\mathbf{Rin}(\otimes^* \mathcal{S}_i) = \bar{\otimes}^* \mathcal{S}_i.$$

Starting from $\mathcal{S}$, we shall now define by transfinite recursion a class $\mathcal{S}_\alpha$ for each ordinal number $\alpha<\Omega$. Thus putting $\mathcal{S}_0 = \mathcal{S}$, write

$$\mathcal{S}_\alpha = \Big(\bigcup_{\beta<\alpha} \mathcal{S}_\beta\Big)_{r\sigma} \qquad (\alpha<\Omega).$$

Observe that the $\mathcal{S}_\alpha$ increase with α.

Lemma 2 We have

$$\sigma\text{-}\mathbf{Rin}(\mathcal{S}) = \bigcup_{\alpha<\Omega} \mathcal{S}_\alpha.$$

To see what the rationale behind the construction is, replace σ by s -- then $\mathcal{S}_0 = \mathcal{S}$, $\mathcal{S}_1 = \mathcal{S}_{rs}$, $\mathcal{S}_2 = \mathcal{S}_{rsrs}$, the ring generated by $\mathcal{S}$ (cf. Exer. 3).

[Note: Trivially, every element of $\sigma\text{-}\mathbf{Rin}(\mathcal{S})$ is contained in some element of $\mathcal{S}_\sigma$ (cf. Exer. 8).]

Proof of Lemma 2 There are two steps in the argument.

(1) $\bigcup_{\alpha<\Omega}\mathcal{S}_\alpha$ is contained in $\sigma\text{-}\mathbf{Rin}(\mathcal{S})$.

(2) $\bigcup_{\alpha<\Omega}\mathcal{S}_\alpha$ is a σ-ring.

Re (1) By definition, $\mathcal{S}_0 = \mathcal{S} \subset \sigma\text{-}\mathbf{Rin}(\mathcal{S})$; in addition, $\emptyset \in \mathcal{S}_1$. Proceeding by transfinite induction, assume that $\mathcal{S}_\beta \subset \sigma\text{-}\mathbf{Rin}(\mathcal{S})$ for every $\beta<\alpha$ and consider a typical element $S \in \mathcal{S}_\alpha$ -- then S is a countable union, say $\cup S_i$, where each S_i has the form A_i or $A_i - B_i$, with

$$A_i, B_i \in \bigcup_{\beta<\alpha}\mathcal{S}_\beta \subset \sigma\text{-}\mathbf{Rin}(\mathcal{S}).$$

Thus $S_i \in \sigma\text{-}\mathbf{Rin}(\mathcal{S})$ and so $S = \cup S_i \in \sigma\text{-}\mathbf{Rin}(\mathcal{S})$, which implies that $\mathcal{S}_\alpha \subset \sigma\text{-}\mathbf{Rin}(\mathcal{S})$. This completes the proof of (1).

Re (2) Let $\{S_i\}$ be a sequence in $\bigcup_{\alpha<\Omega}\mathcal{S}_\alpha$ -- then we claim that $\cup S_i \in \bigcup_{\alpha<\Omega}\mathcal{S}_\alpha$. To prove it, note that for each i there is an α_i such that $S_i \in \mathcal{S}_{\alpha_i}$. Select, as is possible, an $\alpha<\Omega$ such that $\alpha_i<\alpha \forall i$ -- then

$$\cup S_i \in \left(\bigcup_{i=1}^{\infty} \mathcal{S}_{\alpha_i}\right)_{r\sigma} \subset \mathcal{S}_\alpha \subset \bigcup_{\alpha<\Omega} \mathcal{S}_\alpha,$$

as claimed. In an entirely analogous manner, one can show that if $S, T \in \bigcup_{\alpha<\Omega} \mathcal{S}_\alpha$, then $S - T \in \bigcup_{\alpha<\Omega} \mathcal{S}_\alpha$. This completes the proof of (2).

Hence the lemma. //

The transfinite description of $\sigma\text{-}\mathbf{Rin}(\mathcal{S})$ provided by Lemma 2 carries with it an added bonus in that an estimate for the cardinality of $\sigma\text{-}\mathbf{Rin}(\mathcal{S})$ can be easily obtained. To this end, we can suppose that $\text{card}(\mathcal{S}) \geq 2$ since

$$\sigma\text{-}\mathbf{Rin}(\{S\}) = \begin{cases} \{S, \emptyset\} & \text{if } S \neq \emptyset \\ \{\emptyset\} & \text{if } S = \emptyset. \end{cases}$$

Our estimate then reads:

$$\text{card}(\sigma\text{-}\mathbf{Rin}(S)) \leq \text{card}(\mathcal{S})^{\aleph_0} .$$

Indeed, the assumption that $\text{card}(\mathcal{S}) \geq 2$, in conjunction with consideration of the ways in which the sets $\cup S_i \in \mathcal{S}_1$ can be formed (at most $\text{card}(\mathcal{S})^2$ choices for each S_i), leads at once to the conclusion that $\text{card}(\mathcal{S}_1) \leq (\text{card}(\mathcal{S})^2)^{\aleph_0} = \text{card}(\mathcal{S})^{\aleph_0}$. Utilizing now transfinite induction, suppose that

$\text{card}(\mathcal{S}_\beta) \leq \text{card}(\mathcal{S})^{\aleph_0}$ for all β such that $1 \leq \beta < \alpha$, where $1 < \alpha < \Omega$ -- then

$$\text{card}(\bigcup_{\beta<\alpha} \mathcal{S}_\beta) \leq \aleph_0 \cdot \text{card}(\mathcal{S})^{\aleph_0} = \text{card}(\mathcal{S})^{\aleph_0}$$

and so, arguing as above, it follows that $\text{card}(\mathcal{S}_\alpha) \leq \text{card}(\mathcal{S})^{\aleph_0}$. Consequently, for every α with $0 \leq \alpha < \Omega$, $\text{card}(\mathcal{S}_\alpha) \leq \text{card}(\mathcal{S})^{\aleph_0}$. All told, therefore,

$$\begin{aligned}&\text{card}(\sigma\text{-}\mathbf{Rin}(\mathcal{S}))\\ &\quad = \text{card}(\bigcup_{\alpha<\Omega} \mathcal{S}_\alpha) \leq \aleph_1 \cdot \text{card}(\mathcal{S})^{\aleph_0}\\ &\quad \leq 2^{\aleph_0} \cdot \text{card}(\mathcal{S})^{\aleph_0} = \text{card}(\mathcal{S})^{\aleph_0}.\end{aligned}$$

[Note: If $\mathcal{S}$ is finite, then, of course, $\sigma\text{-}\mathbf{Rin}(\mathcal{S})$ is finite, there being the estimate

$$\#(\sigma\text{-}\mathbf{Rin}(\mathcal{S})) \leq 2^{2^{\#(\mathcal{S})}}$$

which is even attainable under the obvious conditions.]

Example Let X be a topological space with weight $\aleph_0$ -- then the cardinality of the class of Borel sets in X cannot exceed the cardinality of the continuum. In fact, the cardinality in question is the same as that of the σ-ring generated by the open sets and this cannot exceed $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. Specialize and suppose in addition that X is a metric space which is complete and perfect, so that $\text{card}(X) = \mathfrak{c}$. Because there are then $\mathfrak{c}$ open sets, the cardinality of the class of Borel sets in X is exactly $\mathfrak{c}$, thus is $< 2^{\mathfrak{c}}$, the cardinality of $\mathcal{P}(X)$.

Let ★ be the property : $? = ?_\sigma$ and $? = ?_\delta$. It is clear that ★ is extensionally attainable. Given any nonempty $\mathcal{S}$, we then write $\mathcal{S}_B$ for ★($\mathcal{S}$) and refer to $M_★$ as operation B. Obviously, $\mathcal{S}_B = \mathcal{S}_{BB}$ and

$$\begin{cases} \mathcal{S}_\sigma \subset \mathcal{S}_B \\ \mathcal{S}_\delta \subset \mathcal{S}_B \end{cases} \qquad \begin{cases} \mathcal{S}_B = \mathcal{S}_{B\sigma} = \mathcal{S}_{\sigma B} \\ \mathcal{S}_B = \mathcal{S}_{B\delta} = \mathcal{S}_{\delta B} \end{cases} .$$

The topological interpretation of $\mathcal{S}_B$ is very simple. Indeed, $\mathcal{S}_B$ is nothing more nor less than the closure in $\mathcal{P}(X)_s$ of $\mathcal{S}_{sd} = \mathcal{S}_{ds}$, thus, in particular, is the closure of $\mathcal{L}at(\mathcal{S})$ in $\mathcal{P}(X)_s$ if $\emptyset \in \mathcal{S}$.

[Note: The reader will agree that the closure of $\mathcal{S}$ itself in $\mathcal{P}(X)_s$ will, in general, be a proper subset of $\mathcal{S}_B$.]

There is an equally straightforward transfinite description of $\mathcal{S}_B$. Namely, put $\mathcal{B}^{(0)}(\mathcal{S}) = \mathcal{S}$, $\mathcal{B}_{(0)}(\mathcal{S}) = \mathcal{S}$ and define via transfinite recursion the classes $\mathcal{B}^{(\alpha)}(\mathcal{S})$, $\mathcal{B}_{(\alpha)}(\mathcal{S})$ by writing

$$\begin{cases} \mathfrak{B}^{(\alpha)}(\mathfrak{S}) = [\bigcup_{\beta<\alpha} \mathfrak{B}_{(\beta)}(\mathfrak{S})]_\sigma \\ \mathfrak{B}_{(\alpha)}(\mathfrak{S}) = [\bigcup_{\beta<\alpha} \mathfrak{B}^{(\beta)}(\mathfrak{S})]_\delta \end{cases} \quad (\alpha<\Omega) \quad .$$

The $\mathfrak{B}^{(\alpha)}(\mathfrak{S})$, $\mathfrak{B}_{(\alpha)}(\mathfrak{S})$ evidently increase with α and for $\begin{cases} \alpha \geqq 1 \\ \alpha > 1 \end{cases}$

$$\begin{cases} [\mathfrak{B}^{(\alpha)}(\mathfrak{S})]_\sigma = \mathfrak{B}^{(\alpha)}(\mathfrak{S}) \\ [\mathfrak{B}^{(\alpha)}(\mathfrak{S})]_d = \mathfrak{B}^{(\alpha)}(\mathfrak{S}), \end{cases}$$

$$\begin{cases} [\mathfrak{B}_{(\alpha)}(\mathfrak{S})]_\delta = \mathfrak{B}_{(\alpha)}(\mathfrak{S}) \\ [\mathfrak{B}_{(\alpha)}(\mathfrak{S})]_s = \mathfrak{B}_{(\alpha)}(\mathfrak{S}). \end{cases}$$

In addition, if

$$\mathfrak{B}(\mathfrak{S}:\alpha) = \mathfrak{B}^{(\alpha)}(\mathfrak{S}) \cap \mathfrak{B}_{(\alpha)}(\mathfrak{S}),$$

then

$$\mathfrak{B}^{(\alpha)}(\mathfrak{S}) \cup \mathfrak{B}_{(\alpha)}(\mathfrak{S}) \subset \mathfrak{B}(\mathfrak{S}:\alpha+1).$$

Our hierarchy may be visualized as follows:

$$\mathfrak{S} \subset \mathfrak{B}(\mathfrak{S}:1) \begin{array}{c} \subset \; \mathfrak{B}^{(1)}(\mathfrak{S}) \; \subset \\ \\ \subset \; \mathfrak{B}_{(1)}(\mathfrak{S}) \; \subset \end{array} \mathfrak{B}(\mathfrak{S}:2) \begin{array}{c} \subset \; \mathfrak{B}^{(2)}(\mathfrak{S}) \\ \\ \subset \; \mathfrak{B}_{(2)}(\mathfrak{S}) \end{array} \quad \cdots$$

Lemma 3 *We have*

$$\mathcal{S}_B = \begin{cases} \bigcup_{\alpha<\Omega} \mathcal{B}^{(\alpha)}(\mathcal{S}) \\ \\ \bigcup_{\alpha<\Omega} \mathcal{B}_{(\alpha)}(\mathcal{S}). \end{cases}$$

[One need only imitate the argument used in the proof of Lemma 2.]

There is a variant on the preceding definitions which is frequently encountered in the literature. To describe it, let us recall that any ordinal α can be written uniquely in the form $\alpha = \lambda+n$, where λ is a limit ordinal or zero and n is a nonnegative integer (α then being termed *odd* or *even* according to the parity of n). This being so, put $\mathcal{B}^{[0]}(\mathcal{S}) = \mathcal{S}$, $\mathcal{B}_{[0]}(\mathcal{S}) = \mathcal{S}$, and define via transfinite recursion the classes $\mathcal{B}^{[\alpha]}(\mathcal{S})$, $\mathcal{B}_{[\alpha]}(\mathcal{S})$ by writing

$$\mathcal{B}^{[\alpha]}(\mathcal{S}) = \begin{cases} \mathcal{B}^{(\alpha)}(\mathcal{S}) & \text{if } \alpha \text{ is odd} \\ \mathcal{B}_{(\alpha)}(\mathcal{S}) & \text{if } \alpha \text{ is even,} \end{cases} \qquad (\alpha<\Omega)$$

$$\mathcal{B}_{[\alpha]}(\mathcal{S}) = \begin{cases} \mathcal{B}^{(\alpha)}(\mathcal{S}) & \text{if } \alpha \text{ is even} \\ \mathcal{B}_{(\alpha)}(\mathcal{S}) & \text{if } \alpha \text{ is odd.} \end{cases} \qquad (\alpha<\Omega)$$

Then it is again the case that

$$\mathcal{S}_{B} = \begin{cases} \bigcup_{\alpha<\Omega} \mathcal{B}^{[\alpha]}(\mathcal{S}) \\ \bigcup_{\alpha<\Omega} \mathcal{B}_{[\alpha]}(\mathcal{S}). \end{cases}$$

Note too that if for some $\alpha \geqq 1$, $\mathcal{B}^{[\alpha]}(\mathcal{S}) = \mathcal{B}^{[\alpha+1]}(\mathcal{S})$ (or $\mathcal{B}_{[\alpha]}(\mathcal{S}) = \mathcal{B}_{[\alpha+1]}(\mathcal{S})$), then $\mathcal{B}^{[\alpha]}(\mathcal{S}) = \mathcal{S}_{B}$ (or $\mathcal{B}_{[\alpha]}(\mathcal{S}) = \mathcal{S}_{B}$). For of the two classes $\mathcal{B}^{[\alpha]}(\mathcal{S})$ and $\mathcal{B}^{[\alpha+1]}(\mathcal{S})$ (or $\mathcal{B}_{[\alpha]}(\mathcal{S})$ and $\mathcal{B}_{[\alpha+1]}(\mathcal{S})$), one is closed under countable unions while the other is closed under countable intersections, hence, when they coincide, $\mathcal{B}^{[\alpha]}(\mathcal{S})$ (or $\mathcal{B}_{[\alpha]}(\mathcal{S})$) must give $\mathcal{S}_{B}$.

Example. By the *Kolmogoroff number* $K(\mathcal{S})$ of $\mathcal{S}$, we understand the smallest ordinal α such that $\mathcal{B}^{[\alpha]}(\mathcal{S}) = \mathcal{S}_{B}$. The apparent asymmetry in the definition is, of course, essentially illusory. There are initial and terminal possibilities, namely, if $\mathcal{S} = \mathcal{S}_{B}$ to begin with, then $K(\mathcal{S}) = 0$, whereas, if $\mathcal{B}^{[\alpha]}(\mathcal{S}) \neq \mathcal{S}_{B}$ $\forall \alpha<\Omega$, then we agree to take $K(\mathcal{S}) = \Omega$. Two problems can then be posed.

(1) Given $\mathcal{S}$, determine $K(\mathcal{S})$.

(2) Given α, find an $\mathcal{S}$ such that $K(\mathcal{S}) = \alpha$.

Here, we shall deal with the second, setting aside systematic consideration of the first for now. Let us mention in passing, however, that examples for which

$K(\mathcal{S}) = \Omega$ do in fact abound, the simplest instance being the case when $\mathcal{S}$ is the class of all open (or closed) subintervals of the line. In Exer. 14 (§1), it was pointed out that there exist easy examples of classes $\mathcal{S}$ such that $K(\mathcal{S}) = 0, 1,$ and 2, but to get an example when $K(\mathcal{S}) = 3$ turned out to be surprisingly difficult, at least if one works on the line, the point being that the classical solution utilizes the continuum hypothesis (but see the paper of Malyšev referenced below). Actually, operating within ZFC alone, it is possible to give a complete answer to (2) in that $\forall \alpha < \Omega$, there exists a nonempty set X and a nonempty class $\mathcal{S}$ contained in $\mathcal{P}(X)$ such that $K(\mathcal{S}) = \alpha$. While interesting, we shall forgo the details, settling instead for an indication. To begin with, it is best to generalize the problem, replacing $\mathcal{P}(X)$ by a complete Boolean algebra $\mathbf{A}$ and then introducing a notion of Kolmogoroff number $K(\mathbf{A})$ for $\mathbf{A}$. This done, the crucial step in the argument consists in proving that $\forall \alpha < \Omega$, there exists a complete Boolean algebra $\mathbf{A}$ satisfying the countable chain condition with $K(\mathbf{A}) = \alpha$. Thanks to the Loomis-Sikorski theorem, any σ-complete Boolean algebra is isomorphic to a σ-algebra of subsets of some set X modulo a σ-ideal. Accordingly, $\mathbf{A}$ can be represented as a certain quotient per a certain X and finally, using the fact that $K(\mathbf{A}) = \alpha$, one produces without difficulty a subset $\mathcal{S}$ of $\mathcal{P}(X)$ with the property that $K(\mathcal{S}) = \alpha$.

[This result is due to Kunen; cf. A. Miller, Ann. Math. Logic, **16**(1979), pp. 233-267.]

For a fairly simple example of a class $\mathcal{S}$ such that $K(\mathcal{S}) = 3$ (and not involving the continuum hypothesis), see V. Malyšev [В. Мальшев], Vestnik Moskov. Univ. Ser.I Mat. Meh., 1965(no.6), pp.8-10.

On the basis of the definitions,

$$\mathcal{S}_B \subset \sigma\text{-}\mathbf{Rin}(\mathcal{S}),$$

the containment being strict in general. Indeed, $\sigma\text{-}\mathbf{Rin}(\mathcal{S}) = \mathcal{S}_{rB}$ but it need not be true that $\mathcal{S}_{rB} = \mathcal{S}_{Br}$, say. For example, take $X = \{1,2,3\}$ and let $\mathcal{S} = \{\{1\}, \{1,2\}, \{1,2,3\}\}$ -- then $\{1,3\} \in \mathcal{S}_{rB} - \mathcal{S}_{Br}$.

[Note: It can even happen that all the classes $\mathcal{S}_B$, $\mathcal{S}_{Br}$, $\mathcal{S}_{Brr}$, . . . are distinct.]

Lemma 4 $\mathcal{S}_B = \sigma\text{-}\mathbf{Rin}(\mathcal{S})$ iff $\mathcal{S}_r \subset \mathcal{S}_B$.

[The necessity is clear. As for the sufficiency, observe that

$$\mathcal{S}_r \subset \mathcal{S}_B \Rightarrow \mathcal{S}_{rB} \subset \mathcal{S}_{BB} = \mathcal{S}_B$$
$$\Rightarrow \sigma\text{-}\mathbf{Rin}(\mathcal{S}) \subset \mathcal{S}_B.]$$

Example Let X be a topological space -- then, traditionally, one writes

$$\begin{cases} \mathcal{F} = \text{closed subsets of } X \\ \mathfrak{G} = \text{open subsets of } X, \end{cases}$$

the classical resolutions

$$\begin{cases} \mathcal{F}_B = \bigcup_{\alpha<\Omega} B^{[\alpha]}(\mathcal{F}) \equiv \mathcal{F}_\alpha \\ \mathfrak{G}_B = \bigcup_{\alpha<\Omega} B_{[\alpha]}(\mathfrak{G}) \equiv \mathfrak{G}_\alpha \end{cases}$$

then being

$$\begin{cases} \mathcal{F} \subset \mathcal{F}_\sigma \subset \mathcal{F}_{\sigma\delta} \cdots \\ \mathfrak{G} \subset \mathfrak{G}_\delta \subset \mathfrak{G}_{\delta\sigma} \cdots \end{cases}.$$

The associated σ-rings (actually σ-algebras)

$$\begin{cases} \sigma\text{-}\mathfrak{Rin}(\mathcal{F}) \\ \sigma\text{-}\mathfrak{Rin}(\mathfrak{G}) \end{cases}$$

are equal, yielding, by definition, the Borel subsets of X. We then ask: Is

$$\begin{cases} \mathcal{F}_B = \sigma\text{-}\mathfrak{Rin}(\mathcal{F})? \\ \mathfrak{G}_B = \sigma\text{-}\mathfrak{Rin}(\mathfrak{G})? \end{cases}$$

Thanks to Lemma 4, these questions are equivalent, i.e.,

$$\mathcal{F}_B = \sigma\text{-}\mathfrak{Rin}(\mathcal{F}) \Longleftrightarrow \mathfrak{G}_B = \sigma\text{-}\mathfrak{Rin}(\mathfrak{G}).$$

To be specific, we shall work with $\mathfrak{G}$ -- then, in decreasing order of strength, the relation

$$\mathfrak{G}_B = \sigma\text{-}\mathfrak{Rin}(\mathfrak{G})$$

is forced by the following conditions.

(C_1) Every closed subset of X is in $\mathfrak{G}_\delta$.

(C_2) Every closed subset of X is in $\mathfrak{G}_\alpha$ for some fixed α.

(C_3) Every closed subset of X is in some $\mathfrak{G}_\alpha$ for some α, but no fixed α suffices.

Are there topological spaces X satisfying these conditions? It is easy to meet (C_1): Simply require that X be perfectly normal (in particular, a metric space); Exer. 28 is also relevant. Turning to (C_2), we claim that $\forall\alpha(1<\alpha<\Omega)$ there exists a topological space X_α such that α is the smallest ordinal for which every closed subset of X_α is in $\mathfrak{G}_\alpha$. Here is the construction. Fix α, $1<\alpha<\Omega$ -- then, as a consequence of certain generalities established in § , there exists a subset S_α of $X = \underset{\sim}{R}$ (usual topology) which is in $\mathfrak{G}_\alpha$ but is not in $\mathfrak{G}_\beta$ for any $\beta<\alpha$. This being so, take for our space X_α the real line topologized by specifying that the open sets are to be all sets of the form $U \cup V$, where U is open in the usual topology and V is any subset of $X_\alpha - S_\alpha$. It is not difficult to see that X_α is normal and Hausdorff, and has the required properties. As for (C_3), it is in fact possible to construct an example having the requisite property, at least if the continuum hypothesis is admitted (cf. Problem. (§)). There is one final point to be considered: Do there exist examples of topological spaces X such that

$$\mathfrak{G}_B \neq \sigma\text{-}\mathfrak{Rin}(\mathfrak{G})?$$

The answer is an emphatic "yes"! Consider

$$X = [0,\Omega] \quad \text{in the order topology}$$

or

$$X = [0,1]^{[0,1]} \quad \text{in the product topology.}$$

In the first case, $\{\Omega\}$, while closed, is not in $\mathfrak{G}_B$; in the second case, $\{c\}$ (c a constant), while closed, is not in $\mathfrak{G}_B$. Note that in both cases, X is a compact Hausdorff space.

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$ -- then by $\mathcal{S}_{s_d}$ ($\mathcal{S}_{\sigma_d}$) we understand the class of subsets of X comprised of all nonempty finite (finite or countable) disjoint unions of sets in $\mathcal{S}$.

Example Let $\mathcal{S}$ be a semiring -- then

$$\mathbf{Rin}(\mathcal{S}) = \mathcal{S}_{s_d},$$

but $\mathcal{S}_{\sigma_d}$ need not be a ring (cf. Exer. 6 (§4)).

Given any nonempty $\mathcal{S}$, the notions of generated algebra and σ-algebra are clear, as are the notations $\mathbf{Alg}(\mathcal{S})$ and $\sigma\text{-}\mathbf{Alg}(\mathcal{S})$. We have

$$\begin{cases} \mathbf{Rin}(\mathcal{S}) \subset \mathbf{Alg}(\mathcal{S}) \\ \sigma\text{-}\mathbf{Rin}(\mathcal{S}) \subset \sigma\text{-}\mathbf{Alg}(\mathcal{S}), \end{cases}$$

with

$$\begin{cases} \mathbf{Alg}(\mathcal{S}) = \{S : S \in \mathbf{Rin}(\mathcal{S}) \quad \text{or} \quad X - S \in \mathbf{Rin}(\mathcal{S})\} \\ \sigma\text{-}\mathbf{Alg}(\mathcal{S}) = \{S : S \in \sigma\text{-}\mathbf{Rin}(\mathcal{S}) \quad \text{or} \quad X - S \in \sigma\text{-}\mathbf{Rin}(\mathcal{S})\}, \end{cases}$$

that is,

$$\begin{cases} \mathbf{Alg}(\mathcal{S}) = \mathbf{Rin}(\mathcal{S}, \{X\}) \\ \sigma\text{-}\mathbf{Alg}(\mathcal{S}) = \sigma\text{-}\mathbf{Rin}(\mathcal{S}, \{X\}). \end{cases}$$

On algebraic grounds alone, it is plain that

$$\mathbf{Alg}(\mathcal{S}) = [\mathcal{S} \cup \mathcal{S}_c]_{ds}.$$

Actually, slightly more is true, viz.

$$\mathbf{Alg}(\mathcal{S}) = [\mathcal{S} \cup \mathcal{S}_c]_{ds_d}.$$

Topologically, σ-$\mathbf{Alg}(\mathcal{S})$ can be viewed as the closure of $\mathbf{Alg}(\mathcal{S})$ in $\mathcal{P}(X)_s$. On the other hand, thanks to Lemma 4,

$$[\mathcal{S} \cup \mathcal{S}_c]_B = \sigma\text{-}\mathbf{Alg}(\mathcal{S}),$$

leading, thereby, to the attendent transfinite descriptions.

Let $\star$ be the property $:? = ?_{\sigma_d}$ and $? = ?_\delta$. It is clear that $\star$ is extensionally attainable. Given any nonempty $\mathcal{S}$, we then write $\mathcal{S}_{B_d}$ for $\star(\mathcal{S})$ and refer to $M_\star$ as operation B_d. Obviously, $\mathcal{S}_{B_d} = \mathcal{S}_{B_d B_d}$ and

$$\begin{cases} \mathcal{S}_{\sigma_d} \subset \mathcal{S}_{B_d} \\ \mathcal{S}_\delta \subset \mathcal{S}_{B_d} \end{cases} \qquad \begin{cases} \mathcal{S}_{B_d} = \mathcal{S}_{B_d \sigma_d} = \mathcal{S}_{\sigma_d B_d} \\ \mathcal{S}_{B_d} = \mathcal{S}_{B_d \delta} = \mathcal{S}_{\delta B_d} \end{cases},$$

with

$$\mathcal{S}_{B_d} \subset \mathcal{S}_B,$$

the containment being strict in general, as can be seen by taking $X = \{1,2,3\}$ and letting $\mathcal{S} = \{\{1\}, \{1,2\}, \{1,3\}\}$. We shall leave it up to the reader to discuss the transfinite aspects of operation B_d.

<u>Lemma 5</u> $\mathcal{S}_{B_d} = \sigma\text{-Alg}(\mathcal{S})$ <u>iff</u> $\mathcal{S}_c \subset \mathcal{S}_{B_d}$.

[The necessity is clear. As for the sufficiency, observe that

$$\begin{aligned} &\mathcal{S}_c \subset \mathcal{S}_{B_d} \\ &\implies [\mathcal{S}_{B_d} \cap \mathcal{S}_{B_d}{}^c]_B = \mathcal{S}_{B_d} \cap \mathcal{S}_{B_d}{}^c \\ &\implies [\mathcal{S} \cup \mathcal{S}_c]_B \subset \mathcal{S}_{B_d} \\ &\implies \sigma\text{-Alg}(\mathcal{S}) \subset \mathcal{S}_{B_d}.] \end{aligned}$$

<u>Example</u> Let X be a topological space -- then by a <u>zero set</u> in X we mean any set of the form $f^{-1}(0)$, where $f:X \to \mathbb{R}$ is continuous. In this connection, observe that it is not restrictive to suppose that $f(X) \subset [0,1]$. The complements in X of the zero sets are called the <u>cozero sets</u>. Agreeing to write

$$\mathcal{Z} = \text{zero sets in } X,$$

we have $\mathcal{Z} \subset \mathcal{F}$, the containment being strict in general (cf. Exer. 30), but there being coincidence if, e.g., X is perfectly normal. Note that

(1) $\emptyset, X \in \mathcal{Z}$, (2) $\mathcal{Z} = \mathcal{Z}_s$,

(3) $\mathcal{Z} = \mathcal{Z}_\delta$, (4) $\mathcal{Z}_c \subset \mathcal{Z}_\sigma$.

In addition, given disjoint $Z_1, Z_2 \in \mathcal{Z}$, there exist disjoint $U_1, U_2 \in \mathcal{Z}_c$ such that

$$Z_1 \subset U_1, \quad Z_2 \subset U_2 \quad .$$

This said, the Baire sets in X are by definition the elements of the σ-algebra $\mathbf{Ba}(X)$ generated by $\mathcal{Z}$. Every Baire set is a Borel set but, in general, not vice-versa (cf. Exer. 32). Owing to Lemma 4 and property (4) supra,

$$\mathcal{Z}_B = \mathbf{Ba}(X).$$

Because

$$\mathcal{Z}_c \subset \mathcal{Z}_\sigma \quad \text{iff} \quad \mathcal{Z} \subset \mathcal{Z}_{c\delta},$$

it follows from Lemma 5 that

$$(\mathcal{Z}_c)_{B_d} = \mathbf{Ba}(X).$$

It is also true that

$$\mathcal{Z}_{B_d} = \mathbf{Ba}(X),$$

although this is not immediate. On the basis of Lemma 5 again, our assertion is equivalent to the statement that $\mathcal{Z}_c \subset \mathcal{Z}_{B_d}$.

Claim Take $X = \underline{R}$ -- then

$$\mathcal{Z}_c \subset \mathcal{Z}_{\sigma_d \delta \sigma_d} \; .$$

[To appreciate the subtlety of this point, the reader may find it instructive to prove directly that $]a,b[$ $(a<b)$ does not belong to $\mathcal{Z}_{\sigma_d \delta}$!]

Admit the claim -- then, for any topological space X,

$$\mathcal{Z}_c \subset \mathcal{Z}_{\sigma_d \delta \sigma_d}$$

and, consequently, $\mathcal{Z}_c \subset \mathcal{Z}_{B_d}$, as desired. Indeed, if $U \in \mathcal{Z}_c$, then there exists a continuous function $f:X \to [0,1[$ such that

$$U = \{x \in X: f(x) \in]0,1[\}.$$

Now, in view of the claim, $]0,1[\in \mathcal{Z}_{\sigma_d \delta \sigma_d}$ (per $\underset{\sim}{R}$), and so

$$U = f^{-1}(]0,1[) \in \mathcal{Z}_{\sigma_d \delta \sigma_d} .$$

Proof of Claim Let $U \in \mathcal{Z}_c$ -- then U is open, hence is a finite or countable union of open, pairwise disjoint intervals. Accordingly, there is no loss of generality in supposing that $U =]a,b[$ $(a<b)$. Let $\{I_m\}$ be a sequence of closed, pairwise disjoint intervals in $\underset{\sim}{R}$ whose union is dense in $]a,b[$.

Put

$$S =]a,b[- \cup I_m.$$

Then the closure $\bar{S}$ of S in $\underset{\sim}{R}$ is a closed, nowhere dense subset of $[a,b]$, and $\bar{S} - S$ is a countable set $\{x_n\}$ consisting of a,b and the endpoints of the I_m. Since $]a,b[$ is

$$S \cup I_1 \cup I_2 \cdot \cdot \cdot,$$

the union being countable and disjoint, and

$$S = \cap(\bar{S} - \{x_1,\ldots,x_n\}),$$

it will be enough to prove that $\bar{S} - \{x_1,\ldots,x_n\} \in \mathcal{Z}_{\sigma_d}$. However, because $\bar{S}$ is nowhere dense, thus 0-dimensional, one can certainly write $\bar{S} - \{x_1,\ldots,x_n\}$ as a countable disjoint union of sets which are closed in $\bar{S}$, and so too in $\underset{\sim}{R}$.

[Note: Suppose that X is a perfectly normal topological space -- then, of course,

$$\mathcal{F} = \mathcal{G}_c \subset \mathcal{G}_\delta \Longrightarrow \mathcal{G}_{B_d} = \sigma\text{-}\mathfrak{Alg}(\mathcal{G}).$$

Furthermore, in this case,

$$\mathcal{G} = \mathcal{F}_c = \mathcal{Z}_c \subset \mathcal{Z}_{\sigma_d\delta\sigma_d} = \mathcal{F}_{\sigma_d\delta\sigma_d}$$
$$\Longrightarrow \mathcal{F}_{B_d} = \sigma\text{-}\mathfrak{Alg}(\mathcal{F}).$$

Here, therefore,

$$\mathfrak{Ba}(X) = \begin{cases} \mathcal{F}_{B_d} \\ \\ \mathcal{G}_{B_d} \end{cases}.$$

We remark, in passing, that perfect normality, while sufficient, is not necessary in order to draw these conclusions (cf. Exer. 33).]

Suppose that $\mathcal{S}$ is a σ-lattice containing X. Put

$$\Sigma_0(\mathcal{S}) = \mathcal{S}, \qquad \Pi_0(\mathcal{S}) = \mathcal{S}_c$$

and define via transfinite recursion the classes $\Sigma_\alpha(\mathcal{S})$, $\Pi_\alpha(\mathcal{S})$ by writing

$$\begin{cases} \Sigma_\alpha(\mathcal{S}) = [\bigcup_{\beta<\alpha} \Pi_\beta(\mathcal{S})]_\sigma \\ \Pi_\alpha(\mathcal{S}) = [\bigcup_{\beta<\alpha} \Sigma_\beta(\mathcal{S})]_\delta \end{cases} \quad (\alpha<\Omega) .$$

If $\beta<\alpha$, then

$$\begin{cases} \Sigma_\beta(\mathcal{S}) \subset \Pi_\alpha(\mathcal{S}) \\ \Pi_\beta(\mathcal{S}) \subset \Sigma_\alpha(\mathcal{S}), \end{cases}$$

and if $\alpha>1$ and $\beta<\alpha$, then

$$\begin{cases} \Sigma_\beta(\mathcal{S}) \subset \Sigma_\alpha(\mathcal{S}) \\ \Pi_\beta(\mathcal{S}) \subset \Pi_\alpha(\mathcal{S}). \end{cases}$$

Therefore

$$\bigcup_{\alpha<\Omega} \Sigma_\alpha(\mathcal{S}) = \bigcup_{\alpha<\Omega} \Pi_\alpha(\mathcal{S}),$$

the σ-algebra generated by $\mathcal{S}$. Note too that

$$\begin{cases} \Sigma_\alpha(\mathcal{S}) \text{ is a } \sigma\text{-lattice} \\ \Pi_\alpha(\mathcal{S}) \text{ is a } \delta\text{-lattice} \end{cases}$$

with

$$\begin{cases} \Sigma_\alpha(\mathcal{S}) = [\Pi_\alpha(\mathcal{S})]_c \\ \Pi_\alpha(\mathcal{S}) = [\Sigma_\alpha(\mathcal{S})]_c. \end{cases}$$

It is customary to refer to the sets in

$$\begin{cases} \Sigma_\alpha(\mathcal{S}) \\ \\ \Pi_\alpha(\mathcal{S}) \end{cases} \quad \text{as} \quad \begin{cases} \text{\underline{additive of class } } \alpha \\ \\ \text{\underline{multiplicative of class } } \alpha, \end{cases}$$

the sets in the intersection

$$\Delta_\alpha(\mathcal{S}) = \Sigma_\alpha(\mathcal{S}) \cap \Pi_\alpha(\mathcal{S})$$

then being <u>ambiguous of class α</u>. Evidently, $\Delta_\alpha(\mathcal{S})$ is an algebra.

Our hierarchy may be visualized as follows:

$$\Delta_1(\mathcal{S}) \begin{array}{c} \subset \\ \\ \subset \end{array} \begin{array}{c} \Sigma_1(\mathcal{S}) \\ \\ \Pi_1(\mathcal{S}) \end{array} \begin{array}{c} \subset \\ \\ \subset \end{array} \Delta_2(\mathcal{S}) \begin{array}{c} \subset \\ \\ \subset \end{array} \begin{array}{c} \Sigma_2(\mathcal{S}) \\ \\ \Pi_2(\mathcal{S}) \end{array} \quad \cdots$$

[Note: It need not be true, of course, that

$$\begin{cases} \Sigma_0(\mathcal{S}) \subset \Sigma_1(\mathcal{S}) \\ \\ \Pi_0(\mathcal{S}) \subset \Pi_1(\mathcal{S}) \end{cases} .$$

However, the assumption $\mathcal{S} \subset \mathcal{S}_{c\sigma}$ would guarantee this.]

<u>Examples</u> (1) Let X be a topological space -- then the preceding considerations are applicable with $\mathcal{S} = \mathfrak{G}$, the associated σ-algebra being $\mathfrak{Bo}(X)$.

(2) Let X be a topological space -- then the preceding considerations are applicable with $\mathcal{S} = \mathfrak{Z}_c$, the associated σ-algebra being $\mathfrak{Ba}(X)$.

For each $\alpha<\Omega$, put

$$\Lambda_\alpha(\mathcal{S}) = u_\alpha(\mathcal{S}).$$

Lemma 6 Suppose that $\mathcal{S} \subset \mathcal{S}_{c\sigma}$ -- then

$$\begin{cases} \Lambda_{2n}(\mathcal{S}) = \Sigma_{2n}(\mathcal{S}) & (n = 0,1,2,\ldots) \\ \Lambda_{2n+1}(\mathcal{S}) = \Pi_{2n+1}(\mathcal{S}) & (n = 0,1,2,\ldots) \end{cases}$$

and

$$\Lambda_\alpha(\mathcal{S}) = \Delta_{\alpha+1}(\mathcal{S}) \quad (\omega \leqq \alpha < \Omega).$$

[Note: There is also a dual result whereby, working with $\mathcal{S}_c$ (instead of $\mathcal{S}$), one picks off $\Sigma_{odd}(\mathcal{S})$ and $\Pi_{even}(\mathcal{S})$, the contention as regards the $\Delta_{\alpha+1}(\mathcal{S})$ being unchanged.]

The proof, while not difficult, is a bit lengthy.

We shall deal first with the case of finite n. If $n = 0$, then, by definition, $\Lambda_0(\mathcal{S}) = \mathcal{S} = \Sigma_0(\mathcal{S})$. On the other hand, if $S \in \Lambda_1(\mathcal{S})$, then $S = \lim S_i$, where $\{S_i\}$ is a sequence in $\mathcal{S}$. In particular:

$$S = \overline{\lim}\, S_i = \bigcap_{i=1}^{\infty} \Big(\bigcup_{m=i}^{\infty} S_m\Big).$$

Because $\mathcal{S}$ is a σ-lattice, $\bigcup_{m=i}^{\infty} S_m \in \mathcal{S}$ $\forall i$, hence $S \in \Pi_1(\mathcal{S})$. To go the other way, take an $S \in \Pi_1(\mathcal{S})$ -- then

$$S = \cap S_i = \lim(S_1 \cap \ldots \cap S_i) \qquad (S_i \in \mathcal{S})$$

belongs to $\Lambda_1(\mathcal{S})$. Proceeding by induction, suppose now that $n \geq 0$ and that our assertion is true for n -- then it must be shown that

$$\begin{cases} \Lambda_{2n+2}(\mathcal{S}) = \Sigma_{2n+2}(\mathcal{S}) \\ \Lambda_{2n+3}(\mathcal{S}) = \Pi_{2n+3}(\mathcal{S}). \end{cases}$$

Let us consider the first of these relations, the argument for the second being similar. If $S \in \Lambda_{2n+2}(\mathcal{S})$, then $S = \lim S_i$, where $\{S_i\}$ is a sequence in

$$\Lambda_0(S) \cup \ldots \cup \Lambda_{2n+1}(\mathcal{S})$$

$$= \Lambda_{2n+1}(\mathcal{S}) = \Pi_{2n+1}(\mathcal{S}) \quad \text{(by induction)}.$$

In particular:

$$S = \underline{\lim}\, S_i = \bigcup_{i=1}^{\infty} \Big(\bigcap_{m=i}^{\infty} S_m\Big).$$

Because $\Pi_{2n+1}(\mathcal{S})$ is a δ-lattice, $\bigcap_{m=i}^{\infty} S_m \in \Pi_{2n+2}(\mathcal{S})$ $\forall i$, hence $S \in \Sigma_{2n+1}(\mathcal{S})$. To go the other way, take an $S \in \Sigma_{2n+2}(\mathcal{S})$ -- then

$$S = \cup S_i = \lim(S_1 \cup \ldots \cup S_i),$$

where

$$S_i \in \bigcup_{j<2n+2} \Pi_j(\mathcal{S})$$
$$= \Pi_{2n+1}(\mathcal{S}) = \Lambda_{2n+1}(\mathcal{S}) \qquad \text{(by induction)},$$

that is, S belongs to $\Lambda_{2n+2}(\mathcal{S})$.

Passing to the transfinite assertion, suppose initially that $\alpha = \omega$. If $S \in \Lambda_\omega(\mathcal{S})$, then $S = \lim S_i$, where $S_i \in \Lambda_{m_i}(\mathcal{S})$, say. The claim is that

$$\begin{cases} S \in \Sigma_{\omega+1}(\mathcal{S}) = [\Pi_\omega(\mathcal{S})]_\sigma \\ S \in \Pi_{\omega+1}(\mathcal{S}) = [\Sigma_\omega(\mathcal{S})]_\delta \end{cases}.$$

This, however, is immediate provided we take into account the relations

$$\begin{cases} \Lambda_{2n}(\mathcal{S}) = \Sigma_{2n}(\mathcal{S}) \subset \Pi_{2n+1}(\mathcal{S}) \\ \Lambda_{2n+1}(\mathcal{S}) = \Pi_{2n+1}(\mathcal{S}) \subset \Sigma_{2n+2}(\mathcal{S}) \end{cases}$$

and the fact that here

$$\bigcap_{i=1}^{\infty} \left(\bigcup_{m=i}^{\infty} S_m\right) = \bigcup_{i=1}^{\infty} \left(\bigcap_{m=i}^{\infty} S_m\right).$$

The other direction is slightly more complicated. Take an

$S \in \Delta_{\omega+1}(\mathcal{S})$ -- then there exist sequences $\{S'_{i,j}\}$, $\{S''_{i,j}\}$ with

$$\begin{cases} S'_{i,j} \in \Sigma_0(\mathcal{S}) \cup \Sigma_1(\mathcal{S}) \quad \ldots \\ S''_{i,j} \in \Pi_0(\mathcal{S}) \cup \Pi_1(\mathcal{S}) \cup \ldots \end{cases}$$

such that

$$S = \bigcup_i \bigcap_j S'_{i,j}$$
$$S = \bigcap_i \bigcup_j S''_{i,j} \quad .$$

Evidently, without loss of generality, it can be assumed that

$$S'_{i,j} \supset S'_{i,j+1}, \quad S''_{i,j} \subset S''_{i,j+1} \ .$$

Consequently (cf. Prob. I (§1)),

$$S = \lim((S'_{1,j} \cap S''_{1,j}) \cup (S'_{2,j} \cap S''_{1,j} \cap S''_{2,j}) \cup \ldots \cup (S'_{j,j} \cap S''_{1,j} \cap \ldots \cap S''_{j,j})).$$

Each term inside the limit sign belongs to $\Lambda_0(\mathcal{S}) \cup \Lambda_1(\mathcal{S}) \cup \ldots$, implying, therefore, that $S \in \Lambda_\omega(\mathcal{S})$. Proceeding by transfinite induction, suppose now that α is $>\omega$ and $<\Omega$ and that our assertion is true for $\omega \leq \beta < \alpha$. If $S \in \Lambda_\alpha(\mathcal{S})$, then $S = \lim S_i$, where $S_i \in \Lambda_{\alpha_i}(\mathcal{S})$, say $(\omega \leq \alpha_i < \alpha)$. Because

$$\Lambda_{\alpha_i}(\mathcal{S}) = \Delta_{\alpha_i+1}(\mathcal{S}) \qquad \text{(by induction)},$$

and $\alpha_i + 1 \leq \alpha$, each S_i belongs to $\Delta_\alpha(\mathcal{S})$, so the usual $\overline{\lim}, \underline{\lim}$

representation forces S into $\Delta_{\alpha+1}(\mathcal{S})$. To finish up, take an $S \in \Delta_{\alpha+1}(\mathcal{S})$ -- then, as above, there exist sequences $\{S'_{i,j}\}$, $\{S''_{i,j}\}$ with

$$\begin{cases} S'_{i,j} \in \Sigma_{\xi_{i,j}}(\mathcal{S}) & (\omega \leq \xi_{i,j} < \alpha) \\ S''_{i,j} \in \Pi_{\eta_{i,j}}(\mathcal{S}) & (\omega \leq \eta_{i,j} < \alpha) \end{cases}$$

such that

$$\begin{cases} S = \bigcup_i \bigcap_j S'_{i,j} \\ S = \bigcap_i \bigcup_j S''_{i,j}, \end{cases}$$

it not being restrictive to assume that

$$S'_{i,j} \supset S'_{i,j+1}, \quad S''_{i,j} \subset S''_{i,j+1}.$$

Let us distinguish two cases.

(A) α is an ordinal of the first kind, i.e., α possesses an immediate predecessor, say $\alpha = \beta+1$ -- then

$$\begin{cases} \omega \leq \xi_{i,j} \leq \beta \\ \omega \leq \eta_{i,j} \leq \beta \end{cases} \Longrightarrow S'_{i,j},\ S''_{i,j} \in \Delta_{\beta+1}(\mathcal{S})$$

$$\Longrightarrow$$

$$(S'_{1,j} \cap S''_{1,j}) \cup (S'_{2,j} \cap S''_{1,j} \cap S''_{2,j}) \cup \ldots \cup (S'_{j,j} \cap S''_{1,j} \cap \ldots \cap S''_{j,j}) \in \Delta_{\beta+1}(\mathcal{S}).$$

But

$$\Delta_\alpha(\mathcal{S}) = \Delta_{\beta+1}(\mathcal{S}) = \Lambda_\beta(\mathcal{S}) \qquad \text{(by induction)},$$

and so $S \in \Lambda_\alpha(\mathcal{S})$, as desired.

(B) α is an ordinal of the second kind, i.e., α possesses no immediate predecessor, thus is a limit ordinal, say $\alpha = \lambda$. Put

$$\zeta_j = \sup \begin{cases} \xi_{1,j}, \ldots, \xi_{j,j} \\ \eta_{1,j}, \ldots, \eta_{j,j} \end{cases} .$$

Then $\zeta_j < \lambda \ \forall j$

$\Longrightarrow$

$(S'_{1,j} \cap S''_{1,j}) \cup (S'_{2,j} \cap S''_{1,j} \cap S''_{2,j}) \cup \ldots \cup (S'_{j,j} \cap S''_{1,j} \cap \ldots \cap S''_{j,j}) \in \Delta_{\zeta_j + 1}(\mathcal{S})$.

However, as λ is a limit ordinal, $\zeta_j + 1 < \lambda \ \forall j$, hence

$$\Delta_{\zeta_j + 1}(\mathcal{S}) = \Lambda_{\zeta_j}(\mathcal{S}) \qquad \text{(by induction)}$$

from which it follows that $S \in \Lambda_\alpha(\mathcal{S})$, as desired.

The proof of Lemma 6 is therefore complete.

[Note: It must be stressed that the assumption $\mathcal{S} \subset \mathcal{S}_{c\sigma}$ is crucial for the validity of this result.]

Suppose still that $\mathcal{S} \subset \mathcal{S}_{c\sigma}$ -- then, thanks to Lemma 5,

$$\mathcal{S}_{B_d} = \sigma\text{-}\mathbf{Alg}(\mathcal{S}).$$

Furthermore, $\forall\alpha>0$:

$$\begin{cases} \mathcal{B}^{(\alpha)}(\Delta_1(\mathcal{S})) = \Sigma_\alpha(\mathcal{S}) \\ \mathcal{B}_{(\alpha)}(\Delta_1(\mathcal{S})) = \Pi_\alpha(\mathcal{S}). \end{cases}$$

We shall conclude this § with a brief discussion of relativization and localization.

Suppose that $\bigstar$ is extensionally attainable. Let X_0 be a subset of X -- then, given any nonempty $\mathcal{S}$, we ask: Is

$$\bigstar(\mathrm{tr}_{X_0}(\mathcal{S})) = \mathrm{tr}_{X_0}(\bigstar(\mathcal{S}))?$$

Generally, this need not be the case. But it will be true under the following assumptions:

(1) $\bigstar(\mathrm{tr}_{X_0}(\bigstar(\mathcal{S}))) = \mathrm{tr}_{X_0}(\bigstar(\mathcal{S}))$;

(2) $\{S \subset X : S \cap X_0 \in \bigstar(\mathrm{tr}_{X_0}(\mathcal{S}))\}$ is a $\bigstar$-class.

Indeed, from (1) we get that

$$\mathrm{tr}_{X_0}(\bigstar(\mathcal{S})) \supset \bigstar(\mathrm{tr}_{X_0}(\mathcal{S}))$$

whereas from (2) we get that

$$\mathrm{tr}_{X_0}(\bigstar(\mathcal{S})) \subset \bigstar(\mathrm{tr}_{X_0}(\mathcal{S})).$$

Evidently, the properties

$$\begin{cases} ? \text{ is a lattice} \\ ? \text{ is a ring } (\sigma\text{-ring, } \delta\text{-ring}) \end{cases}$$

are instances where conditions (1) and (2) are met.

Example Borel sets relativize. Thus, suppose that X is a topological space with ambient topology $\mathfrak{T}$. Let X_0 be a subset of X -- then, by definition, the class $\text{tr}_{X_0}(\mathfrak{T})$ is the relative topology on X_0, and, by the above, we have

$$\text{tr}_{X_0}(\mathfrak{Bo}(X)) = \sigma\text{-}\mathfrak{Rin}(\text{tr}_{X_0}(\mathfrak{T})) = \mathfrak{Bo}(X_0).$$

Example Baire sets need not relativize. To produce an example, we shall work within the Stone-Čech compactification $\beta\underset{\sim}{N}$ of $\underset{\sim}{N}$. Choose, as is possible, a class $\{S_i\}$ of $\mathfrak{c}$ infinite subsets of $\underset{\sim}{N}$ such that

$$\text{card}(S_i \cap S_j) < +\infty \qquad \forall i \neq j.$$

This done, call $\bar{S}_i$ the closure of S_i in $\beta\underset{\sim}{N}$ -- then the $\bar{S}_i - \underset{\sim}{N}$ are pairwise disjoint, open and closed subsets of $\beta\underset{\sim}{N} - \underset{\sim}{N}$. Put

$$S = \bigcup_i (\bar{S}_i - \underset{\sim}{N})$$

and consider the subspace $X = \underset{\sim}{N} \cup S$ of $\beta\underset{\sim}{N}$. Since $\beta\underset{\sim}{N} - \underset{\sim}{N}$ is a zero set in $\beta\underset{\sim}{N}$, S is a zero set, hence a Baire set in X. Now

$$\text{card}(\mathfrak{Ba}(X)) \leq 2^{\aleph_0},$$

X being separable. On the other hand, it is clear that

$$\text{card}(\mathcal{Ba}(S)) \geq 2^{2^{\aleph_0}} .$$

Accordingly, not every Baire set of S is a Baire set of X, and so here Baire sets do not relativize.

Under certain conditions, however, Baire sets will relativize. Thus, suppose that X is a topological space -- then a subspace X_0 of X is said to be *$\mathcal{Z}$-embedded* in X if $\forall$ zero set Z_0 in X_0 $\exists$ a zero set Z in X such that $Z_0 = Z \cap X_0$, i.e., if, in an obvious notation,

$$\text{tr}_{X_0}(\mathcal{Z}) = \mathcal{Z}_0.$$

But then

$$\text{tr}_{X_0}(\mathcal{Ba}(X)) = \sigma\text{-}\mathcal{Rin}(\text{tr}_{X_0}(\mathcal{Z})) = \mathcal{Ba}(X_0).$$

For orientation, let us consider some specific instances of $\mathcal{Z}$-embeddings.

(1) Let X be a completely regular, Hausdorff topological space -- then X is $\mathcal{Z}$-embedded in its Stone-Čech compactification βX.

[This follows from the definitions.]

(2) Let X be a normal topological space -- then every closed subset X_0 of X is $\mathcal{Z}$-embedded in X.

[Bear in mind the Tietze extension theorem.]

(3) Let X be a compact Hausdorff space -- then every Baire set X_0 of X is $\mathcal{Z}$-embedded in X.

[In fact, X_0 is necessarily Lindelöf.]

[Note: A systematic discussion of $\mathcal{Z}$-embedding may be found in R. Blair and A. Hager, Math. Z., **136**(1974), pp. 41-52.]

Let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$ -- then by the *localization* $\mathcal{S}_{loc}$ of $\mathcal{S}$ we mean the class consisting of all $X_0 \subset X$ for which

$$tr_{X_0}(\mathcal{S}) \subset \mathcal{S}.$$

Obviously, $X \in \mathcal{S}_{loc}$, so $\mathcal{S}_{loc}$ is nonempty. In addition, if $\mathcal{S}$ is multiplicative, then $\mathcal{S} \subset \mathcal{S}_{loc}$.

Suppose that $\mathcal{S}$ is a ring (σ-ring, δ-ring) -- then $\mathcal{S}_{loc}$ is an algebra (σ-algebra, δ-algebra).

Example Let X be a Hausdorff topological space. Let $\mathcal{K}$ be the class of all compact subsets of X -- then it is easy to see that

$$S \in [\mathcal{Ba}_b(X)]_{loc} \text{ iff } S \cap K \in \mathcal{Ba}(K) \ \forall K \in \mathcal{K}.$$

Consequently,

$$\mathcal{Ba}(X) \subset [\mathcal{Ba}_b(X)]_{loc} \quad ,$$

the containment being strict in general (cf. Exer. 40), but there being coincidence if, e.g., X is σ-compact.

Localization need not commute with generation.

Example In general,

$$\sigma\text{-}\mathbf{Rin}(\mathcal{S}_{loc}) \neq [\sigma\text{-}\mathbf{Rin}(\mathcal{S})]_{loc} .$$

Thus, take $X = \mathbf{N}$ and let $\mathcal{S} = \{\{n\}:n\in\mathbf{N}\}$ -- then $\mathcal{S}_{loc} = \{X\}$, hence, in this case,

$$\begin{aligned} \sigma\text{-}\mathbf{Rin}(\mathcal{S}_{loc}) &= \{\emptyset,X\} \\ &\neq \mathbf{P}(X) = [\sigma\text{-}\mathbf{Rin}(\mathcal{S})]_{loc} . \end{aligned}$$

Notes and Remarks

The term "extensionally attainable" has been borrowed from T. Hildebrandt, Introduction to the Theory of Integration, Academic Press, New York, 1963 (see p. 148). If $\star$ is an extensionally attainable property, then some authors would refer to $\star(\mathcal{S})$ as the $\star$-stabilization of $\mathcal{S}$. The generation of lattices and rings was discussed already by F. Hausdorff, Grundzüge der Mengenlehre, Veit & Comp., Leipzig, 1914 (see pp. 14-16). The transfinite approach to operation B has its origins in E. Borel's Lecons sur la Théorie des Fonctions, Gauthier-Villars, Paris, 1898 (see pp. 46-48), although this author evidently did not believe in transfinite numbers. The general formulation is due to F. Hausdorff (op. cit. pp. 304-306), further details and refinements being presented by him in Math. Ann., **77**(1916), pp. 430-437 and later on in his famous Mengenlehre, Walter de Gruyter, Berlin, 1927 (see pp. 85-90). The axiomatic approach to Borel sets in terms of a generated σ-ring was stressed by W. Sierpiński, <u>Bull. Acad. Sci. Cracovie</u>, **A**(1918), pp. 29-34. Lemmas 4 and 5 are results of Sierpiński; cf., respectively, <u>Annales Soc. Polon. Math.</u>, **6**(1927), pp. 50-53 and <u>Fund. Math</u>, **12**(1928), pp. 206-210. For an excellent account of the theory as it stood around 1930 and which is still very readable even now, consult H. Hahn, <u>Reelle Funktionen</u>, Akademische Verlagsgesellschaft M.B.H., Leipzig, 1932 (see pp. 258-276). Given $\alpha(1<\alpha<\Omega)$, the existence of a topological space X_α such that $\mathcal{F} \subset \mathcal{G}_\alpha$ was first noted by S. Willard, <u>Fund. Math.</u>, **71**(1971), pp. 187-191. The definition in the text of a Baire set is apparently due to E. Hewitt, <u>Fund. Math.</u>, **37**(1950), pp. 161-187. The reader is warned that while we consider the definitions in the text of Borel set and Baire set to be the most natural, other writers might use these terms for very different entities. Eg: In some treatments, the Borel sets in a Hausdorff topological space are taken to be the σ-ring generated by the compact sets, the Baire sets then being the σ-ring generated by the compact G_δ's. The fact that $\mathbf{Ba}(X)$ can be produced from $\mathcal{Z}$ by operation B_d was established by J. Jayne, <u>Mathematika</u>, **24**(1977), pp. 241-256. In this connection, it should be kept in mind that there is a theorem in general topology which says that no nonempty, open subset of a connected compact Hausdorff space X can be written as a countable disjoint union of nonempty, closed subsets of X; cf. K. Kuratowski, <u>Topology</u>, Vol. II, Academic Press, New York, 1968 (see p. 173). The origin of the notation $\Sigma_\alpha(\mathcal{S})$, $\Pi_\alpha(\mathcal{S})$ lies in recursive function theory; it was introduced by J. Addison, <u>Fund. Math.</u>, **46**(1959), pp. 123-135. The procedure itself, however, can be traced back to F. Hausdorff, <u>Math. Z.</u>, **5**(1919), pp. 292-309. Emphasis on the $\Lambda_\alpha(\mathcal{S})$ was placed by Ch. de la Vallée Poussin, <u>Intégrales de Lebesgue</u>, <u>Fonctions d'Ensemble</u>, <u>Classes de Baire</u>, Gauthier-Villars, Paris, 1916 (see p. 37). The connection between the two, i.e., Lemma 6, was found by W. Sierpiński, <u>Fund. Math.</u>, **19**(1932), pp. 257-264; see also J. Albuquerque, <u>Portugual. Math.</u>, **4**(1943-1945), pp. 161-198, pp. 217-224. The notion of localization appears explicitly in I. Segal, <u>Amer. J. Math.</u>, **73**(1951), pp. 275-313, although it is implicit in earlier writings. N. Dinculeanu, <u>Vector Measures</u>, Pergamon Press, London, 1967, defines the Borel sets in a locally compact Hausdorff space as the localization of the δ-ring generated by the compact sets, Baire sets being defined similarly as the localization of the δ-ring generated by the compact G_δ's.

Exercises

(1) Let ★ be the property :? is a topology. Verify that ★ is extensionally attainable. Given any nonempty $\mathfrak{S}$, ★($\mathfrak{S}$) is called the topology generated by $\mathfrak{S}$ and is denoted by $\mathfrak{Top}(\mathfrak{S})$. Verify that $\mathfrak{Top}(\mathfrak{S}) = \mathfrak{S}_{d\Sigma}$ with, if necessary, $\emptyset$ and X adjoined.

(2) Given a ring $\mathfrak{S}$, a ring with unit containing $\mathfrak{S}$ is the class

$$\hat{\mathfrak{S}} = \mathfrak{Rin}(\mathfrak{S},\{\cup\mathfrak{S}\}).$$

If $\mathfrak{T}$ is a ring with unit containing $\mathfrak{S}$, then $\cup\mathfrak{T} \supset \cup\mathfrak{S}$. Nevertheless, show by example that there exists a ring $\mathfrak{S}$ and a ring with unit $\mathfrak{T}$ such that

$$\begin{cases} \mathfrak{T} \supset \mathfrak{S} \\ \mathfrak{T} \not\supset \hat{\mathfrak{S}}. \end{cases}$$

[Take X = [0,2]. Let $\mathfrak{S}$ be the class consisting of all first category subsets of [0,1]. Consider

$$\mathfrak{T} = \mathfrak{Rin}(\mathfrak{S},\{[0,2]\}).]$$

(3) Let $\mathfrak{S}$ be nonempty -- then we have:

(i) $\mathfrak{Rin}(\mathfrak{S}) = \mathfrak{S}_{rds} = \mathfrak{S}_{rsd}$;

(ii) $\mathfrak{Rin}(\mathfrak{S}) = \mathfrak{S}_{dsrs} = \mathfrak{S}_{sdrs}$;

(iii) $\mathfrak{Rin}(\mathfrak{S}) = \mathfrak{S}_{rsrs}$.

Show by example that $\mathfrak{S}_{rsrs} \neq \mathfrak{S}_{rsr}$ in general.

[Take X = {1,2,3,4,5} and let $\mathfrak{S}$ = {{2,4}, {1,2,3}, {1,4,5}.]

(4) Let $\bigstar$ be the property: $? = ?_r$ and $? = ?_\Sigma$. Verify that $\bigstar$ is extensionally attainable. Given any nonempty $\mathcal{S}$, show that

$$\bigstar(\mathcal{S}) = \mathcal{S}_{r\Sigma r\Sigma}.$$

[It is enough to prove that

$$\mathcal{S}_{r\Sigma r\Sigma r} = \mathcal{S}_{r\Sigma r\Sigma}.$$

Incidentally, observe that Σ cannot, in general, be replaced by σ here; on the other hand, in view of Exer. 3(iii), the substitution of s for Σ does lead to a true statement.]

(5) True or False? Suppose that $\emptyset \in \mathcal{S}$, $\mathcal{S} = \mathcal{S}_d$, and $\mathcal{S}_s = \mathfrak{Rin}(\mathcal{S})$ -- then $\mathcal{S}$ is a semiring.

[Compare with Exer. 5 (§4).]

(6) Let X be a topological space -- then the ring generated by the open subsets of X is called the class of *constructible sets* in X. Verify that $S \subset X$ is constructible iff S can be written as a finite union of locally closed subsets of X.

(7) Let $\mathcal{S}$ be nonempty -- then $\mathfrak{Rin}(\mathcal{S})$ $(\sigma\text{-}\mathfrak{Rin}(\mathcal{S}))$ is the union of the rings (σ-rings) generated by the subsets of $\mathcal{S}$ of cardinality $<\aleph_0$ $(\leq\aleph_0)$.

(8) Let $\mathcal{S}$ be nonempty -- then every set in $\mathfrak{Rin}(\mathcal{S})$ $(\sigma\text{-}\mathfrak{Rin}(\mathcal{S}))$ can be covered by a finite (countable) union of sets in $\mathcal{S}$.

[The class of all sets which can be covered by a finite (countable) union of sets in $\mathcal{S}$ is a ring (σ-ring).]

(9) Let X be a nonempty set. Suppose that $\mathcal{S}$ is a σ-algebra in X admitting a generating subclass $\mathcal{S}_0$ of cardinality $\leq\aleph_0$ with the property that for all $x \neq y$ there exists an $S_0 \in \mathcal{S}_0$ such that either $x \in S_0$ and $y \notin S_0$ or $x \notin S_0$ and $y \in S_0$. Under these conditions, prove that X can be equipped with the structure of a separable metric space in which the Borel sets are precisely the elements of $\mathcal{S}$.

[Let $\mathcal{S}_0 = \{S_1, S_2, \ldots\}$ be an enumeration of $\mathcal{S}_0$. Consider the metric d defined by the rule

$$d(x,y) = \Sigma(|\chi_{S_i}(x) - \chi_{S_i}(y)|/2^i).]$$

(10) Let $X = [0,\Omega]$, equipped with the order topology -- then the Borel sets in X consist of those subsets S of X such that either S or $X - S$ contains an unbounded, closed subset of $[0,\Omega[$. Is every subset of X a Borel set?

[The class of unbounded, closed subsets of $[0,\Omega[$ is closed under countable intersections; accordingly, the class in question is a σ-ring containing the Borel sets. To obtain equality, let S be an unbounded, closed subset of $[0,\Omega[$ -- then it need only be shown that every subset T of $X - S$ is Borel. There is no loss of generality in supposing that $0 \in S$, $\Omega \notin T$. Given $\alpha \in S$, let α' be the first successor to α in S. Define a set-valued function f on S by the prescription

$$f(\alpha) = \{\beta \in T : \alpha < \beta < \alpha'\}.$$

Then $f(S) = T$. For each α such that $f(\alpha) \neq \emptyset$, fix an enumeration $\{f(\alpha)_n\}$ of the elements of $f(\alpha)$. Write

$$T_n = \bigcup_{\alpha \in S} \{f(\alpha)_n\}.$$

The T_n are Borel and $T = \cup T_n$.]

(11) Let X be a topological space -- then every Borel set in X has the property of Baire.

(12) Let X be a metric space -- then X is separable iff $\forall \varepsilon > 0$, $\mathcal{B}\mathfrak{a}(X)$, is generated by the open balls of radius $\leqq \varepsilon$. Show by example that there exists a nonseparable metric space X in which the open balls

$$\begin{cases} \text{do generate } \mathcal{B}\mathfrak{a}(X) \\ \text{do not generate } \mathcal{B}\mathfrak{a}(X). \end{cases}$$

(13) Let X be a topological space, all of whose points are closed; let S be a discrete subspace of X -- then S is a Borel subset of X.

[In fact, S is constructible.]

(14) Let X be a Hausdorff topological space -- then the σ-ring generated by the compact subsets of X is, by definition, the class of *σ-bounded Borel sets* in X. Justify this terminology by proving that a Borel set in X is σ-bounded iff it is contained in a countable union of compact sets. Hence or otherwise, infer that if X is

$$\begin{cases} \sigma\text{-compact} \\ \text{compact,} \end{cases}$$

then

$$\begin{cases} \mathcal{B}a(X) = [\mathcal{B}a_b(X)]_\sigma \\ \mathcal{B}a(X) = \mathcal{B}a_b(X). \end{cases}$$

(15) Let $X = [0,\Omega[$, equipped with the order topology. Characterize explicitly the elements of the δ-ring of bounded Borel sets in X.

(16) Let X be a Hausdorff topological space. Give a transfinite description of $\mathcal{B}a_b(X)$.

(17) Let X be a Hausdorff topological space. Let X_0 be a compact subset of X -- then the bounded Borel sets in X, when relativized to X_0, give the bounded Borel sets in X_0, i.e.,

$$\mathrm{tr}_{X_0}(\mathcal{B}a_b(X)) = \mathcal{B}a_b(X_0).$$

Is this true if X_0 is not compact?

(18) Let X be a Hausdorff topological space. Let $\mathcal{K} = \{K\}$ be a class of compact subsets of X such that

$$\begin{cases} \forall K_1, K_2 \in \mathcal{K},\ \exists K_3 \in \mathcal{K} \text{ st } \begin{cases} K_1 \subset K_3 \\ K_2 \subset K_3 \end{cases} \\ \forall \text{ compact } C \subset X,\ \exists K \in \mathcal{K} \text{ st } C \subset K. \end{cases}$$

Then

$$\mathcal{B}a_b(X) = \bigcup_{K \in \mathcal{K}} \mathcal{B}a_b(K).$$

[Show that the union in question is a δ-ring.]

(19) True or False? Let X be a Hausdorff topological space -- then the bounded Borel sets in X are precisely the relatively compact Borel sets in X.

(20) Let $\mathfrak{S}$ be a σ-ring in X; let $\mathfrak{T}$ be a σ-ring in Y -- then any $E \in \mathfrak{S} \bar{\otimes} \mathfrak{T}$ has at most $\mathfrak{c}$ distinct horizontal or vertical sections.

[Fix $E \in \mathfrak{S} \bar{\otimes} \mathfrak{T}$ -- then there exist σ-rings $\mathfrak{S}_E \subset \mathfrak{S}$ and $\mathfrak{T}_E \subset \mathfrak{T}$ such that $E \in \mathfrak{S}_E \bar{\otimes} \mathfrak{T}_E$ and such that both $\mathfrak{S}_E$ and $\mathfrak{T}_E$ are generated by no more than $\aleph_0$ elements (cf. Exer. 7). Owing to Lemma 2 (§5),

$$\begin{cases} E_x \in \mathfrak{T}_E & \forall x \in X \\ E^y \in \mathfrak{S}_E & \forall y \in Y . \end{cases}$$

On the other hand,

$$\begin{cases} \operatorname{card}(\mathfrak{S}_E) \leq \mathfrak{c} \\ \operatorname{card}(\mathfrak{T}_E) \leq \mathfrak{c} \text{ .]} \end{cases}$$

(21) Let $\mathfrak{S}$ be a σ-ring in X. Suppose that $\operatorname{card}(X) > \mathfrak{c}$ -- then the diagonal D in $X \times X$ does not belong to $\mathfrak{S} \bar{\otimes} \mathfrak{S}$.

[This follows from Exer. 20.]

(22) Let X and Y be topological spaces -- then

$$\mathcal{B}a(X) \bar{\otimes} \mathcal{B}a(Y) \subset \mathcal{B}a(X \times Y),$$

the containment being strict in general, but there being coincidence if the weights of X and Y are both $\leq \aleph_0$. Does coincidence obtain if X and Y are arbitrary Lindelöf spaces?

[Note: Do Baire sets "multiply"? While the answer is, of course, "no" in general, an important sufficient condition is this. Suppose that X and Y are completely regular, Hausdorff topological spaces for which $X \times Y$ is $\mathcal{Z}$-embedded in $\beta X \times \beta Y$, the product of the Stone-Čech compactifications of X and Y -- then

$$\mathrm{Ba}(X) \bar{\otimes} \mathrm{Ba}(Y) = \mathrm{Ba}(X \times Y).$$

For the details and further results, see R. Blair and A. Hager, Set-Theoretic Topology, Academic Press, New York, 1977, pp. 47-72.]

(23) Let X and Y be Hausdorff topological spaces -- then

$$[\mathrm{Ba}_b(X)]_\sigma \bar{\otimes} [\mathrm{Ba}_b(Y)]_\sigma \subset [\mathrm{Ba}_b(X \times Y)]_\sigma,$$

the containment being strict in general, but there being coincidence if the weights of X and Y are both $\leq \aleph_0$. Does coincidence obtain if X and Y are arbitrary metric spaces?

(24) Take for X the Sorgenfrey line $\underset{\sim}{E}$, i.e., $\underset{\sim}{E}$ is the real line equipped with the topology generated by the $[a,b[$ -- then

$$\mathrm{Ba}(\underset{\sim}{E}) = \mathrm{Ba}(\underset{\sim}{R})$$

but

$$\mathrm{Ba}(\underset{\sim}{E} \times \underset{\sim}{E}) \neq \mathrm{Ba}(\underset{\sim}{R} \times \underset{\sim}{R}).$$

[To establish the second point, consider the line $L: x + y = 0$ -- then, in the relative topology per $\underset{\sim}{E} \times \underset{\sim}{E}$, L is discrete. Use now the fact that Borel sets relativize.]

Is

$$\mathrm{Ba}(\underset{\sim}{E}) = \mathrm{Ba}(\underset{\sim}{R})?$$

Is

$$\mathrm{Ba}(\underset{\sim}{E}\times\underset{\sim}{E}) = \mathrm{Ba}(\underset{\sim}{R}\times\underset{\sim}{R})?$$

(25) Give an example of an infinite class $\mathcal{S}$ of subsets of $\underset{\sim}{R}$ such that

$$\underset{\sim}{R}\in\mathcal{S} \quad \text{and} \quad \mathcal{S} = \mathcal{S}_B$$

but such that $\mathcal{S}$ is not a σ-algebra.

(26) Estimate the cardinality of $\mathcal{S}_B$. Can the same be done of $\mathcal{S}_{B_d}$?

(27) True or False? Let $\mathcal{S}$ be a ring. Suppose that for some limit ordinal $\lambda<\Omega$,

$$\mathcal{S}_B = \bigcup_{\alpha<\lambda} B^{[\alpha]}(\mathcal{S}).$$

Then there is an $\alpha<\lambda$ such that

$$\mathcal{S}_B = B^{[\alpha]}(\mathcal{S}).$$

(28) There exists a completely regular, nonnormal, Hausdorff topological space X for which $\mathcal{F}\subset\mathfrak{G}_\delta$.

[The classical example is the so-called Moore plane $\underset{\sim}{\Gamma}$, i.e., $\underset{\sim}{\Gamma}$ is the closed upper half-plane $\{(x,y)\in\underset{\sim}{R}^2 : y\geq 0\}$, topologized by specifying local open neighborhoods: The open neighborhoods of (x,y) $(y>0)$ are to be the usual open neighborhoods but the open neighborhoods of $(x,0)$ are to be the sets $\{x\}\cup U$, where U is an open disk in the upper half-plane tangent to the x-axis at x.]

(29) Let $\ast$ be the property: $? = ?_{\sigma_d}$ and $? = ?_c$. Verify that $\ast$ is extensionally attainable. Given any nonempty $\mathcal{S}$, we then write $\mathcal{S}_{B_c}$ for $\ast(\mathcal{S})$ and refer to $M_\ast$ as operation B_c. Determine the properties of this operation. Show by example that $\mathcal{S}_{B_c}$ need not coincide with $\sigma\text{-}\mathfrak{Alg}(\mathcal{S})$. Prove that

$$\mathcal{S}_{B_c} = \sigma\text{-}\mathfrak{Alg}(\mathcal{S})$$

iff

$$\mathcal{S}_r \subset \mathcal{S}_{B_c} \quad \text{or} \quad \mathcal{S}_d \subset \mathcal{S}_{B_c} .$$

[So, in particular, if X is a topological space, then

$$\mathfrak{Ba}(X) = \begin{cases} \mathcal{F}_{B_c} \\ \\ \mathcal{G}_{B_c} \end{cases} .]$$

(30) Let X be a nonnormal, Hausdorff topological space -- then $\mathcal{Z}$ is properly contained in $\mathcal{F}$.

(31) A compact Hausdorff space X is 0-dimensional iff $\mathcal{Z}_\sigma = \mathcal{Z}_{\sigma_d}$.

(32) Let $X = [0,\Omega]$, equipped with the order topology -- then the Baire sets in X consist of those subsets S such that either

$$\text{card}(S) \leq \aleph_0 \quad \text{or} \quad \text{card}(X - S) \leq \aleph_0.$$

Thus, in this case, $\mathbf{Ba}(X)$ is strictly contained in $\mathfrak{Ba}(X)$ (cf. Exer. 10).

(33) Take for X the real line topologized by specifying that the open sets are to be all sets of the form $U \cup V$, where U is open in the usual topology and V is any subset of $\underset{\sim}{P} = X - \underset{\sim}{Q}$ -- then

$$\mathcal{B}\mathfrak{o}(X) = \begin{cases} \mathcal{F}_{B_d} \\ \\ \mathfrak{G}_{B_d} \end{cases} .$$

However, X, while normal and Hausdorff, is not perfectly normal. Is $\mathcal{B}\mathfrak{a}(X) = \mathcal{B}\mathfrak{o}(X)$?

(34) There exists a compact Hausdorff space X for which $\mathcal{F}_{B_d} \neq \mathcal{F}_B$.

[Let $A = D \cup \{\infty\}$ be the Alexandroff compactification of an uncountable discrete set D. Form the product $A \times \underset{\sim}{N}$ and let S be the set obtained by identifying the $(\{\infty\},n)\,(n \in \underset{\sim}{N})$. Equip S with the quotient topology -- then S is a completely regular, σ-compact, Hausdorff topological space. Let $X = \beta S$, the Stone-Čech compactification of S -- then $S \in \mathcal{F}_B$ but $S \notin \mathcal{F}_{B_d}$.]

(35) Consider $X = [0,1]^{[0,1]}$ in the product topology. Is the subspace of all continuous $f : [0,1] \rightarrow [0,1]$ a Borel (Baire) set in X?

(36) Take $X = \underset{\sim}{R}$ -- then

$$\underset{\sim}{P} \in \mathcal{F}_{\sigma_d \delta \sigma_d \delta} .$$

(37) Let $\mathcal{S}$ be nonempty -- then we have:

(i) $\mathcal{S}_B = \sigma\text{-}\mathbf{Alg}(\mathcal{S})$ iff $\mathcal{S}_c \subset \mathcal{S}_B$;

(ii) $\mathcal{S}_{B_d} = \sigma\text{-}\mathbf{Rin}(\mathcal{S})$ iff $\mathcal{S}_r \subset \mathcal{S}_{B_d}$.

[Compare these statements with Lemmas 4 and 5.]

(38) Let $\mathcal{S}$ be nonempty -- then

$$\delta\text{-}\mathbf{Rin}(\mathcal{S}) = \bigcup_{S \in \mathbf{Rin}(\mathcal{S})} tr_S(\sigma\text{-}\mathbf{Rin}(\mathcal{S})).$$

(39) True or False? Let $\mathcal{S}$ be a σ-ring in X; let $\mathcal{T}$ be a σ-ring in Y -- then

$$\mathcal{S}_{loc} \mathbin{\bar{\otimes}} \mathcal{T}_{loc} = (\mathcal{S} \mathbin{\bar{\otimes}} \mathcal{T})_{loc} \quad .$$

Retaining the given hypotheses, determine the validity of the relation

$$tr_{X_0 \times Y_0}(\mathcal{S} \mathbin{\bar{\otimes}} \mathcal{T}) = tr_{X_0}(\mathcal{S}) \mathbin{\bar{\otimes}} tr_{Y_0}(\mathcal{T}).$$

(40) Let $X = [0,\Omega[$, equipped with the order topology -- then

$$[\mathbf{Ba}_b(X)]_{loc} = \mathbf{P}(X).$$

Therefore, in this case, $\mathbf{Ba}(X)$ is strictly contained in $[\mathbf{Ba}_b(X)]_{loc}$ (cf. Exer. 10 and 15).

[For a somewhat different example, discuss $X = \underset{\sim}{R} \times \underset{\sim}{R}$, where, in the first factor, $\underset{\sim}{R}$ has the usual topology and, in the second factor, $\underset{\sim}{R}$ has the discrete topology.]

Problems

1. DYNKIN CLASSES

Let X be a nonempty set; let $\mathcal{S}$ be a nonempty subset of $\mathcal{P}(X)$ -- then $\mathcal{S}$ is said to be a *Dynkin class* if $\mathcal{S} = \mathcal{S}_{\sigma_d}$ and

$$S,T \in \mathcal{S},\ S \supset T \Longrightarrow S - T \in \mathcal{S}.$$

Take $\bigstar$ to be the property: ? is a Dynkin class. It is clear that $\bigstar$ is extensionally attainable. Given any nonempty $\mathcal{S}$, we then call $\bigstar(\mathcal{S})$ the Dynkin class generated by $\mathcal{S}$ and denote it by $\mathfrak{Dyn}(\mathcal{S})$.

Every σ-ring is a Dynkin class but a Dynkin class is a σ-ring iff it is closed under the formation of finite intersections.

[For a simple example of a class which is a Dynkin class but is not a σ-ring, take $X = \{1,2,3,4\}$ and consider

$\mathcal{S} = \{\emptyset,\{1,2\},\{1,3\},\{2,4\},\{3,4\},\{1,2,3,4\}\}$.]

If $\mathcal{S} = \mathcal{S}_d$, then

$$\sigma\text{-}\mathfrak{Rin}(\mathcal{S}) = \mathfrak{Dyn}(\mathcal{S}).$$

Ref E. Dynkin, Die Grundlagen der Theorie der Markoffschen Prozesse, Springer-Verlag, Berlin, 1961 (see pp. 1-2).

[Note: Results substantially the same as these were obtained many years earlier by W. Sierpiński, Fund. Math., **12**(1928), pp. 206-210.]

There is a variant on the preceding theme which is sometimes useful. Consider the following properties of a nonempty $\mathfrak{S}$:

(1) $\mathfrak{S} = \mathfrak{S}_{s_d}$;

(2) $\forall S \in \mathfrak{S}: \forall S_i \in \mathfrak{S}:$

$$S_1, S_2, \ldots \subset S, \ S_i \cap S_j = \emptyset \ (i \neq j)$$

$$\Longrightarrow \ \cup S_i \in \mathfrak{S};$$

(3) $S, T \in \mathfrak{S}, \ S \supset T \Longrightarrow S - T \in \mathfrak{S}$.

Let $\bigstar$ be the conjunction of (1), (2), and (3) -- then $\bigstar$ is extensionally attainable and the above results on Dynkin classes can be carried over to this setting in the obvious way. In particular, observe that if $\mathfrak{S} = \mathfrak{S}_d$, then $\bigstar(\mathfrak{S})$ is simply $\delta\text{-}\mathfrak{Rin}(\mathfrak{S})$.

II. STABILITY OF SECTIONS

If

$$\begin{cases} \mathfrak{S} \subset \mathfrak{P}(X) \\ \mathfrak{S} \subset \mathfrak{P}(Y) \end{cases}$$

both contain $\emptyset$, then

$$\begin{cases} \forall x: \mathcal{B}^{(\alpha)}(\mathfrak{S} \boxtimes \mathfrak{T})_x \subset \mathcal{B}^{(\alpha)}(\mathfrak{T}) \ \text{ and } \ \mathcal{B}_{(\alpha)}(\mathfrak{S} \boxtimes \mathfrak{T})_x \subset \mathcal{B}_{(\alpha)}(\mathfrak{T}) \\ \forall_y: \mathcal{B}^{(\alpha)}(\mathfrak{S} \boxtimes \mathfrak{T})^y \subset \mathcal{B}^{(\alpha)}(\mathfrak{S}) \ \text{ and } \ \mathcal{B}_{(\alpha)}(\mathfrak{S} \boxtimes \mathfrak{T})_y \subset \mathcal{B}_{(\alpha)}(\mathfrak{S}). \end{cases} \quad (\alpha < \Omega)$$

[This follows by an easy transfinite induction on α.]

Take now $\mathcal{S} = \mathcal{P}(X)$ and suppose that $\text{card}(\mathfrak{T}) \leq \aleph_0$. Let E be a nonempty subset of $X \times Y$ -- then, given $\alpha (0 < \alpha < \Omega)$,

$$E \in \mathcal{B}^{[\alpha]}(\mathcal{P}(X) \boxtimes \mathfrak{T}) \quad \text{iff} \quad E_x \in \mathcal{B}^{[\alpha]}(\mathfrak{T}) \quad (\forall x \in \pi_X(E)).$$

[To discuss the nontrivial point, viz. that

$$E_x \in \mathcal{B}^{[\alpha]}(\mathfrak{T}) \ \ (\forall x \in \pi_X(E)) \Longrightarrow E \in \mathcal{B}^{[\alpha]}(\mathcal{P}(X) \boxtimes \mathfrak{T}),$$

one can argue by transfinite induction on α, treating first the case when $\alpha = 1$ and then looking at the cases when α is odd or even separately. Here is the proof for $\alpha = 1$. Let $\mathfrak{T} = \{T_1, T_2, \ldots\}$ be an enumeration of $\mathfrak{T}$. Put

$$S_i = \{x \in \pi_X(E) : T_i \subset E_x\}.$$

Then

$$E = \cup(S_i \times T_i) \in \mathcal{B}^{[1]}(\mathcal{P}(X) \boxtimes \mathfrak{T}).]$$

Ref. R. Bing, W. Bledsoe, and R. Mauldin, Pacific J. Math., 51 (1974), pp. 27-36.

III. SETS GENERATED BY RECTANGLES

Let X be a nonempty set -- then, in §5, we discussed the question: Is $\mathcal{P}(X) \otimes \mathcal{P}(X)$ dense in $\mathcal{P}(X \times X)_s$? As has been seen there, the answer depends on the cardinality of X, the case of mystery being when $\aleph_1 < \text{card}(X) \leq \mathfrak{c}$.

If $\text{card}(X) \leq \aleph_1$, then it is actually true that

$$\mathfrak{P}(X \times X) = \mathcal{B}^{[2]}(\mathfrak{P}(X) \boxtimes \mathfrak{P}(X)),$$

i.e., each subset of $X \times X$ can be generated from the rectangles in just two steps. Assuming Martin's axiom, this conclusion remains in force if only $\mathrm{card}(X) \leq \mathfrak{c}$.

On the other hand, the density of $\mathfrak{P}(X) \otimes \mathfrak{P}(X)$ in $\mathfrak{P}(X \times X)_s$ or still, the relation

$$\mathfrak{P}(X \times X) = \mathfrak{P}(X) \bar{\otimes} \mathfrak{P}(X),$$

is equivalent to the existence of a countable ordinal $\alpha \geq 2$ such that

$$\mathfrak{P}(X \times X) = \mathcal{B}^{[\alpha]}(\mathfrak{P}(X) \boxtimes \mathfrak{P}(X)).$$

Ref R. Bing, W. Bledsoe, and R. Mauldin, Pacific J. Math., **51**(1974), pp. 27-36.

[Note: One could ask: Does

$$\mathfrak{P}(X \times X) = \mathfrak{P}(X) \bar{\otimes} \mathfrak{P}(X)$$

$$\Longrightarrow$$

$$\mathfrak{P}(X \times X) = \mathcal{B}^{[2]}(\mathfrak{P}(X) \boxtimes \mathfrak{P}(X))?$$

For a discussion of this question, see A. Miller, Ann. Math. Logic, **16**(1979), pp. 233-267. Consequences and implications may be found in R. Mauldin, Fund. Math., **95**(1977), pp. 129-139.]

IV. POINT-FINITE CLASSES

Let X be a nonempty set. Fix a subset $\mathcal{S}$ of $\mathfrak{P}(X)$ containing $\emptyset$ and X.

A nonempty class $\mathfrak{X} \subset \mathfrak{P}(X)$ is said to be **point-finite** if each point of X belongs to at most a finite number of elements of $\mathfrak{X}$.

(H) Suppose that $\mathfrak{X}$ is a point-finite class in X such that $\mathfrak{X}_\Sigma \subset \mathfrak{S}_B$ -- then, for some $\alpha < \Omega$,

$$\mathfrak{X} \subset \mathbf{B}^{[\alpha]}(\mathfrak{S}).$$

It will be simplest to examine first a special case.

(P) Suppose that $\mathfrak{X}$ is a disjoint class in X such that $\mathfrak{X}_\Sigma \subset \mathfrak{S}_B$ -- then, for some $\alpha < \Omega$,

$$\mathfrak{X} \subset \mathbf{B}^{[\alpha]}(\mathfrak{S}).$$

[Proceed by contradiction -- then there exist $\aleph_1$ disjoint subclasses $\mathfrak{X}_\beta$ of $\mathfrak{X}$ such that for all $\alpha,\beta < \Omega$, $\mathfrak{X}_\beta \not\subset \mathbf{B}^{[\alpha]}(\mathfrak{S})$. Because $\mathfrak{X}_\Sigma \subset \mathfrak{S}_B$, there is a function $f : [0,\Omega[\rightarrow [0,\Omega[$ such that $\cup\mathfrak{X}_\beta \in \mathbf{B}^{[f(\beta)]}(\mathfrak{S})$ $(\beta \leq f(\beta))$. Choose $X_\beta \in \mathfrak{X}_\beta$, $X_\beta \notin \mathbf{B}^{[f(\beta)]}(\mathfrak{S})$. Put $A = \bigcup_{\beta<\Omega} X$ -- then, for some α, $A \in \mathbf{B}^{[\alpha]}(\mathfrak{S})$. But now $X_\alpha = A \cap \cup\mathfrak{X}_\alpha \in \mathbf{B}^{[f(\alpha)]}(\mathfrak{S})$, a contradiction.]

Ref D. Preiss, Comment. Math. Univ. Carolinae, 15(1972), pp. 341-344.

[The above proof is due to W. Fleissner, Trans. Amer. Math. Soc., 251(1979), pp. 309-328.]

In order to deduce (H) from (P), the following artifice will be needed.

Lemma Let Y be a separable metric space with topology $\mathfrak{T}$. Suppose that $\{X(y) : y \in Y\}$ is a point-finite class in X such that

$$\{X(y) : y \in Y\}_\Sigma \subset \mathfrak{S}_B.$$

Then

$$\{X(y) \times \{y\} : y \in Y\}_\Sigma \subset (\mathcal{S} \boxtimes \mathcal{T})_B.$$

[Choose, as is possible, a basis $N_n (n \in \mathbf{N})$ for Y satisfying the diameter condition, i.e., $\operatorname{diam}(N_n) \to 0$ and with the property that each point of Y belongs to N_n for arbitrarily large values of n. Given a nonempty subset Y_0 of Y, put

$$M_n = \cup\{X(y) : y \in N_n \cap Y_0\}.$$

Then

$$\cup\{X(y) \times \{y\} : y \in Y_0\} = \overline{\lim} \, (M_n \times N_n),$$

hence is in $(\mathcal{S} \boxtimes \mathcal{T})_B$.]

Proof of (H) Proceed by contradiction -- then

$$\mathcal{X} \not\subset \mathcal{B}^{[\alpha]}(\mathcal{S}) \qquad \forall \alpha < \Omega.$$

Accordingly, one may select sets

$$X_\alpha \in \mathcal{X} - (\mathcal{B}^{[\alpha]}(\mathcal{S}) \cup \{X_\beta : \beta < \alpha\}) \qquad (\alpha < \Omega).$$

Viewing $Y = \{\alpha : \alpha < \Omega\}$ as a subspace of $\mathbf{R}$, statement (P), in conjunction with the lemma supra, allows one to conclude that

$$A = \cup\{X_\alpha \times \{\alpha\} : \alpha < \Omega\} \in \mathcal{B}^{[\beta+1]}(\mathcal{S} \boxtimes \mathcal{T})$$

for some $\beta > 1$. Since

$$X \times \{\alpha\} \in \mathcal{B}^{[2]}(\mathcal{S} \boxtimes \mathcal{T}),$$

it follows that

$$A \cap (X \times \{\alpha\}) = X_\alpha \times \{\alpha\} \in \mathcal{B}^{[\beta+1]}(\mathcal{S} \boxtimes \mathcal{T}).$$

However (cf. Prob. II), this implies that $X_\alpha \in \mathcal{B}^{[\beta+1]}(\mathcal{S})$ $\forall \alpha < \Omega$, a contradiction. //

It can be easily shown by example that statement (H) is no longer true if "point-finite" is replaced by "point-countable" (defined in the obvious way).

Ref R. Hansell, Proc. Amer. Math. Soc., 83(1981), pp. 375-378.

V. THEOREMS OF MILLER AND KUNEN

Suppose that X is a topological space for which $\mathrm{Ba}(X) = \mathcal{P}(X)$. Does $\exists$ an $\alpha < \Omega$ such that

$$\mathrm{Ba}(X) = \Sigma_\alpha(\mathcal{G})?$$

The answer, in general, is unknown. However, if X is a metric space, then the response is positive.

Theorem (Miller) Suppose that X is a separable metric space for which $\mathrm{Ba}(X) = \mathcal{P}(X)$ -- then $\exists$ an $\alpha < \Omega$ such that

$$\mathrm{Ba}(X) = \Sigma_\alpha(\mathcal{G}).$$

[First note that the cardinality of X is necessarily $< \mathfrak{c}$. For otherwise,

$$\mathrm{card}(\mathfrak{Ba}(X)) \geqq 2^{\mathfrak{c}} > \mathfrak{c},$$

which is impossible as there can be at most $\mathfrak{c}$ Borel sets in a separable metric space. If $\mathrm{card}(X) \leqq \aleph_0$, then the assertion is clear. Let us consider the simplest nontrivial case, viz. when $\mathrm{card}(X) = \aleph_1$, referring the reader to the paper infra for the details when $\aleph_1 < \mathrm{card}(X) < \mathfrak{c}$. Write $X = \{x_\alpha : \alpha < \Omega\}$ and proceed by contradiction. For each $\alpha < \Omega$, let $A_\alpha \in \Sigma_{\alpha+1}(\mathfrak{G}) - \Sigma_\alpha(\mathfrak{G})$ and put $A = \{(x_\alpha, a) : a \in A_\alpha\}$ -- then it need only be shown that $A \in \Sigma_\beta(\mathfrak{G} \times \mathfrak{G})$ for some $\beta < \Omega$ as this would entail

$$A_{\beta+1} = A \cap (\{x_{\beta+1}\} \times X) \in \Sigma_\beta(\mathfrak{G}).$$

But, in view of the fact that X is of cardinality $\aleph_1$ and of weight $\aleph_0$, we have

$$\begin{array}{ccccc} \mathfrak{P}(X) & \bar{\otimes} & \mathfrak{P}(X) & = & \mathfrak{P}(X \times X) \\ \| & & \| & & \| \\ \mathfrak{Ba}(X) & \bar{\otimes} & \mathfrak{Ba}(X) & = & \mathfrak{Ba}(X \times X), \end{array}$$

making the contention plain enough.]

Ref A. Miller, Ann. Math. Logic, 16(1979), pp. 233-267.

[Note: Observe that the continuum hypothesis denies the existence of an uncountable separable metric space all of whose subsets are Borel. On the other hand, in the presence of Martin's axiom and the negation of the continuum hypothesis, it can be shown that there exists an uncountable set $X \subset \mathbb{R}$ in which every subset is an F_σ (or, equivalently, G_δ); cf. F. Tall, Dissertationes Math., 148(1977), pp. 1-57.]

<u>Theorem</u> (Kunen) <u>Suppose that</u> X <u>is a metric space for which</u> $\mathbf{Ba}(X) = \mathbf{P}(X)$ -- <u>then</u> $\exists$ <u>an</u> $\alpha < \Omega$ <u>such that</u>

$$\mathbf{Ba}(X) = \Sigma_\alpha(\mathfrak{G}).$$

[Kunen's proof is given in the paper of Miller cited above. It runs as follows. Because X is a metric space, X admits a σ-discrete basis $\mathcal{N} = \cup\{\mathcal{N}_n : n \in \underset{\sim}{N}\}$. For each $N \in \mathcal{N}$, let $\alpha(N)$ be the smallest ordinal α such that $\mathbf{P}(N) = \Sigma_\alpha(\mathrm{tr}_N(\mathfrak{G}))$. Given $n \in \underset{\sim}{N}$ and $\alpha < \Omega$, let

$$C_{n,\alpha} = \{N \in \mathcal{N}_n : \alpha(N) < \alpha\}.$$

Claim: $\forall n \, \exists \alpha(n)$ such that

$$\mathrm{card}(\mathcal{N}_n - C_{n,\alpha(n)}) \leq \aleph_0.$$

Indeed, if not, then for some n it would be possible to find $A_\alpha, N_\alpha (\alpha < \Omega)$ with:

(1) $N_\alpha \in \mathcal{N}_n$;

(2) $N_\alpha \neq N_\beta \ (\forall \alpha \neq \beta)$;

(3) $A_\alpha \in \Sigma_{\alpha+1}(\mathrm{tr}_{N_\alpha}(\mathfrak{G})) - \Sigma_\alpha(\mathrm{tr}_{N_\alpha}(\mathfrak{G}))$.

Since the union $\cup A_\alpha$ cannot be Borel under these circumstances, we have a contradiction. The claim established, let $\alpha^* = \sup\{\alpha(n)\}$. Put

$$X_0 = X - \cup\{N \in \mathcal{N} : \alpha(N) < \alpha^*\}.$$

Thanks to the claim, X_0 is a separable subspace of X, so, by Miller's theorem, $\exists \alpha_0 < \Omega$ such that $\mathbf{Ba}(X_0) = \Sigma_{\alpha_0}(\mathfrak{G}_0)$. If now $\alpha = \sup\{\alpha_0, \alpha^*+1\}$, then $\mathbf{Ba}(X) = \Sigma_\alpha(\mathfrak{G})$.]

VI. POINT-FINITE CLASSES (BIS)

As in Prob. IV, let X be a nonempty set. Fix a subset $\mathcal{S}$ of $\mathcal{P}(X)$ containing $\emptyset$ and X.

Suppose that $\mathcal{X}$ is a point-finite class in X such that $\mathcal{X}_\Sigma \subset \mathcal{S}_B$ -- then, as has been seen above, $\mathcal{X}$ is contained in $B^{[\alpha]}(\mathcal{S})$ for some $\alpha < \Omega$. We now ask: Does there exist an $\alpha < \Omega$ such that $\mathcal{X}_\Sigma \subset B^{[\alpha]}(\mathcal{S})$?

To give an answer, write $\mathcal{X} = \{X_i : i \in I\}$ -- then there will be an α with the stated property if $\exists$ an uncountable set J such that

$$\mathcal{P}(I \times J) = \mathcal{P}(I) \bar{\otimes} \mathcal{P}(J).$$

[The proof is similar to that of statement (H) in Prob. IV, modulo an appropriate variant of the lemma appearing there.]

The question of the equality

$$\mathcal{P}(I \times J) = \mathcal{P}(I) \bar{\otimes} \mathcal{P}(J)$$

has been considered in Exer. 5 (§5). Recall that it will hold if both card(I) and card(J) are $\leq \aleph_1$ (or even $\leq \mathfrak{c}$ if Martin's axiom is assumed). Consequently, the answer to the question supra is affirmative if $\text{card}(I) \leq \aleph_1$.

There is another condition on J which will force the equality

$$\mathcal{P}(I \times J) = \mathcal{P}(I) \bar{\otimes} \mathcal{P}(J),$$

namely that $\mathcal{P}(J)$ be generated as a σ-algebra by a set of cardinality $\leq \aleph_0$.

[Use Exer. 9, Prob., V, and Prob. II.]

Example Let X be a metric space. Take, in this context, $\mathcal{S} = \mathcal{G}$. Suppose that $\mathcal{X} = \{\{x\} : x \in X\}$. If $\mathcal{X}_\Sigma \subset \mathcal{S}_B$, then $\mathrm{Ba}(X) = \mathcal{P}(X)$, so in this case we are back in the setting of Prob. V. Assume now that X is, in addition, separable. Let $\mathcal{X}$ be a point-finite class in X such that $\mathcal{X}_\Sigma \subset \mathcal{S}_B$ -- then there exists an $\alpha < \Omega$ such that $\mathcal{X}_\Sigma \subset \mathcal{B}^{[\alpha]}(\mathcal{S})$. This, of course, is obvious if $\mathrm{card}(I) \leq \aleph_0$. On the other hand, if $\mathrm{card}(I) > \aleph_0$, fix a point x_i in each X_i -- then $X_I = \{x_i : i \in I\}$ is an uncountable separable metric space all of whose subsets are Borel, hence

$$\mathcal{P}(I \times X_I) = \mathcal{P}(I) \bar{\otimes} \mathcal{P}(X_I).$$

Ref R. Hansell, General Topology and Modern Analysis, 1981, pp. 405-416.

VII. ZERO SETS IN UNIFORM SPACES

Let X be a uniform space -- then the class $\mathcal{Z}$ of zero sets of the bounded uniformly continuous functions $f : X \to \mathbb{R}$ has the following properties:

(1) $\emptyset, X \in \mathcal{Z}$, (2) $\mathcal{Z} = \mathcal{Z}_s$,

(3) $\mathcal{Z} = \mathcal{Z}_\delta$, (4) $\mathcal{Z}_c \subset \mathcal{Z}_\sigma$.

In addition, given disjoint $Z_1, Z_2 \in \mathcal{Z}$, there exist disjoint $U_1, U_2 \in \mathcal{Z}_c$ such that

$$Z_1 \subset U_1, \quad Z_2 \subset U_2 .$$

One has:

$$\mathcal{Z}_{B_d} = \sigma\text{-}\mathrm{Alg}(\mathcal{Z}).$$

[This can be seen by repeating the argument for its topological analogue virtually word-for-word.]

Ref J. Jayne, Proc. Prague Symp. General Topology, Part B, 1976, pp. 187-194.

Let X be a nonempty set; let $\mathcal{Z}$ be a class of subsets of X possessing the five properties supra -- then X can be equipped with the structure of a uniform space with respect to which $\mathcal{Z}$ is precisely the class of zero sets of the bounded uniformly continuous functions $f : X \to \underline{R}$. Consequently,

$$\mathcal{Z}_{B_d} = \sigma\text{-}\mathrm{Alg}(\mathcal{Z}).$$

[In this connection, recall that a topology $\mathfrak{T}$ on X is the uniform topology for some uniformity on X iff the topological space $(X,\mathfrak{T})$ is completely regular.]

Ref H. Gordon, Pacific J. Math., 36(1971), pp. 133-157.

VIII. DISJOINT GENERATION

Let X be a nonempty set; let $\mathcal{S}$ be a nonempty class of subsets of X such that

$$\mathcal{S} = \mathcal{S}_s , \qquad \mathcal{S}_c \subset \mathcal{S}_\sigma.$$

Suppose in addition that given disjoint $S_1, S_2 \in \mathcal{S}$, there exist disjoint $C_1, C_2 \in \mathcal{S}_c$ such that

$$S_1 \subset C_1, \quad S_2 \subset C_2 .$$

Then

$$\mathcal{S}_{B_d} = \sigma\text{-}\mathrm{Alg}(\mathcal{S}).$$

[According to Lemma 5, it suffices to prove that $\mathcal{S}_c \subset \mathcal{S}_{B_d}$. For this purpose, show by a direct set-theoretic construction that

$$\mathcal{S}_c \subset \mathcal{S}_{\delta\sigma_d\delta\sigma_d} \quad .]$$

Ref J. Jayne, Mathematika, 24(1977), pp. 241-256.

IX. INCREASING AND DECREASING LIMITS

Let $\mathcal{S} \subset \mathcal{P}(X)$ be nonempty. Write

$$\begin{cases} (\uparrow)(\mathcal{S}) \\ (\downarrow)(\mathcal{S}) \end{cases}$$

for the class of all subsets of X which are the limit of an

$$\begin{cases} \text{increasing} \\ \text{decreasing} \end{cases}$$

sequence of sets in $\mathcal{S}$.

Suppose now that $\mathcal{S}$ is a lattice. Put

$$\Xi^0(\mathcal{S}) = (\uparrow)(\mathcal{S}), \ \Xi_0(\mathcal{S}) = (\downarrow)(\mathcal{S})$$

and define via transfinite recursion the classes $\Xi^\alpha(\mathcal{S})$, $\Xi_\alpha(\mathcal{S})$ by writing

$$\begin{cases} \Xi^\alpha(\mathcal{S}) = (\uparrow)(\bigcup_{\beta<\alpha} \Xi_\beta(\mathcal{S})) \\ \Xi_\alpha(\mathcal{S}) = (\downarrow)(\bigcup_{\beta<\alpha} \Xi^\beta(\mathcal{S})). \end{cases} \qquad (\alpha < \Omega)$$

Investigate these classes.

Ref W. Sierpiński, Fund. Math., 18 (1932), pp. 1-22.

X. $\aleph$-OPERATIONS

Let $\aleph$ be an infinite cardinal. Consider a map

$$M : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$$

with the following properties:

(1) If $f : X \to X$ is a function and if $\mathcal{S} \subset \mathcal{P}(X)$ is a class, then

$$f^{-1}(M(\mathcal{S})) \subset M(f^{-1}(\mathcal{S}));$$

(2) If $\mathcal{S}', \mathcal{S}'' \subset \mathcal{P}(X)$ are classes, if $M(\mathcal{S}') \subset M(\mathcal{S}'')$, and if $S'' \in M(\mathcal{S}'')$, then

$$M(\mathcal{S}' \cup \{S''\}) \subset M(\mathcal{S}'').$$

Under these circumstances, M is said to be an *$\aleph$-operation* if for every initial ordinal ζ with $\mathrm{card}(\zeta) \leq \aleph$ and if for any increasing transfinite ζ-sequence $\{\mathcal{S}_\alpha : \alpha < \zeta\}$, the inclusions

$$M(\mathcal{S}_\alpha) \subset M(\mathcal{S}) \ (\alpha < \zeta) \Rightarrow M\Big(\bigcup_{\alpha<\zeta} \mathcal{S}_\alpha\Big) \subset M(\mathcal{S}).$$

Illustrate this concept by examining the various set-theoretic operations which have been discussed in this §.

If $\ast$ is extensionally attainable, then is it necessarily true that $M_\ast$ is an $\aleph$-operation?

Ref M. Ershov, SLN, 794(1980), pp. 105-111.

[Here also may be found a number of selection theorems of substantial generality.]