Key to Practice Midterm

Mathematics 307
Spring 2008

Problem 1. Solve the differential equation

$$\frac{dy}{dt} + 3y = 4t^2 e^{-3t} - te^{-2t}.$$

Solution: It is a linear equation. The integrating factor is

$$\mu(t) = e^{\int 3dt} = e^{3t}$$

Multiplying $\mu(t)$ to both sides of the differential equation, we have

$$\frac{d}{dt}(e^{3t}y) = 4t^2 - te^t$$

Integrating it, we obtain

$$e^{3t}y = \int (4t^2 - te^t)dt = \frac{4}{3}t^3 - te^t + e^t + C$$

Therefore

$$y = \frac{4}{3}t^3 e^{-3t} - te^{-2t} + e^{-2t} + Ce^{-3t}$$

where C is an arbitrary number.

Problem 2. Find the solution of the initial value problem in explicit form

$$\frac{dy}{dt} = \frac{3t^2 - e^t}{2y - 8},$$

$$y(0) = 1.$$

(You do NOT need to determine the interval in which the solution exists.)

Solution: It is a separable equation. We rewrite it as

$$(2y - 8)dy = (3t^2 - e^t)dt$$

Integrating left side w.r.t y and right side w.r.t. t, we have

$$\int (2y - 8)dy = \int (3t^2 - e^t)dt$$

That is

$$y^2 - 8y = t^3 - e^t + C$$

Using initial condition, plugging in $t = 0$ and $y = 1$, we have

$$1^2 - 8 \cdot 1 = 0^3 - e^0 + C$$

So

$$C = -6$$

and the solution is

$$y^2 - 8y = t^3 - e^t - 6$$

To find the explicit form, we complete square

$$(y - 4)^2 = t^3 - e^t + 10$$

and get

$$y - 4 = \pm\sqrt{t^3 - e^t + 10}$$

Since only the negative square root satisfies the initial condition $y(0) = 1$, we have

$$y = 4 - \sqrt{t^3 - e^t + 10}$$

Problem 3. A tank contains 100 gal of water and 1 lb of salt. Water containing a salt concentration of $(1 - e^{-\frac{1}{50}t})$ lb/gal flows into the tank at a rate of 2 gal/minute, and the well-stirred mixture in the tank flows out at the same rate. Find the amount of salt in the tank at any time.

Solution: Let $y(t)$ be the amount, in lb, of the salt in the tank at time t. The rate of change of salt in the tank, $\frac{dy}{dt}$, is given by

$$\begin{aligned}\frac{dy}{dt} &= \text{rate in} \; - \; \text{rate out} \\ &= \text{rate of water in} \; \cdot \; \text{concentration in} \; - \; \text{rate of water out} \; \cdot \; \text{concentration out} \\ &= 2 \cdot (1 - e^{-\frac{1}{50}t}) - 2 \cdot \frac{y}{100}\end{aligned}$$

Thus the differential equation is

$$\frac{dy}{dt} = 2 \cdot (1 - e^{-\frac{1}{50}t}) - 2 \cdot \frac{y}{100}$$

The initial condition is

$$y(0) = 1$$

The differential equation is linear. Rewriting it in the usual form for a linear equation, we have

$$\frac{dy}{dt} + \frac{y}{50} = 2 - 2e^{-\frac{t}{50}}$$

The integrating factor is

$$\mu(t) = e^{\frac{t}{50}}$$

Multiplying the integrating factor, we have

$$\frac{d}{dt}(e^{\frac{t}{50}}y) = 2e^{\frac{t}{50}} - 2$$

Integrating it, we obtain

$$y = 100 - 2te^{-\frac{t}{50}} + Ce^{-\frac{t}{50}}$$

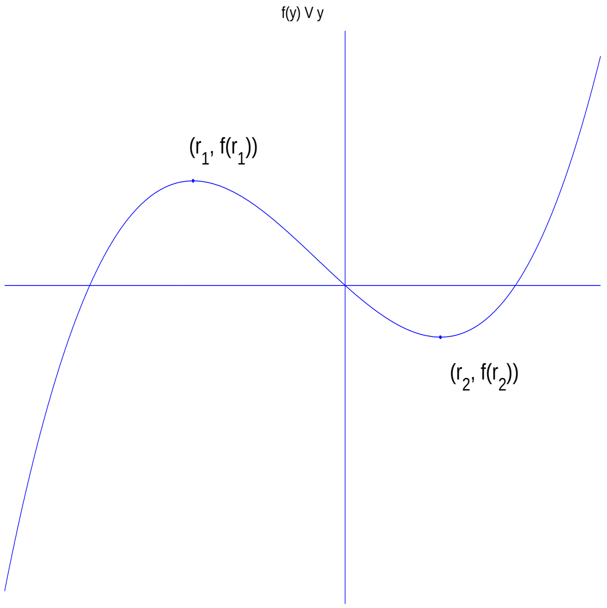
From initial condition, we know $C = -99$.
Therefore we find the solution

$$y(t) = 100 - 2te^{-\frac{t}{50}} - 99e^{-\frac{t}{50}}$$

Problem 4. Consider the differential equation

$$\frac{dy}{dt} = (y+3)y(y-2).$$

Determine the equilibrium solutions, classify each one as asymptotically stable, unstable or semistable, and sketch several graphs of solutions in the ty-plane. (You might need the graph of $f(y) = (y+3)y(y-2)$ below, where $r_1 = \frac{-1-\sqrt{19}}{3}$ and $r_2 = \frac{-1+\sqrt{19}}{3}$.)



Solution: (1) Equilibrium solutions are constant solutions, and they must satisfy the equation

$$(y+3)y(y-2) = 0$$

The roots are

$$y = -3, \quad y = 0, \quad y = 2.$$

So, the equilibrium solutions are

$$y = -3, \quad y = 0, \quad y = 2.$$

(2) Let $f(y) = (y+3)y(y-2)$. We have

$$\frac{dy}{dt} = f(y)$$

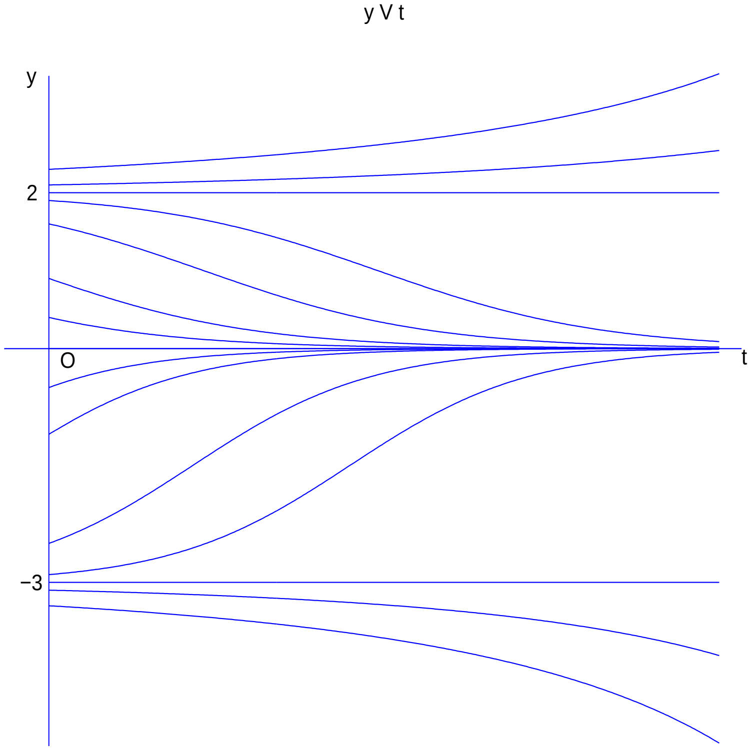
and

$$\frac{d^2y}{dt^2} = f'(y)f(y)$$

From the graph of $f(y)$ V y, we know

when $y < -3$, $f(y) < 0$ and $f'(y) > 0$, so $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} < 0$
when $-3 < y < r_1$, $f(y) > 0$ and $f'(y) > 0$, so $\frac{dy}{dt} > 0$ and $\frac{d^2y}{dt^2} > 0$
when $r_1 < y < 0$, $f(y) > 0$ and $f'(y) < 0$, so $\frac{dy}{dt} > 0$ and $\frac{d^2y}{dt^2} < 0$
when $0 < y < r_2$, $f(y) < 0$ and $f'(y) < 0$, so $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} > 0$
when $r_2 < y < 2$, $f(y) < 0$ and $f'(y) > 0$, so $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} < 0$
when $y > 2$, $f(y) > 0$ and $f'(y) > 0$, so $\frac{dy}{dt} > 0$ and $\frac{d^2y}{dt^2} > 0$

And then, we can sketch several graphs of solutions in the ty-plane.



(3) From the graph, it is easy to see that

$$y = -3 \text{ is unstable, } y = 0 \text{ is stable, } y = 2 \text{ is unstable.}$$

Problem 5. Consider the initial value problem

$$\frac{dy}{dt} = cos(10\pi t + 5\pi y) - y, \quad y(0) = 0.$$

Use Euler's method with step size $h = 0.1$ to approximate $y(0.1)$ and $y(0.2)$.

Solution: To use Euler's method, we know the formula are

$$y_{n+1} = y_n + f(t_n, y_n) \cdot h$$

$$t_{n+1} = t_n + h$$

where $n = 0, 1, \cdots$.
In our case

$$f(t, y) = cos(10\pi t + 5\pi y) - y, \quad h = 0.1$$

We start with $t_0 = 0$ and $y_0 = 0$.
The first step,

$$y_1 = y_0 + f(t_0, y_0) \cdot h = 0 + (\cos(10\pi \cdot 0 + 5\pi \cdot 0) - 0) \cdot 0.1 = 0.1$$

$$t_1 = t_0 + h = 0 + 0.1 = 0.1$$

So

$$y(0.1) \approx 0.1$$

The second step,

$$y_2 = y_1 + f(t_1, y_1) \cdot h = 0.1 + (\cos(10\pi \cdot 0.1 + 5\pi \cdot 0.1) - 0.1) \cdot 0.1 = 0.09$$

$$t_2 = t_1 + h = 0.1 + 0.1 = 0.2$$

So

$$y(0.2) \approx 0.09$$

Problem 6. Find the general solutions of the following differential equations.

(1) $y'' + y' - 6y = 0$.

Solution: The characteristic equation is

$$r^2 + r - 6 = 0$$

It has two different real roots

$$r_1 = 2, \quad r_2 = -3$$

Therefore the general solution is

$$y = c_1 e^{2t} + c_2 e^{-3t}$$

where c_1 and c_2 are two arbitrary numbers.

(2) $y'' + 4y' + 5y = 0$.

Solution: The characteristic equation is

$$r^2 + 4r + 5 = 0$$

It has two complex conjugate roots

$$r = -2 \pm i$$

Therefore the general solution is

$$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$$

where c_1 and c_2 are two arbitrary numbers.

Problem 7. Solve the initial value problem

$$y'' - 6y' + 9y = 0$$

$$y(0) = 2, \quad y'(0) = 2$$

Solution: The characteristic equation is

$$r^2 - 6r + 9 = 0$$

It has two same real roots

$$r = 3$$

So the general solution is

$$y = c_1 t e^{3t} + c_2 e^{3t} \tag{1}$$

Next we use the initial conditions to determine c_1 and c_2.
Using $y(0) = 2$, we have

$$2 = c_1 \cdot 0 \cdot e^{3\cdot 0} + c_2 e^{3\cdot 0} \tag{2}$$

Taking derivative w.r.t. t of equation (1), we have

$$y' = c_1 e^{3t} + 3c_1 t e^{3t} + 3c_2 e^{3t}$$

Using $y'(0) = 2$, we have

$$2 = c_1 e^{3\cdot 0} + 3c_1 \cdot 0 \cdot e^{3\cdot 0} + 3c_2 e^{3\cdot 0} \tag{3}$$

Solving system (2)(3), we find

$$c_1 = -4, \quad c_2 = 2$$

Therefore the solution to this initial value problem is

$$y = -4te^{3t} + 2e^{3t}$$