$$\int \frac{1}{x^3+1}\,dx$$

sol. Partial Fraction

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$x=-1$ $\quad 1 = A(1+1+1) \Rightarrow A = \frac{1}{3}$

$x=0$ $\quad 1 = \frac{1}{3} + C \Rightarrow C = 2/3$

$x=1$ $\quad 1 = \frac{1}{3} + (B+\frac{2}{3})2 \Rightarrow B = 1/3$

$$\frac{1}{x^3+1} = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1}$$

Substitution/ Manipulation

$$\frac{-\frac{1}{3}x+\frac{2}{3}}{x^2-x+1} = \frac{-\frac{1}{6}(2x-1) - \frac{1}{6} + \frac{2}{3}}{x^2-x+1}$$

$$= \frac{-\frac{1}{6}(2x-1)}{x^2-x+1} + \frac{\frac{1}{2}}{x^2-x+1}$$

$$\frac{\frac{1}{2}}{x^2-x+1} = \frac{\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}}\,\frac{1}{[\frac{2}{\sqrt{3}}(x-\frac{1}{2})]^2+1} = \frac{\frac{1}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}}{[\frac{2}{\sqrt{3}}(x-\frac{1}{2})]^2+1}$$

So

$$\frac{1}{x^3+1} = \frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{6}(2x-1)}{x^2-x+1} + \frac{\frac{1}{\sqrt{3}}\,\frac{2}{\sqrt{3}}}{[\frac{2}{\sqrt{3}}(x-\frac{1}{2})]^2+1}$$

Integrate

$$\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}(x-\frac{1}{2}) + C$$

$\square$