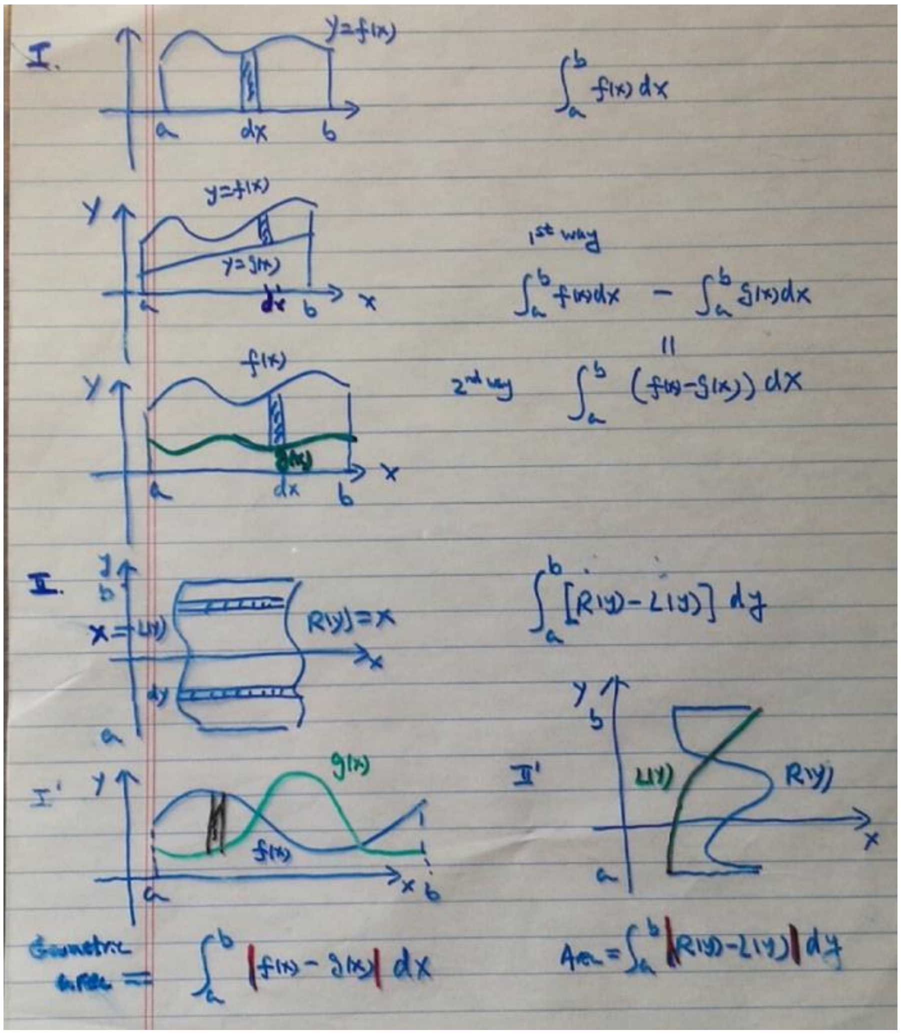
Sec6.1 Area between curves

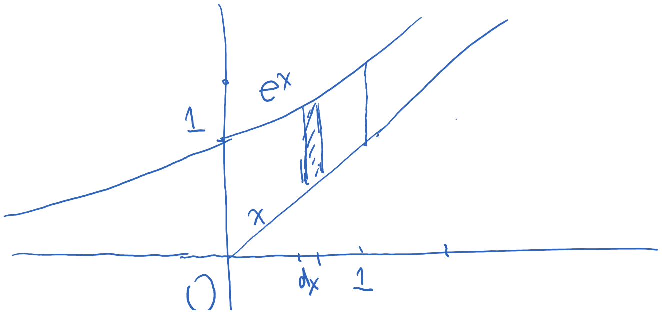
I Cut along horizontal direction(each slice is along v. dir.) $\int_a^b T(x) - B(x)dx$

II Cut along vertical direction(each slice is along h. dir.) $\int_c^d R(y) - L(y)dy$

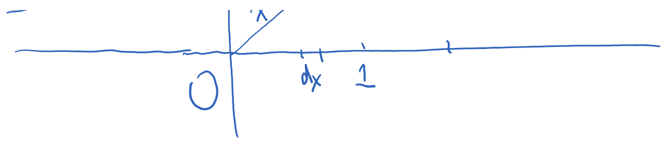
I'/II' geometric area $\int_a^b |T(x) - B(x)|dx$, $\int_c^d |R(y) - L(y)|dy$.



eg1. Find the area of region bounded above by $y = e^x$, below by $y = x$, and bounded on two sides by $x = 0$ and $x = 1$.



$$A = \int_0^1 (e^x - x)\,dx$$
$$= \left(e^x - \tfrac{1}{2}x^2\right)\Big|_0^1$$
$$= \left(e^1 - \tfrac{1}{2}\right) - \left(e^0 - 0\right)$$
$$= e - \tfrac{3}{2}$$



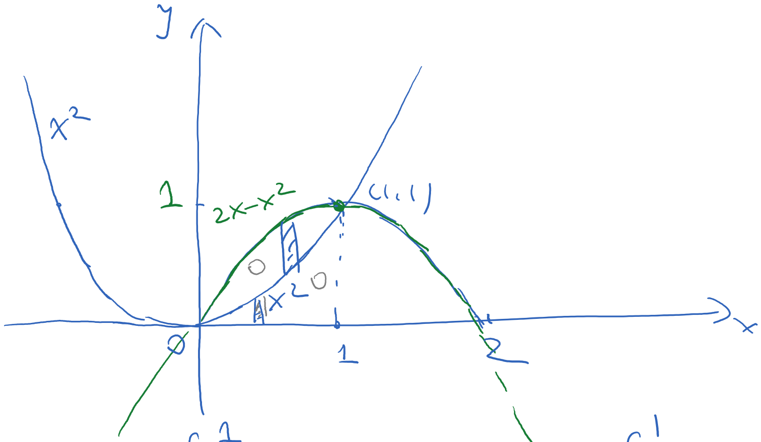
$= (e^1 - \frac{1}{2}) - (e^0 - 0)$

$= e - \frac{3}{2}$

eg2. Find the area of the region enclosed by the parabola $y = x^2$ and $y = 2x - x^2$.

$= x(2-x)$

$x = 0,\ x = 2$ x-intercept.



$\begin{cases} y = x^2 \\ y = 2x - x^2 \end{cases}$

$x^2 = 2x - x^2$

$0 = 2x^2 - 2x = 2x(x-1)$

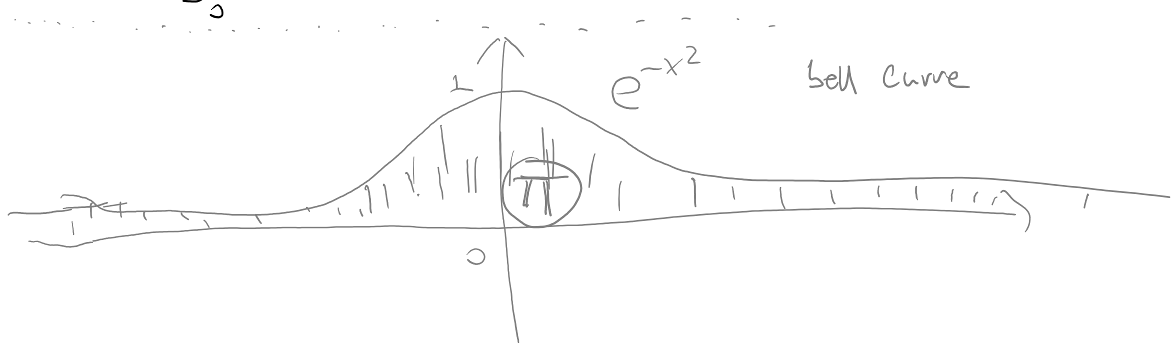
$\begin{cases} x = 0 \\ y = 0 \end{cases}$ $\begin{cases} x = 1 \\ y = 1 \end{cases}$

$A = \int_0^1 (2x - x^2 - x^2)\,dx = \int_0^1 2x - 2x^2\,dx = (x^2 - \frac{2}{3}x^3)\Big|_0^1$

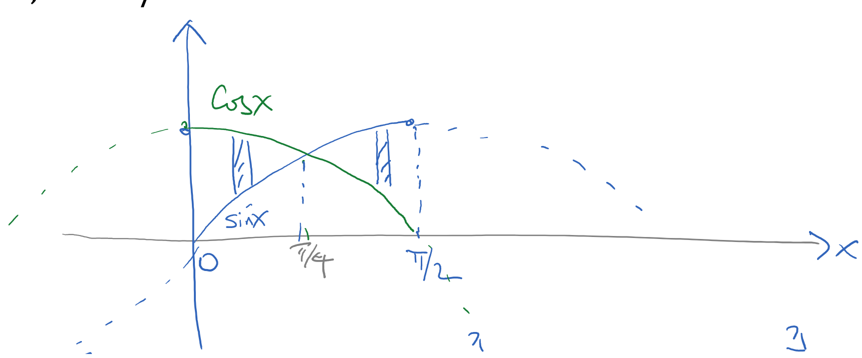
$= \frac{1}{3}$

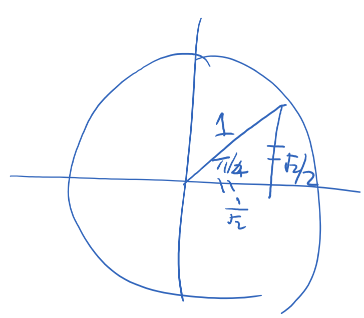
Comparison. Area between x-axis & $y = x^2$ from 0, to 1.

$A = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$



eg5. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0, x = \pi/2$.





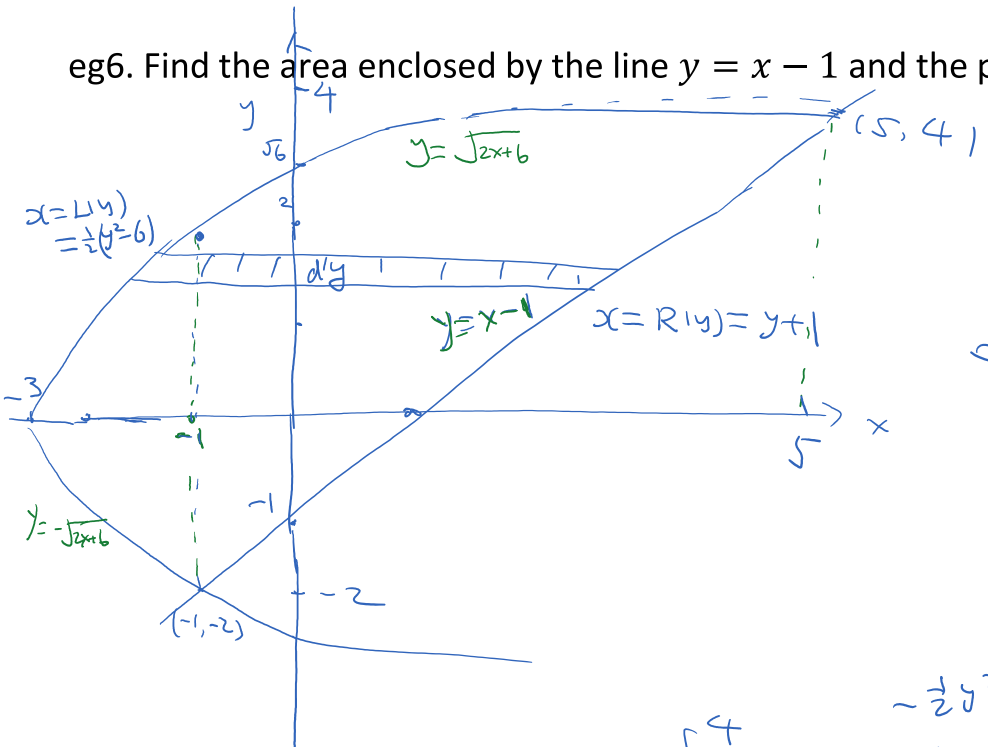
$$A = \int_0^{\frac{\pi}{2}} |\cos x - \sin x|\,dx = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -(\cos x - \sin x)\,dx$$

$$= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + \left[-(\sin x + \cos x)\right]\Big|_{\frac{\pi}{4}}^{\pi/2}$$

$$= \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) - (\sin 0 + \cos 0) + \left[-\left(\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) + \underbrace{\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}\right]$$

$$= 2\sqrt{2} - 2$$

eg6. Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$x = \frac{1}{2}(y^2 - 6)$

$\begin{cases} y = x - 1 \\ y^2 = 2x + 6 \end{cases}$ $\quad x = y + 1$

$y^2 = 2(y+1) + 6$

$0 = y^2 - 2y - 8 = (y-4)(y+2)$

$\begin{cases} y = 4 \\ x = 5 \end{cases}$ $\quad \begin{cases} y = -2 \\ x = -1 \end{cases}$

$$A = \int_{-2}^{4} [R(y) - L(y)]\,dy = \int_{-2}^{4} \Big[y + 1 \underbrace{- \tfrac{1}{2}(y^2 - 6)}_{-\frac{1}{2}y^2 + 3}\Big]\,dy$$

$$= \int_{-2}^{4} \left(y - \tfrac{1}{2}y^2 + 4\right)dy = \left(\tfrac{1}{2}y^2 - \tfrac{1}{6}y^3 + 4y\right)\Big|_{-2}^{4}$$

$$= \left(\tfrac{1}{2}4^2 - \tfrac{1}{6}4^3 + 4\cdot 4\right) - \left[\tfrac{1}{2}(-2)^2 - \tfrac{1}{6}(-2)^3 + 4(-2)\right]$$

$$= 18$$

Horizontal way.

$$A = \int_{-3}^{5} T(x) - B(x)\,dx = \int_{-3}^{-1} \underbrace{\left[\sqrt{2x+6} - (-\sqrt{2x+6})\right]}_{2\sqrt{2x+6}}\,dx + \int_{-1}^{5} \sqrt{2x+6} - (x+1)\,dx$$

$$= \tfrac{2}{2}(2x+6)^{\frac{3}{2}}\Big|^{-1} + \left[\tfrac{1}{3}(2x+6)^{\frac{3}{2}} - \tfrac{1}{2}x^2 - x\right]\Big|^{5} = \cdots = 18$$

-3 -3 -1

$$= \frac{2}{3}(2x+6)^{\frac{3}{2}}\Big|_{-3}^{-1} + \left[-\frac{1}{3}(2x+6)^{\frac{3}{2}} - \frac{1}{2}x^2 - x\right]\Big|_{-1}^{5} = \cdots = 18$$