Lecture 2 Laplace equations

∘ invariance, explicit solutions
∘ mean value formula
maximum principle,
(higher order) derivative estimates,
regularity–analyticity
Harnack
Liouville
∘ Green's functions for balls & half space
∘ weak formulations
mean value
weak or weaker/distribution
viscosity
∘ Energy method, uniqueness, Dirichlet principle, Neumann version

Invariance for Harmonic functions, solutions to $\triangle u = 0$
$\cdot\ u(x + x_0)$
$\cdot\ u(Rx)$
$\cdot\ u(tx)$
RMK. Equations don't know/care which coordinates they are in.
$\cdot\ u + v$, au, where $\triangle v = 0$
$\cdot \int u(x-y)\,\varphi(y)\,dy$
$\cdot \frac{u(x+\varepsilon e)-u(x)}{\varepsilon} \longrightarrow D_e u$, so is $D^k u$
$\cdot \frac{u(R\varepsilon x)-u(x)}{\varepsilon} \longrightarrow D_\theta u = x_i u_j - x_j u_i$
$\cdot\ \frac{u((1+\varepsilon)x)-u(x)}{\varepsilon} \longrightarrow Du(x)\cdot x = r u_r$, so are $r\partial_r(ru_r) = ru_r + \underline{r^2 u_{rr}}$, $r^3 u_{rrr}, \cdots$
$\cdot\ |x|^{2-n}\, u\left(\frac{x}{|x|^2}\right)$ Kelvin transformation

RNK. "Kelvin" transformation for the heat equation $u_t - \triangle u = 0$, $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} u\left(\frac{x}{t}, \frac{-1}{t}\right)$.

Example of harmonic functions
$\cdot\ \sigma_1(x) = x_1 + \cdots + x_n,\ \sigma_2(x) = x_1 x_2 + \cdots, \cdots, \sigma_n(x) = x_1 \cdots x_n$.
$\cdot$ Re / Im $f(z)$ on C^1 or C^n, $\ln z = \ln r + i\theta$
$\cdot$ radial ones

$$\begin{cases} \frac{1}{r^{n-2}} & n \geq 3 \qquad = |x|^{2-n}\ \mathbf{1}\left(x/|x|^2\right) \\ \ln r & n = 2 \qquad \partial_n r^{2-n}\big|_{n=2} = -\ln r \\ r & n = 1 \end{cases}$$

$u(x) = u(r)$
$D_1 u = u_r \frac{x_1}{r}$
$D_{11} u = \frac{1}{r} u_r + \left(\frac{u_r}{r}\right)_r \frac{x_1}{r} x_1 = u_{rr}\frac{x_1^2}{r^2} - u_r \frac{x_1^2}{r^3} + \frac{1}{r} u_r$
$\left(D_{12} u = \left(\frac{u_r}{r}\right)_r \frac{x_2}{r} x_1 = u_{rr}\frac{x_1 x_2}{r^2} - u_r \frac{x_1 x_2}{r^3}\right)$

[0] October 28, 2016

$$\triangle u = u_{rr} - \frac{u_r}{r} + n\frac{u_r}{r} = u_{rr} + (n-1)\frac{u_r}{r}$$
$$= \frac{1}{r^{n-1}}\left(r^{n-1}u_r\right)_r \qquad = 0 \text{ for the above.}$$

RMK. Fourier way: $\triangle u = \delta \quad -|\xi|^2\hat{u} = 1$

$$u(x) = -\int_{R^n}\frac{1}{|\xi|^2}e^{ix\cdot\xi}d\xi$$

satisfies

$$u(Rx) = u(x) \quad \text{radial}$$
$$u(tx) = -\int_{R^n}\frac{t^2}{|t\xi|^2}e^{itx\cdot\xi}\frac{d(t\xi)}{t^n} = \frac{1}{t^{n-2}}u(x) \quad \text{homogeneous order } 2-n.$$

So $u = c_n r^{2-n} + 9.9999$ $n \neq 3$; $n = 2$ is a different story.

More harmonic functions.

eg1.

$$D_1 r^{2-n} = (2-n)\,r^{1-n}\frac{x_1}{r} = (2-n)\,r^{-n}x_1 = (2-n)\frac{x_1}{r^n}$$
$$D_{11}r^{2-n} = (2-n)\left[-nr^{-n-1}\frac{x_1}{r}x_1 + r^{-n}\right] = (2-n)\frac{r^2-nx_1^2}{r^{n+2}}$$
$$D_{12}r^{2-n} = (2-n)\frac{-nx_1x_2}{r^{n+2}}$$

Let $P_k(x)$ be any homogeneous polynomial of degree k, $P_k(D)\,r^{2-n} = \frac{H_k(x)}{r^{n-2-2k}}$. For example, $\sigma_k(D)\,r^{2-n} = \frac{\sigma_k(x)}{r^{n-2-2k}}$. Note $H_k \neq P_k$ in general, but $H_k(x) = r^{2-n}\frac{H_k\left(\frac{x}{r^2}\right)}{\left|\frac{x}{r^2}\right|^{n-2-2k}}$ is the Kelvin transform of harmonic function $P_k(D)\,r^{2-n}$, thus harmonic.

Exercise: $H_k(x)$ are ALL harmonic polynomials of degree k.

$n = 2$, $z^k \dashrightarrow r^k\cos k\theta$, $r^k\sin k\theta$

$n = 3$ $k = 2$ $\quad r^4\left(\frac{xy}{r^4}, \frac{yz}{r^4}, \frac{zx}{r^4}, \frac{x^2-y^2}{r^4}, \frac{y^2-z^2}{r^4}\right)$ spherical harmonics named by Lord Kelvin.

eg2. Harmonic function

$$|x-x_0|^{2-n} - |x|^{2-n}\left|\frac{x}{|x|^2} - x_0\right|^{2-n} \overset{|x|=1}{=} |x-x_0|^{2-n} - |x-x_0|^{2-n} = 0,$$

is Green's function (up to a multiple) for the unit ball.

<u>Mean value equality</u>

Recall the divergence formula (the fundamental theorem of calculus)

$$\int_\Omega div\left(\vec{V}\right)dx = \int_{\partial\Omega}\left\langle\vec{V},\gamma\right\rangle dA.$$

$\vec{V} = Du$, then $0 = \int_{\partial\Omega} u_\gamma dA$.
$\vec{V} = vDu$, then $\int_\Omega \langle Dv, Du\rangle + v \triangle u = \int_{\partial\Omega} vu_\gamma dA$.
$\vec{V} = uDv$, then $\int_\Omega \langle Du, Dv\rangle + u \triangle v = \int_{\partial\Omega} uv_\gamma dA$.

$$\int_\Omega v \triangle u - u \triangle v = \int_{\partial\Omega} vu_\gamma - uv_\gamma dA.$$

Mean value case. For harmonic function u

$$u(0) = \frac{1}{|\partial B_l|}\int_{\partial B_l} ud.$$

Now $\triangle u = 0$ in B_1, take $v = |x|^{2-n}$, $\Omega = B_1 \backslash B_\varepsilon$,

$$B_1 \backslash B_\varepsilon \ figure$$

we then have $0 = \int_{\partial\Omega} vu_\gamma - uv_\gamma dA$, or

$$\overbrace{\int_{\partial(B_1\backslash B_\varepsilon)} vu_\gamma dA}^{0} = \int_{\partial(B_1\backslash B_\varepsilon)} uv_\gamma dA = \int_{\partial B_1} u\frac{(2-n)}{r^{n-1}}dA - \int_{\partial B_\varepsilon} u\frac{(2-n)}{r^{n-1}}dA. \qquad (*)$$

We get $\int_{\partial B_1} udA = \int_{\partial B_\varepsilon} u\frac{1}{\varepsilon^{n-1}}dA \overset{\varepsilon\to 0}{\to} |\partial B_1|\, u(0)$. So $u(0) = \frac{1}{|\partial B_1|}\int_{\partial B_1} udA$. Also for all other radius l by scaling $w(x) = u(lx)$.

RMK. In hindsight one just takes $v = \frac{-1}{(n-2)|\partial B_1|}\frac{1}{|x|^{n-2}} \overset{\text{def}}{=} \Gamma$.

"Solid" formulations:

$$u(0) = \frac{1}{|B_1|}\int_{B_1(0)} udx$$

Take a weight function nr^{n-1}, from $1 = \int_0^1 nr^{n-1}dr$ we have

$$u(0) = \int_0^1 nr^{n-1}\left[\frac{1}{|\partial B_1|\, r^{n-1}}\int_{\partial B_r(0)} udA\right]dr = \frac{1}{|B_1|}\int_0^1\int_{\partial B_r(0)} udAdr = \frac{1}{|B_1|}\int_{B_1(0)} udx.$$

Also scaled version

$$u(0) = \frac{1}{|B_r|}\int_{B_r(0)} udx.$$

Take another weight $2nr^{2n-1}$, from $1 = \int_0^1 2nr^{2n-1} dr$ we have

$$
\begin{aligned}
u(0) &= \int_0^1 2nr^{2n-1} \left[\frac{1}{|\partial B_1| \, r^{n-1}} \int_{\partial B_r(0)} u dA \right] dr = \frac{2}{|B_1|} \int_0^1 \int_{\partial B_r(0)} r^n u dA dr \\
&= \frac{2}{|B_1|} \int_{B_1(0)} |x|^n u dx.
\end{aligned}
$$

Take a radial weight $\varphi(x) = \varphi(r) \in C_0^\infty(B_1(0))$ such that $1 = \int_{R^n} \varphi dx = \int_0^1 \varphi(r) |\partial B_1| r^{n-1} dr$, then we have

$$
u(0) = \int_0^1 \varphi(r) |\partial B_1| r^{n-1} \left[\frac{1}{|\partial B_1| \, r^{n-1}} \int_{\partial B_r(0)} u dA \right] dr = \int_0^1 \int_{\partial B_r(0)} \varphi u dA dr = \frac{1}{|B_1|} \int_{B_1(0)} \varphi u dx.
$$

RMK1. Tracing the sign of $\triangle u$ and noticing $\displaystyle\int_{\partial B_\varepsilon} v u_\gamma dA \overset{\varepsilon \to 0}{\to} 0$, one gets mean value inequalities for superharmonic functions $\triangle u \leq 0 : u(0) \leq ⨍ u$ and subharmonic functions $\triangle u \geq 0 : u(0) \leq ⨍ u$.

RMK2. "$\cdots$ all the women are strong, all the men are good-looking, and all the children are above average." –A Prairie Home Companion with Garrison Keillor.

RMK3. Also for $u \in C_0^\infty(\mathbb{R}^n)$, after applying Green's formula,

$$
u(x) = \int_{\mathbb{R}^n} \frac{-1}{(n-2) n |B_1|} \frac{1}{|x-y|^{n-2}} \triangle u(y) dy = \Gamma * u.
$$

Here the singular integral $\int_{R^n} |x-y|^{2-n} f(y) dy$ is defined as $\lim_{\varepsilon \to 0} \int_{R^n \backslash B_\varepsilon(x)} |x-y|^{2-n} f(y) dy$ for any $f \in C_0^\infty(\mathbb{R}^n)$.

Green case. Still $\triangle u = 0$ in B_1, but

$$
v = G(x, x_0) = \frac{-1}{(n-2) |\partial B_1|} \left(|x - x_0|^{2-n} - |x|^{2-n} \left| \frac{x}{|x|^2} - x_0 \right|^{2-n} \right),
$$

$\Omega = B_1 \backslash B_\varepsilon(x_0)$.

$\mathrm{B}_1 \backslash \mathrm{B}_\varepsilon(x_0)$ figure

Taking limits on two ends of (*), we get Poisson formula

$$
u(x_0) = \int_{\partial B_1} \frac{\partial G(x, x_0)}{\partial \gamma_x} u(x) dA = \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1 - |x_0|^2}{|x - x_0|^n} u(x) dA.
$$

Note $u(x) = \int_{\partial B_1} \frac{\partial G(y,x)}{\partial \gamma_y} \varphi(y) dA_y$, as sum of harmonic functions $\frac{1-|x|^2}{|y-x|^n}$, is harmonic–smooth, analytic in terms regularity—-for $\varphi \in C^0, L^1, \cdots$.

Application 1. Strong maximum principle (No toughing).

$$
\begin{aligned}
\triangle u_1 = \triangle u_2 = 0 \\
u_1 \geq u_2, \text{ ``}=\text{''} \text{ at } 0
\end{aligned}
$$

then

$$0 = u_1(0) - u_2(0) = \frac{1}{|B_r|} \int_{B_r} (u_1 - u_2)\, dx \geq 0.$$

It follows that $u_1 \equiv u_2$.

Application 2. Smooth effect and derivative test.

Take radial weight $\varphi(y) = \varphi(|y|) \in C_0^\infty(R^n)$ such that $1 = \int \varphi(y)\, dy = \int_0^\infty \varphi(r)\, |\partial B_r|\, dr$. Then

$$\begin{aligned}
\int_{R^n} u(y)\, \varphi(x-y)\, dy &= \int_0^\infty \int_{\partial B_r(x)} u(y)\, \varphi(x-y)\, dA dr \\
&= \int_0^\infty u(x)\, \varphi(r)\, |\partial B_r|\, dr = u(x) \int \varphi(y)\, dy \\
&= u(x).
\end{aligned}$$

Consequence $u(x) = \int_{R^n} u(y)\, \varphi(x-y)\, dy$ is smooth for continuous "initial" $u(y)$, and

$$D^k u(0) = \int u(y)\, D_x^k \varphi(x-y)\, dy = (-1)^k \int u(y)\, D_y^k \varphi(x-y)\, dy.$$

Thus

$$\left| D^k u(0) \right| \leq C(k, n, \varphi)\, \|u\|_{L^1(B_1)} \leq C(k, n, \varphi)\, |B_1|\, \|u\|_{L^\infty(B_1)}$$

Scaled and positive versions

$$\begin{aligned}
v : B_R &\to R \ \text{ harmonic} \\
u(x) = v(Rx) : B_1 &\to R
\end{aligned}$$

$$|D^\alpha u(0)| = \left| D^\alpha v(0)\, R^{|\alpha|} \right| \leq C(k, n, \varphi) \frac{\|v\|_{L^1(B_R)}}{R^n} \leq C(k, n, \varphi)\, |B_1|\, \|v\|_{L^\infty(B_R)},$$

that is

$$\left| D^k v(0) \right| \leq \left\{ \begin{array}{l} \frac{C(k,n,\varphi)\|v\|_{L^1(B_R)}}{R^{n+k}}, \quad \text{also } \frac{C(k,n,\varphi)|B_1|}{R^k} v(0), \text{ provided } u \geq 0 \\ \frac{C(k,n,\varphi)\|v\|_{L^\infty(B_R)}}{R^k} \end{array} \right. .$$

RMK. The larger the domain, the flatter the harmonic graph.

Sharper estimates for analyticity

"Direct" way: try concrete φ, calculate its derivatives, to have

$$C(n, |\alpha|) \leq |\alpha|! C^{|\alpha|}, \quad \text{have not found any } \varphi \text{ yet.}$$

"Canonical" way:

$$|D^\alpha u(0)| \leq (|\alpha|\, n)^{|\alpha|}\, \|u\|_{L^\infty(B_1)}.$$

Step 1.

$$\begin{aligned}
D_1 v(0) &= \frac{1}{|B_R|} \int_{B_R} D_1 v dx = \frac{1}{|B_R|} \int_{B_R} \operatorname{div}(v, 0, \cdots, 0)\, dx = \frac{1}{|B_R|} \int_{\partial B_R} v \gamma_1 dA \\
&\leq \frac{|\partial B_R|}{|B_R|} \|u\|_{L^\infty(B_R)},
\end{aligned}$$

then, say $D_1 v(0) = |Dv(0)|$

$$|Dv(0)| \le \frac{n}{R} \|u\|_{L^\infty(B_R)}.$$

Step 2. Fill in the unit ball with balls with radii $\frac{1}{k}, \frac{2}{k}, \cdots, \frac{k-1}{k}, 1,$

figure

we have

$$\begin{aligned}\left|D^k v(0)\right| &\le kn \left\|D^{k-1}v\right\|_{L^\infty\left(B_{1/k}\right)} \le (kn)^2 \left\|D^{k-1}v\right\|_{L^\infty\left(B_{2/k}\right)} \le \cdots \\ &\le (kn)^k \|v\|_{L^\infty(B_1)} \le k!\,(en)^k \|v\|_{L^\infty(B_1(0))},\end{aligned}$$

where we used $k!e^k = k!\left(1 + \cdots + \frac{k^k}{k!} + \cdots\right) \ge k^k$. Generally we have

$$|D^\alpha v(0)| \le \frac{(|\alpha|\, n)^{|\alpha|}}{R^{|\alpha|}} \|v\|_{L^\infty(B_R)} \le \frac{|\alpha|!\,(en)^{|\alpha|}}{R^{|\alpha|}} \|v\|_{L^\infty(B_R)}.$$

Step 3. For now smooth harmonic function u, we have its Taylor expansion

$$u(x) = u(0) + Du(0)\cdot x + \cdots + \frac{1}{(N-1)!} \sum_{|\alpha|=N-1} \frac{D^\alpha u(0)\, x^\alpha}{\alpha!} + R_N$$

$$R_N = \frac{1}{N!} \sum_{|\alpha|=N} D^\alpha u(*)\, x^\alpha = \frac{1}{N!} |x|^N D^N_\omega u(*) \qquad \text{with } \omega = \frac{x}{|x|}.$$

This can be shown via integration by parts. Setting $f(t) = u(tx)$, then

$$\begin{aligned} f(1) &= f(0) + \int_0^1 f'(s)\, ds \\ &= f(0) + (s-1) f'(s)\big|_0^1 - \int_0^1 (s-1) f''(s)\, ds \\ &\cdots \\ &= f(0) + f'(0) + \frac{1}{2!} f''(0) + \cdots + \frac{1}{(N-1)!} f^{(N-1)}(0) + \frac{(-1)^N}{(N-1)!} \int_0^1 (s-1)^{N-1} f^{(N)}(s)\, ds \\ &= f(0) + f'(0) + \frac{1}{2!} f''(0) + \cdots + \frac{1}{(N-1)!} f^{(N-1)}(0) + \frac{1}{N!} f^{(N)}(s^*).\end{aligned}$$

Now we use the polarized expression $|x|^N D^N_\omega u(*)$ to get a better estimate, at the same time, with simpler argument. Note $f'(t) = x \cdot Du(tx) = |x|\, D_\omega u(tx)$ and also $f^{(N)}(t) = |x|^N D^N_\omega u(tx)$. We then have from Step 2

$$\begin{aligned}|R_N| &= \frac{1}{N!} |x|^N \left|D^N_\omega u(*)\right| \le \frac{1}{N!} |x|^N N!\,(en)^N \|u\|_{L^\infty\left(B_{|x|+1}(0)\right)} \\ &= (|x|\, en)^N \|u\|_{L^\infty\left(B_{|x|+1}(0)\right)} \overset{N\to\infty}{\to} 0\end{aligned}$$

once $|x| < \frac{1}{en}$.

RMK. The above "canonical" way is a model for analyticity of solutions to analytic elliptic equations.

"Elementary" way.

Starting from Green's formula

$$\int_{B_1} v \triangle u - u \triangle v dx = \int_{\partial B_1} v u_\gamma - u v_\gamma dA$$

with $v(y) = c_n \frac{1}{|x-y|^{n-2}} = K(x-y)$, we have

$$u(x) = \int_{\partial B_1} K(x-y) u_\gamma - y \cdot D_y K(x-y) u dA(y).$$

Let us write the kernels as analytic functions of x :

$$|x-y|^{2-n} = \left(|x|^2 - 2x\cdot y + |y|^2\right)^{1-n/2} = |y|^{2-n}\left(1 + \frac{|x|^2 - 2x \cdot y}{|y|^2}\right)^{1-n/2}$$
$$\overset{|y|=1}{=} 1 + \begin{pmatrix} 1-n/2 \\ 1 \end{pmatrix}\left(|x|^2 - 2x\cdot y\right) + \begin{pmatrix} 1-n/2 \\ 2 \end{pmatrix}\left(|x|^2 - 2x \cdot y\right)^2 + \cdots$$

is analytic in terms x for small x. So is $y \cdot D_y |x-y|^{2-n}$ for small x and $|y| = 1$.

RMK. "Wave" functions are not necessarily smooth, let alone analytic.

Application 3. Harnack inequalities–a quantitative version of the strong maximum principle.

eg. Consider positive harmonic functions r^{2-n}, $x_1 r^{-n}$ on $\{x_1 > 0\}$.

$r^{2-n}, x_1 r^{-n}$ graph $figure$

eg. In general for $\triangle \mathrm{u} = 0,\ u > 0$ in $B_1(0)$, we have

$$u(x) = \frac{1}{\left|B_{1-|x|}\right|}\int_{B_{1-|x|}(x)} u dx \le \frac{1}{\left|B_{1-|x|}\right|}\int_{B_1(0)} u dx = \frac{|B_1|}{\left|B_{1-|x|}\right|} u(0) = \frac{1}{(1-|x|)^n} u(0).$$

RNK. As those two examples suggest, from estimating the kernel of Poisson representation, we have a sharper comparison

$$\frac{(1-|x|)}{2^{n-1}} u(0) \le u(x) \le \frac{2}{(1-|x|)^{n-1}} u(0).$$

Going along a chain of balls with geometrically increasing radii, from the boundary, using the following Harnack, one can get a super linear growth out of the boundary for non-negative harmonic functions. Question: can one refine the Harnack chain argument to make a linear growth from the boundary of the ball? For $C^{1,1}$ boundary, the linear growth is true by simple barrier argument. This is the content of Hopf boundary lemma. For less smooth domains, no linear growth. For example: $y^2 - x^2$ in the domain $\{y > |x|\}$.

Harnack. Suppose $\triangle \mathrm{u} = 0,\ u > 0$ in $B_r(x_0)$. Then we have

$$\sup_{B_{r/4}(x_0)} u \le 3^n \inf_{B_{r/4}(x_0)} u.$$

In fact

4 circle figure $\mathrm{B}_1, B_{1/4}, B_{1/4}(x_{\max}), B_{3/4}(x_{\min})$

$$\begin{aligned}\max_{B_{1/4}} u = u(x_{\max}) &= \frac{1}{|B_{1/4}|}\int_{B_{1/4}(x_{\max})} udx \\ &\le \frac{1}{|B_{1/4}|}\int_{B_{3/4}(x_{\min})} udx \\ &= 3^n u(x_{\min}) = 3^n \min_{B_{1/4}} u.\end{aligned}$$

Consequences $\cdots$, for example one sided Liouville for entire harmonic functions.

RMK. Harnack inequality is in fact a quantitative version of the strong maximum principle. It measures how much the minimum leaps when moving inside, or flipping around how much the maximum drops when moving inside. For example, to move inside $B_{1/4}$ from B_1,

$$\min_{B_{1/4}}(u - m_1) \ge 3^{-n} \max_{B_{1/4}}(u - m_1)$$

or

$$m_{1/4} \ge m_1 + 3^{-n}\left(M_{1/4} - m_1\right).$$

The flip version is

$$\min_{B_{1/4}}(M_1 - u) \ge 3^{-n} \max_{B_{1/4}}(M_1 - u)$$

or

$$M_{1/4} \le M_1 - 3^{-n}\left(M_1 - m_{1/4}\right).$$

(This should be Moser's observation: subtracting the leap from the drop, one has oscillation decay of the "harmonic" function.)

Liouville Theorem. Given $\triangle u = 0$ and $|u| \le M$ in R^n, then $u \equiv$constant.

Differential way. u is smooth, and $|Du(0)| \le \frac{n}{R}\|u\|_{L^\infty(B_R)} \le \frac{n}{R}M \to 0$, as $R \to \infty$. Similarly $Du \equiv 0$ everywhere, and then u is constant.

Harnack way. $\inf_{R^n} u = m$, $u - m \ge 0$, we then have

$$0 \le \sup_{B_R}(u - m) \le \inf_{B_R}(u - m) \to 0, \quad \text{as} \quad R \to \infty.$$

Harnack way also gives the following

One sided Liouville. $\triangle u = 0$ and $u \geq -M$ in R^n, then $u \equiv$constant.

Fundamental theorem of algebra: There is a root for any polynomial equation $P(z) = 0$ in $C = R^2$. Suppose otherwise, then harmonic function $\ln|P(z)| = \mathrm{Re}\ln P(z)$ has a lower bound in R^2. By one sided Liouville, it is constant. An inconsistence. Two sided cut: $1/P(z)$ would be bounded holomorphic, or two bounded harmonic functions. By Liouville, they are constant. A contradiction.

Green's function: Dirichlet and Neumann problem for balls and half space

* "wave" function can be determined by initial position and velocity

* minimal surface is determined by boundary height, nonlinear version of harmonic functions; given pay-off at the boundary, should know expectation of a random walk starting anywhere inside (Dirichlet Problem)

* steady state irrotational/incompressible fluid can be determined by boundary flux (Neumann problem)

Q. Why not Cauchy data for harmonic functions?

A1. No special direction like time

A2. Green's identity

$$\int_\Omega u \triangle v - v \triangle u = \int_{\partial\Omega} uv_\gamma - vu_\gamma dA$$

$\cdot\ v = 1 \quad \int_{\partial\Omega} u_\gamma dA = 0$, cannot arbitrarily assign u_γ

$\cdot\ v = x_1 x_2 \quad \int_{\partial\Omega} uv_\gamma - vu_\gamma dA = 0.$

Q. What can be described then?

A. Dirichlet, Neumann, and mixed for closed domain.

Exercise. [J] Sec3.3 Problem4: Solve for harmonic function with Cauchy data

$$\begin{cases} \triangle u = 0 \ \text{ in } S^1 \times (1-\varepsilon, 1+\varepsilon) \\ u(1,\theta) = \varphi \\ u_r(1,\theta) = \psi \end{cases} \qquad \varphi, \psi \ \text{ analytic} .$$

Dirichlet.

Pick $v(x,y) = G(x,y)$ such that $\triangle_y v = \delta(x)$ and $v(x, \partial\Omega) = 0$ to pick up $u(x)$ and ignore u_γ. We have the fundamental solution for $\delta(x)$. To take care of boundary, we subtract a harmonic function

$$G(x,y) = \frac{c_n}{|x-y|^{n-2}} - h(x,y)\Bigg|_{y\in\partial\Omega} = 0.$$

Now the hard to find h need to satisfy

$$\begin{cases} \triangle_y h(x,y) = 0 \quad \text{in } \Omega \\ h(x,y) = \frac{c_n}{|x-y|^{n-2}} \ \text{ for } y \in \partial\Omega \end{cases} .$$

Then

$$u(x) = \int_\Omega G(x,y) f(y)\, dy + \int_{\partial\Omega} \varphi(y) G_{\gamma(y)}(x,y)\, dA(y), \quad G_{\gamma(y)}(x,y) \text{ harmonic}^{\clubsuit} \text{ in } x$$

solves

$$\begin{cases} \triangle u(x) = f(x) & \text{in } \Omega \\ u(x) = \varphi(x) & \text{on } \partial\Omega \end{cases}.$$

Surprisingly in the sense $\lim_{\Omega\ni x\to y} u(x) = \varphi(y)$ for smooth enough domain and continuous φ.

RMK. The limiting behavior for general domain is hard to justify. For smooth domains $G(a,y)$ is smooth up to the boundary. By the following symmetry, $\triangle_x G(x,y) = 0$ and then $G_{\gamma(y)}(x,y)$ is harmonic$^{\clubsuit}$ in x, $\triangle_x\, G_{\gamma(y)}(x,y) = 0$ for $y \in \Omega$.

Symmetry of Green's function: $G(x,y) = G(y,x)$ for $x \neq y \in \Omega$.

From the ball case, we see this symmetry as follows

$$\begin{aligned} G(x,y) &= \frac{-1}{(n-2)\,|\partial B_1|}\left[|x-y|^{2-n} - |y|^{2-n}\left|x - \frac{y}{|y|^2}\right|^{2-n}\right] = c_n\left[* - \left||y|\,x - \frac{y}{|y|}\right|^{2-n}\right] \\ &= c_n\left[|x-y|^{2-n} - \left||y|^2|x|^2 - 2x\cdot y + 1\right|^{1-n/2}\right]. \end{aligned}$$

In general, the symmetry is still valid, as we readily read off from Green's formula

$$\int_\Omega u \triangle v - v \triangle u = \int_{\partial\Omega} u v_\gamma - v u_\gamma dA.$$

Take $u(y) = G(a,y)$ and $v(y) = G(b,y)$ with $a \neq b \in \Omega$. Recall they both vanish on the boundary of Ω, are smooth (harmonic) away from a and b respectively, and δ_a and δ_b respectively after Laplace. Equivalently and actually what is really going on is the following. The domain we are working on is the one after deleting two small balls around a and b, $\Omega \backslash \{B_\varepsilon(a) \cup B_\varepsilon(b)\}$. Now the Green's formula reads

$$\begin{aligned} 0 &= \int_{\partial B_\varepsilon(a)} u v_\gamma - v u_\gamma dA + \int_{\partial B_\varepsilon(b)} u v_\gamma - v u_\gamma dA \\ &= \int_{\partial B_\varepsilon(a)} O(1)\,\varepsilon^{2-n} - [v(a) + o(1)]\frac{1}{|\partial B_1|\,\varepsilon^{n-1}} dA \\ &+ \int_{\partial B_\varepsilon(b)} [u(b) + o(1)]\frac{1}{|\partial B_1|\,\varepsilon^{n-1}} - O(1)\,\varepsilon^{2-n} dA. \end{aligned}$$

Let ε go to 0, we see $v(a) = u(b)$, that is $G(b,a) = G(a,b)$.

Example: Ball $\Omega = B_1$, $G_{\gamma(y)} = G_r(x,y) = \frac{1}{|\partial B_1|}\frac{1-|x|^2}{|x-y|^n}$, that is, Poisson formula (for all dimension n)

$$u(x) = \frac{1}{|\partial B_1|}\int_{\partial B_1(0)} \frac{1-|x|^2}{|x-y|^n}\varphi(y)\, dA(y)$$

solves

$$\begin{cases} \triangle u(x) = 0 & \text{in } B_1(0) \\ u(x) = \varphi(x) & \text{on } \partial B_1, \quad \text{i.e.} \quad \lim_{B_1 \ni x \to y} u(x) = \varphi(y) \end{cases}.$$

In particular,

$$u(0) = \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} \varphi(y)\, dA(y) \quad \text{and} \quad 1 = \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} \frac{1 - |x|^2}{|x-y|^n} 1\, dA(y).$$

Derivation of Poisson kernel.

By the symmetry

$$\begin{aligned} G(x,y) &= c_n \left[|x-y|^{2-n} - |y|^{2-n} \left| x - \frac{y}{|y|^2} \right|^{2-n} \right] \\ &= c_n \left[|x-y|^{2-n} - \left| |y|^2 |x|^2 - 2x \cdot y + 1 \right|^{1-n/2} \right] \\ &= c_n \left[|x-y|^{2-n} - |x|^{2-n} \left| y - \frac{x}{|x|^2} \right|^{2-n} \right] \end{aligned}$$

we have

$$D_y G(x,y) = -c_n (n-2) \left[\frac{y-x}{|y-x|^n} - \frac{1}{|x|^{n-2}} \frac{y - \bar{x}}{|y-\bar{x}|^n} \right] \quad \text{with } \bar{x} = \frac{x}{|x|^2}$$

and

$$\frac{y}{|y|} \cdot D_y G(x,y) = -c_n (n-2) \frac{1}{|y|} \left[\frac{|y|^2 - x \cdot y}{|y-x|^n} - \frac{|x|^2 \left(|y|^2 - \bar{x} \cdot y \right)}{|x|^n \, |y-\bar{x}|^n} \right]$$

$$\overset{|y|=1}{=} -c_n (n-2) \frac{1 - |x|^2}{|y-x|^n} \qquad \text{recall} \quad |x|^2 |y - \bar{x}|^2 = |x|^2 |y|^2 - 2x \cdot y + 1 = |y|^2 |x - \bar{y}|^2.$$

RMK. Though Poisson formula takes the same form in all dimensions n, the Green's function takes different forms in dimension $n = 2$ and $n = 1$.

$$n = 2: \quad G(x,y) = \frac{1}{2\pi} \left(\ln|x-y| - \ln|y| - \ln\left| x - \frac{y}{|y|^2} \right| \right).$$

Note the corrector $\ln\left| x - y|y|^{-2} \right|$ equals $\ln|x-y|$ for $|y| = 1$, but is not smooth for y near 0. The right choice is $\ln|y| + \ln\left| x - y|y|^{-2} \right|$, which can also be seen by taking the derivative of the Green's function in general dimension with respect to dimension n.

$$n = 1: \quad G(x,y) = \frac{1}{2} \left[|x-y| - |xy - 1| \right] = \begin{cases} \frac{1}{2}(y - x + xy - 1) & \text{for } y > x \\ \frac{1}{2}(x - y + xy - 1) & \text{for } y < x \end{cases}.$$

From Green's formula

$$u(x) = -c_n (n-2) \int_{\partial B_1(0)} \frac{1 - |x|^2}{|y-x|^n} \varphi(y)\, dA(y).$$

Testing with $u \equiv 1$, we see $-c_n (n-2) = |\partial B_1|^{-1}$.

Limit justification.

The intuition is that the positive Poisson kernel $P(x,y)$ approaches the "boundary" delta function $\delta(y_0)$ as x goes to y_0. For any small $\varepsilon > 0$, as φ is continuous, there exists δ such that $|\varphi(y) - \varphi(y_0)| \le \varepsilon$ whenever $|y - y_0| \le \delta$. Take $|x - y_0| \le \eta(\varepsilon)$ to be determined later. Let us measure the difference

$$
\begin{aligned}
u(x) - \varphi(y_0) &= \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1-|x|^2}{|x-y|^n} \left[\varphi(y) - \varphi(y_0)\right] dA(y) \\
&= \underbrace{\frac{1}{|\partial B_1|} \int_{\partial B_1 \cap B_\delta(y_0)}}_{I} + \underbrace{\frac{1}{|\partial B_1|} \int_{\partial B_1 \backslash B_\delta(y_0)}}_{II}.
\end{aligned}
$$

$$
|I| \le \frac{\varepsilon}{|\partial B_1|} \int_{\partial B_1} \frac{1-|x|^2}{|x-y|^n} dA(y) \le \varepsilon
$$

$$
\begin{aligned}
|II| &\le \frac{2\|\varphi\|_{L^\infty(\partial B_1)}}{|\partial B_1|} \int_{\partial B_1 \backslash B_\delta(y_0)} \frac{(y_0+x)\cdot(y_0-x)}{|x-y|^n} dA(y) \\
&\overset{\eta(\delta)\le\delta/2}{\le} \frac{2\|\varphi\|_{L^\infty(\partial B_1)}}{|\partial B_1|} \int_{\partial B_1 \backslash B_\delta(y_0)} \frac{2\eta(\delta)}{|\delta/2|^n} dA(y) \\
&\le 2^{n+2} \|\varphi\|_{L^\infty(\partial B_1)} \frac{\eta(\delta)}{|\delta|^n} \overset{\eta(\delta)=\varepsilon\delta^n}{\le} \|\varphi\|_{L^\infty(\partial B_1)} 2^{n+2}\varepsilon.
\end{aligned}
$$

Therefore $|u(x) - \varphi(y_0)| \le \varepsilon + \|\varphi\|_{L^\infty(\partial B_1)} 2^{n+2}\varepsilon$ if $|x - y_0| \le \eta(\delta)$.

RMK. Recall $G(x,y)$ vanishes for $x \in \partial B_1$ and the integral of $|x-y|^{2-n}$ over small ball is small, then one can show that $\int_{B_1} G(x,y) f(y)\, dy$ goes to 0 as x goes to the boundary ∂B_1 for say, continuous f ($f \in C^a$ is enough to make sense of the equation $\triangle u = f$). Thus the Green-Poisson representation in ♣ for $\triangle u = f$ and $u = \varphi$ on ∂B_1 really takes the boundary value, say when $f \in C^\alpha$ and $\varphi \in C^0$.

Poisson formula scaled version

$$
u(x) = \frac{1}{|\partial B_1|} \int_{\partial B_R} \frac{R^2 - |x|^2}{R\,|x-y|^n} \varphi(y)\, dA(y).
$$

Two ways: guess or scaling $u(x) = v(Rx)$.

Neumann problem

$$
\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u_\gamma = \psi & \text{on } \partial\Omega \end{cases}.
$$

Try to pick $v(x,y) = G(x,y)$ (in Green's formula) such that $\triangle_y v = \delta(x)$ and $v_{\gamma(y)} \overset{y\in\partial\Omega}{=} 0$.

Green function should be like

$$
\begin{cases} G(x,y) = c_n \frac{1}{|x-y|^{n-2}} + h(x,y) \\ G_{\gamma(y)} = 0 \ \text{ for } y \in \partial\Omega \end{cases}.
$$

But

$$\int_{\partial\Omega} G_{\gamma(y)} dA(y) = \int_{\partial B_\varepsilon(x)} G_{\gamma(y)} dA(y) \to 1, \text{ as } \varepsilon \to 0.$$

So we can "only" go with $G_{\gamma(y)} = 1/|\partial\Omega|$ for $y \in \partial\Omega$ (then the representation is up to a constant), and the harmonic h satisfies

$$\begin{cases} \triangle_y h(x,y) = 0 & \text{in } \Omega \\ h_{\gamma(y)} = \frac{1}{|\partial\Omega|} - \partial_{\gamma(y)} \frac{c_n}{|x-y|^{n-2}} & \text{on } \partial\Omega \end{cases}.$$

This harmonic corrector h is hard to come by.

A direct way on the sphere.

$$u(x) = \int_{\partial B_1} K(x,y)\, \psi(y)\, dA(y) + const.$$

where

$$K(x,y) = \int \frac{P(x,y)}{r} dr \quad \text{with } r = |x|,$$

solves

$$\begin{cases} \triangle u = 0 & \text{in } B_1 \\ u_\gamma = \psi & \text{on } \partial B_1 \quad \text{with } \int_{\partial B_1} \psi = 0 \end{cases}.$$

Derivation of K : Look for a kernel $K(x,y)$ so that

$$\begin{aligned} &\triangle_x K(x,y) = 0 \\ &K_r(x,y) = \frac{x}{|x|} \cdot D_x K = P(x,y) \quad \text{Poisson kernel,} \end{aligned}$$

then would pick up the Neumann boundary value. However, K_r is not a harmonic function in terms of x (to match harmonic $P(x,y)$), only rK_r is. Thus we are lead to

$$rK_r = P + const.$$

and

$$K = \int \frac{P(x,y)}{r} dr + c \ln r.$$

Observe, up to a $\ln r$ term, the indefinite integral is harmonic in terms of x. This is because the x-analytic function P's Taylor expansion consists of homogeneous harmonic polynomials, operation $\int \frac{\cdot}{r} dr$ preserves harmonicity up to $\ln r$ (operation rD_r preserves all harmonicity).

We proceed with the integral

$$\begin{aligned} K &= \int \frac{1}{r} \frac{1-r^2}{\left(1 + r^2 - 2r\ \frac{x}{|x|} \cdot y\right)^{n/2}} dr + c \ln r \\ &= \int \frac{1}{r} \frac{1-r^2}{(1 + r^2 - 2r\ \cos)^{n/2}} dr + c \ln r. \end{aligned}$$

$n = 2$

$$\begin{aligned}
&\int \frac{1}{r} \frac{1 + r^2 - 2r \ \cos - 2r^2 + 2r \ \cos}{1 + r^2 - 2r \ \cos} dr + c \ln r \\
&= \int \frac{1}{r} - \frac{2 (r - \ \cos)}{1 + r^2 - 2r \ \cos} dr + c \ln r \\
&= \ln r - \ln \left(1 + r^2 - 2r \ \cos\right) + c \ln r \\
&\overset{c=-1}{=} - \ln \left(1 + r^2 - 2r \ \cos\right).
\end{aligned}$$

And

$$u(x) = \frac{-1}{2\pi} \int_{\partial B_1} \ln \left(1 + |x|^2 - 2x \cdot y\right) \quad \psi(y)\, dA(y)$$

is harmonic and takes the Neumann boundary value $u_\gamma = \psi$.

$n = 3, 5, 7, \cdots$ Question: Can one integrate out an explicit kernel K?

$n = 4, 6, 8, \cdots$ is fine.

RMK. • In complex analysis, the Cauchy integral $f(z) = \frac{1}{2\pi i} \int_C \frac{1}{\xi - z} f(\xi)\, d\xi$ is nice:

$$\frac{1}{2\pi i} \int_C \frac{1}{\xi - z} f(\xi)\, d\xi \overset{z \to \xi_0 \in C}{\longrightarrow} f(\xi_0).$$

But it requires $f(\xi)$ to be holomorphic near contour C.

• Now a variant (for real contour value), Non-Cauchy integral

$$f(z) = \frac{1}{2\pi i} \int_C \left[\frac{1}{\xi - z} + h(\xi, z)\right] f(\xi)\, d\xi$$

still holomorphic in terms of z, no matter what boundary value $f(\xi)$ is, in particular $f(\xi) = \varphi(\xi)$ real. (Splitting Schwarz kernel), we "add" the following

$$\frac{1}{\xi - z} + \frac{1}{\xi - z} - \frac{1}{\xi} = \frac{\xi + z}{\xi - z} \frac{1}{\xi}.$$

Then

$$\begin{aligned}
u + iv = f(z) &= \frac{1}{2\pi i} \int_{\partial B_1} \frac{\xi + z}{\xi - z} \frac{1}{\xi} \varphi(\xi)\, d\xi \qquad \text{Schwarz kernel} \\
&= \frac{1}{2\pi} \int_{\partial B_1} \frac{\xi + z}{\xi - z} \varphi(\phi)\, d\phi \ = \frac{1}{2\pi} \int_{\partial B_1} \frac{|\xi|^2 - |z|^2 - \xi \bar{z} + z \bar{\xi}}{|\xi - z|^2} \varphi(\phi)\, d\phi \\
&= \frac{1}{2\pi} \int_{\partial B_1} \frac{1 - r^2 + i 2r \sin(\theta - \phi)}{1 + r^2 - 2r \cos(\theta - \phi)} \varphi(\phi)\, d\phi.
\end{aligned}$$

Now

$$\begin{aligned}
u = \operatorname{Re} f(z) &= \frac{1}{2\pi} \int_{\partial B_1} \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta - \phi)} \varphi(\phi)\, d\phi \overset{r \to 1}{\longrightarrow} \varphi(\theta) \quad \text{surprisingly,} \\
v = \operatorname{Im} f(z) &= \frac{1}{2\pi} \int_{\partial B_1} \frac{2r \sin(\theta - \phi)}{1 + r^2 - 2r \cos(\theta - \phi)} u(\phi)\, d\phi : \ u \to \text{its conjugate} \\
&\overset{r=1}{=} \frac{1}{2\pi} \int_{\partial B_1} \frac{\sin \phi}{1 - \cos \phi} u(\theta - \phi)\, d\phi.
\end{aligned}$$

Taking derivative

$$u_r(1,\theta) = v_\theta(1,\theta) = \frac{1}{2\pi}\int_{\partial B_1} \frac{\sin\phi}{1-\cos\phi} u_\theta(\theta-\phi)\,d\phi : u_\theta(1,\theta) \overset{\text{Hilbert transform}}{\longrightarrow} u_r(1,\theta).$$

Question: In general dimension, what is the explicit formula of "$\theta \cdot D_\theta u(1,\theta)$" $\to$ $u_r(1,\theta)$ on ∂B_1^n ?

RMK. Schwarz had a cute geometric change of variable from ϕ to ϕ^* with $e^{i\phi}$, z, and $e^{i\phi^*}$ being on the same line for each fixed $z \in B_1$, satisfying $d\phi^* = \frac{|\xi|^2-|z|^2}{|\xi-z|^2}d\phi$. Then $u(z) = \frac{1}{2\pi}\int_{\partial B_1}\varphi(\phi)\,d\phi^*$. As $z \to e^{i\phi_0}$, most of ϕ^* on ∂B_1 corresponds to angles near ϕ_0. Thus the limit is $\varphi(\phi_0)$. Going up to 3-d and above, the straight Schwarz like change of variable would have extra factor in the Jacobian. Other ways of change of variables?

Example: Half space

Kelvin way. $y = x/|x|^2 : B_{1/2}(0.5e_n) \to \{y_n > 1\}$

figure

Take $u(x) = |x|^{2-n} v\left(x/|x|^2\right)$ for x in the ball, the Dirichlet/Nuemann problem on the half space is converted to the ball

$$\begin{cases} \triangle v(y) = 0 & \text{for } y_n > 1 \\ v = \varphi(y') & \text{for } y_n = 1 \end{cases} \to \begin{cases} \triangle u(x) = 0 & \text{in } B_{1/2}(e_n/2) \\ u(x) = |x|^{2-n}\varphi\left(x/|x|^2\right) & \text{on } \partial B_{1/2}(e_n/2) \end{cases}$$

$$\begin{cases} \triangle v(y) = 0 & \text{for } y_n > 1 \\ v_{y_n} = \psi(y') & \text{for } y_n = 1 \end{cases} \to \begin{cases} \triangle u(x) = 0 & \text{in } B_{1/2}(e_n/2) \\ |x|^n\left[(2-n)u(x) - u_{\gamma(x)}\right] = \psi\left(\frac{x}{|x|^2}\right) & \text{on } \partial B_{1/2}(e_n/2) \end{cases},$$

where the Neumann condition is transformed as follows: $v(y) = |y|^{2-n} u\left(y/|y|^2\right)$

$$\begin{aligned} v_{y_n} &= (2-n)\frac{y_n}{|y|^n}u + \frac{1}{|y|^{n-2}}Du \cdot \left[\frac{(0,\cdots,0,1)}{|y|^2} + y\frac{-2y_n}{|y|^4}\right] \\ &= (2-n)\frac{y_n}{|y|^n}u + \frac{y_n}{|y|^n}Du \cdot 2\left[\left(0,\cdots,0,\frac{1}{2y_n}\right) - \frac{y}{|y|^2}\right] \\ &\overset{y_n=1}{=} |x|^n\left[(2-n)u - u_{\gamma(x)}\right]. \end{aligned}$$

A mixed boundary value problem.

Direct way.

Poisson formula for Dirichlet

$$u(x) = \frac{1}{|\partial B_1|}\int_{R^{n-1}} \frac{2x_n}{|x-y|^n}\varphi(y)\,dy$$

solves

$$\begin{cases} \triangle u = 0 & \text{for } x_n > 0 \\ u = \varphi(x') & \text{for } x_n = 0 \quad \text{in the sense } u(x', x_n) \overset{x_n \to 0^+}{\longrightarrow} \varphi(x') \end{cases},$$

say $\frac{1}{1+r^n}\varphi \in L^1\left(R^{n-1}\right)$ and $\varphi \in C^0$.

Derivation

$$G\left(x,y\right)=c_n\left[\frac{1}{\left|x-y\right|^{n-2}}-\frac{1}{\left|x-\bar{y}\right|^{n-2}}\right] \quad \text{with } \bar{y}=\left(y_1,\cdots,y_{n-1},-y_n\right) \text{ and } c_n=\frac{1}{\left(2-n\right)\left|\partial B_1\right|}$$

$$\begin{aligned}-\partial_{y_n}G &= \left(n-2\right)c_n\left[\frac{y_n-x_n}{\left|x-y\right|^n}-\frac{y_n+x_n}{\left|x-\bar{y}\right|^n}\right]\\ &\overset{y_n=0}{=} \frac{1}{\left|\partial B_1\right|}\frac{2x_n}{\left|x-y\right|^n}.\end{aligned}$$

The boundary limit justification is through the following manipulation

$$\begin{aligned}u\left(x\right) &= \frac{1}{\left|\partial B_1\right|}\int_{R^{n-1}}\frac{2x_n}{\left(x_n^2+\left|y\right|^2\right)^{n/2}}\varphi\left(x'-y\right)dy\\ &= \frac{1}{\left|\partial B_1\right|}\int_{R^{n-1}}\frac{2}{\left(1+\left|y\right|^2\right)^{n/2}}\varphi\left(x'-x_ny\right)dy. \qquad \spadesuit\end{aligned}$$

For example, $u\equiv 1$ implies $\frac{1}{\left|\partial B_1^n\right|}\int_{R^{n-1}}\frac{2}{\left(1+|y|^2\right)^{n/2}}dy=1$. (Compare $\int_{R^{n-1}}\frac{1}{\left(1+|y|^2\right)^{(n+1)/2}}dy=\frac{1}{n-1}\int_{R^{n-1}}\text{div}\,\frac{y}{\left(1+|y|^2\right)^{(n-1)/2}}dy=\frac{\left|\partial B_1^{n-1}\right|}{n-1}$.)

RMK. For L^1 boundary data φ, the limit justification is done via Lebesgue dominate convergence theorem.

RMK. From $\spadesuit$, one sees the tangential derivatives $D_{x'}u$, after a convolution–Hilbert transform, determines the normal derivative u_{x_n}.

$$D_{x'}u\left(x\right)==\int_{R^{n-1}}\frac{2}{\left(1+\left|y\right|^2\right)^{n/2}}D_{x'}\varphi\left(x'-x_ny\right)dy.$$

$$\begin{aligned}u_{x_n} &= \frac{1}{\left|\partial B_1\right|}\int_{R^{n-1}}\frac{2}{\left(1+\left|y\right|^2\right)^{n/2}}\left(-y\right)\cdot D_{x'}\varphi\left(x'-x_ny\right)dy\\ &\overset{\text{chang var } x_ny \text{ to } y}{=} \frac{1}{\left|\partial B_1\right|}\int_{R^{n-1}}\frac{2}{\left(x_n^2+\left|y\right|^2\right)^{n/2}}\left(-y\right)\cdot D_{x'}\varphi\left(x'-y\right)dy\\ &= \frac{2}{\left|\partial B_1\right|}\int_{R^{n-1}}\frac{-\left(x'-y\right)}{\left(x_n^2+\left|x'-y\right|^2\right)^{n/2}}\cdot D_{x'}\varphi\left(y\right)dy.\end{aligned}$$

Then we have reached the Poisson formula–Hilbert transform

$$u_{x_n}=\frac{2}{\left|\partial B_1\right|}\frac{-x'}{\left(\left|x'\right|^2+x_n^2\right)^{n/2}}*\nabla_{x'}u: \;\; \nabla_{x'}u\left(x',0\right)\to u_{x_n}\left(x',x_n\right).$$

So the Drichelet data $u\left(x',0\right)$ along the boundary, then $\nabla_{x'}u\left(x',0\right)$ already determines the Neumann boundary data $u_{x_n}\left(x',0\right)$; but only for THE bounded solution. e.g. $u+x_n$ is another unbounded solution.

RMK. One can describe the analytic Cauchy data for harmonic functions near a small neighborhood of the analytic boundary. One cannot describe both the initial SMOOTH position and velocity along the boundary for harmonic functions.

For example $n = 2$. Suppose $\triangle u = 0$ for $y > 0$, and $u(x, 0) = 0$, $u_y(x, 0) = \psi \in C^\infty \backslash C^\omega$, not analytic and u is C^2 up to the boundary Then one makes an odd reflection: $u(x, y) = -u(x, -y)$ for $y < 0$. The C^2 extension is a harmonic, then analytic. A contradiction.

C^2 version: assume u is C^2 up to the boundary, the C^2 extension is harmonic, then analytic. A contradiction.

C^0 version (Schwarz): assume u is continuous up to the boundary. Pick any ball centered at the boundary, let h be the harmonic function by Poisson formula taking the boundary value u.

$$\text{Maximum principle} \Rightarrow \; h(x, y) = -h(x, -y) \overset{y=0}{=} 0$$
$$\text{M.P. again} \Rightarrow u = h.$$

So u is harmonic, then analytic. A contradiction.

RMK. • $n = 2$ Cauchy integral

$$u + iv = \frac{1}{\pi i} \int_{R^1} \frac{1}{\xi - z} \varphi(\xi)\, d\xi = \frac{1}{\pi i} \int_{R^1} \frac{\xi - x + iy}{|\xi - z|^2} \varphi(\xi)\, d\xi$$

$$u(x, y) = \frac{1}{\pi} \int_{R^1} \frac{y}{(\xi - x)^2 + y^2} \varphi(\xi)\, d\xi \overset{y \to 0^+}{\to} \varphi(x)$$
$$v(x, y) = \frac{1}{\pi} \int_{R^1} \frac{x - \xi}{(x - \xi)^2 + y^2} \varphi(\xi)\, d\xi \Big|_{y=0} : u(x, 0) \to \text{conjugate } v(x, y)$$

$$H* : \; u \to \text{conjugate } v$$
$$u_x \to -u_y \text{ and } v_x \to -v_y \quad u_x - iu_y \text{ holomorphic}$$
$$u_y = -v_x \to -v_y = u_x.$$

• Examples:

$$\text{Poisson: } u + iv = \frac{x - i(y + 1)}{x^2 + (y + 1)^2} = \frac{1}{\pi} \int_{R^1} \frac{y}{(x - \xi)^2 + y^2} \frac{\xi - i}{\xi^2 + 1} d\xi$$
$$\text{Hilbert: } v = \frac{-(y + 1)}{x^2 + (y + 1)^2} = \frac{1}{\pi} \int_{R^1} \frac{x - \xi}{(x - \xi)^2 + y^2} \frac{\xi}{\xi^2 + 1} d\xi = H * u(\cdot, 0)$$
$$u_y = \frac{2(y + 1)x}{\left[x^2 + (y + 1)^2\right]^2} = \frac{1}{\pi} \int_{R^1} \frac{x - \xi}{(x - \xi)^2 + y^2} \frac{1 - \xi^2}{(\xi^2 + 1)^2} d\xi = H * u_x(\cdot, 0)$$

• (Communicated by Don Marshall) Let a and b be two points on the real line. The harmonic function, angle $\measuredangle bza$ equals π for z between a and b, and 0 outside a and b. Observe

$$\measuredangle bza = \arg \frac{b - z}{a - z} = \operatorname{Im} \ln \frac{b - z}{a - z} = \operatorname{Im} \int_a^b \frac{1}{t - z} dt = \operatorname{Im} \int_a^b \frac{t - x + iy}{|t - z|^2} dt.$$

Any harmonic function taking piecewise constant values on the real line is just the sum of the corresponding angles time constants, then divide π. The continuous version is just our above Poisson formula.

Poisson formula for Neumann

$$u(x) = \begin{cases} \frac{1}{\pi}\int_{R^1} \ln|x-y| \quad \psi(y)\,dy & n=2 \\ \frac{-2}{(n-2)|\partial B_1|}\int_{R^{n-1}} \frac{1}{|x-y|^{n-2}}\varphi(y)\,dy & n\geq 3 \end{cases}$$

solves/is a solution to

$$\begin{cases} \triangle u = 0 & \text{for } x_n > 0 \\ u_{x_n} = \psi(x') & \text{for } x_n = 0 \ \text{ in the limit sense.} \end{cases}$$

Derivation. Look for kernel $K(x,y)$ such that its boundary normal derivative is the Poisson kernel

$$\partial_{x_n} K(x,y) = \frac{1}{|\partial B_1|}\frac{2x_n}{|x-y|^n} \ \text{ for } y_n = 0.$$

Then

$$K(x,y) = \int \frac{1}{|\partial B_1|}\frac{2x_n}{|x-y|^n}dx_n = \begin{cases} \frac{1}{\pi}\ln|x-y| & n=2 \\ \frac{-2}{(n-2)|\partial B_1|}\frac{1}{|x-y|^{n-2}} & n\geq 3 \end{cases}.$$

One contrasting example: $\int xyzdz = 0.5xyz^2$ is not harmonic anymore. The limit justification is the same as in Dirichlet problem.

RMK. Bounded solution to $\begin{cases} \triangle u = f(x) & x_n > 0, \ \text{ say } \in C_0^\alpha(x_n>0) \\ u = \varphi & x_n = 0, \ \text{ say } \in L^1 \cap C^0 \end{cases}$ has the following Poisson representation:

$$\begin{aligned} u(x) &= G(x,\cdot) * f(\cdot) - G_{y_n}(x,\cdot) * \varphi(\cdot) \\ &= \frac{1}{(2-n)|\partial B_1|}\int_{R_+^n}\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-\bar{y}|^{n-2}}\right) f(y)\,dy \\ &+ \frac{1}{|\partial B_1|}\int_{R^{n-1}} \frac{2}{\left(1+|y|^2\right)^{n/2}}\varphi(x'-x_n y)\,dy. \end{aligned}$$

- Lebesgue dominate convergence theorem
- Schauder $\begin{array}{l} f \in C^a \Longrightarrow D^2u \in C^a \\ f \in C^0 \not\Rightarrow D^2 u \in C^0, \ D^2u \ \text{ could be unbounded, } \ \text{c-eg } u = xy\ln^{1/3} r \end{array}$
- Boundary Schauder $\begin{array}{l} \varphi \in C^{2,a}(\partial\Omega) \Longrightarrow D^2u \in C^a\left(\bar{\Omega}\right) \\ u_{x'} = P * \varphi_{x'}, \ \ u_{x'x'} = P * \varphi_{x'x'} \\ u_{x_n} = R * \varphi_{x'} \ \ u_{x_n x'} = R * \varphi_{x'x'} \ \ \text{and } u_{x_n x_n} = f - P * \triangle_{x'}\varphi \\ \\ \varphi \in C^1 \not\Rightarrow u_{x_n} \in C^0, \ \ u_{x_n} \ \text{ could be unbounded, c-eg } u = \operatorname{Im} z \log^{1/3} z. \end{array}$

Weak formulation for Laplace equation: $\triangle u = 0$.
Mean value formulation.
Suppose $u \in L^1$ satisfy $u(x) = ⨍_{B_r(x)} u(y)\,dy$ for all x and r.

Exercise. Then u is continuous, since

$$u(x) - u(x_0) = \fint_{B_1(x)} u(y)\,dy - \fint_{B_1(x_0)} u(y)\,dy \overset{x\to x_0}{\to} 0.$$

2 minor overlap circle$figure$

In turn, we have $u(x) = \fint_{\partial B_r(x)} u(y)\,dy$. In fact

$$\begin{aligned}
\frac{d}{dr}:\ r^n |B_1| u(x_0) &= \int_{B_r(x_0)} u(y)\,dy \\
nr^{n-1} |B_1| u(x_0) &= \int_{\partial B_r(x_0)} u(y)\,dy \\
|\partial B_r| u(x_0) &= \int_{\partial B_r(x_0)} u(y)\,dy.
\end{aligned}$$

Then

$$u(x) = \int_{R^n} \varphi(x-y) u(y)\,dy \in C^\infty$$

for $\varphi(x) = \varphi(|x|)$ with $\int_{R^n} \varphi(|x|)\,dx = 1$. Let us check $\triangle u = 0$.

$$\begin{aligned}
\int_{\partial B_\varepsilon(0)} u dA &= \int_{\partial B_\varepsilon(0)} u(0) + Du(0)\cdot x + \frac{1}{2}\underbrace{D_{ij}u(0)\,x_i x_j}_{\lambda_1 x_1^2+\cdots+\lambda_n x_n^2} + \varepsilon^3 dA \\
|\partial B_\varepsilon| u(0) &= |\partial B_\varepsilon| u(0) + 0 + \frac{1}{2}\left(\lambda_1\frac{\varepsilon^2}{n} + \cdots + \lambda_n \frac{\varepsilon^2}{n}\right)|\partial B_\varepsilon| + O\left(\varepsilon^3\right)|\partial B_\varepsilon| \\
&\Rightarrow \frac{1}{2n} \triangle u(0) = 0.
\end{aligned}$$

Integration by parts formulation.

For $u \in C^0/L^1/$distribution $\int u \triangle \varphi = 0$ for any $\varphi \in C_0^\infty$. How to move to mean value formulation?

Q. How to find $\varphi \in C_0^\infty$ such that

$$\triangle\varphi = \frac{1}{|B_2|}\chi_{B_2} - \frac{1}{|B_1|}\chi_{B_1}?$$

$C^{1,1}$ approach. $\varphi \sim$ “$\frac{|x|^2}{2n|B_2|}\chi_{B_2}$” $-$ “$\frac{|x|^2}{2n|B_1|}\chi_{B_1}$”.

Analytic way. We just look for those radial ones by solving

$$\varphi_{rr} + \frac{n-1}{r}\varphi_r = \frac{1}{|B_1|}\chi_{B_1} \ \text{ or } \ \frac{1}{|B_2|}\chi_{B_2}.$$

For $r \leq 1$

$$\varphi = \frac{1}{|B_1|}\frac{r^2}{2n}\chi_{[0,1]} + c_1.$$

For $r > 1$

$$\varphi = c_2 r^{2-n} + c_3.$$

After $\mathrm{C}^{1,1}$ matching at $r = 1$, we have

$$\varphi_1 = \begin{cases} \frac{1}{|B_1|}\frac{r^2}{2n}\chi_{B_1} - \frac{1}{|B_1|2n} - \frac{1}{|B_1|(n-2)n} & \text{for } |x| \leq 1 \\ \frac{-1}{|B_1|(n-2)n}\frac{1}{r^{n-2}} & \text{for } |x| > 1 \end{cases}.$$

Similarly

$$\varphi_2 = \begin{cases} \frac{1}{|B_2|}\frac{r^2}{2n}\chi_{B_2} - \frac{2^2}{|B_2|2n} - \frac{1}{|B_1|(n-2)n} & \text{for } |x| \leq 1 \\ \frac{-1}{|B_1|(n-2)n}\frac{1}{r^{n-2}} & \text{for } |x| > 1 \end{cases}.$$

"Incidentally" the gradient matching coefficient c_2 leads exactly the coefficient for the fundamental solution $\Gamma = \frac{-1}{|B_1|(n-2)n}\frac{1}{|x|^{n-2}}$.

Geometric way (Caffarelli).

quadratics drop down to fundamental$figure$

This requires $\varphi_2 = \frac{|x|^2}{2n|B_2|} - A$ to touch r^{2-n}, in fact $\frac{-1}{?r^{n-2}}$ at $|x| = 2$. We have a system $\frac{2^2}{2n|B_2|} - A = \frac{-1}{?2^{n-2}}$ and $\frac{2 \cdot 2}{2n|B_2|} = \frac{(n-2)}{?2^{n-1}}$ which implies $? = n(n-2)|B_1|$ and $A = \frac{2(n-1)}{n(n-2)|B_2|}$. Similarly we get $\varphi_1 = \frac{|x|^2}{2n|B_1|} - A'$ touching $\frac{-1}{?r^{n-2}} = \frac{-1}{n(n-2)|B_1|r^{n-2}}$ at $|x| = 1$. Thus $\varphi = \varphi_2 - \varphi_1 \in C_0^{1,1}$ answers the above question.

$$\text{For } u \in L^1, \ \int u \triangle \varphi = 0 \Rightarrow \fint_{B_2} u = \fint_{B_1} u.$$

Therefore (exercise)

$$u(x) = \lim_{r \to 0} \fint_{B_r(x)} u \ \text{ a.e. at Lebesgue point of } L^1 \ u.$$

Cor. (Weyl) $u \in L^1/C^0$ satisfying $\int u \triangle \varphi = 0$ for any $\varphi \in C_0^\infty$. Then $u \in C^\infty$ and $\triangle u = 0$.

Warning:

$$\int \frac{1}{|x|^{n-2}} \triangle \varphi = c_n \varphi(0) \neq 0 \ !$$

C^∞ approach (Weyl)
Work for $u \in$distribution
$\psi(x) = \psi(|x|) \in C_0^\infty$ with $\int \psi = 1$
$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n} \psi\left(\frac{x}{\varepsilon}\right)$

$\Gamma * \psi_\varepsilon$ graph figure

Step 1. $\varphi_\varepsilon = \Gamma * \psi_\varepsilon = \begin{cases} \Gamma \quad \text{for} \quad |x| \geq \varepsilon \\ \text{smooth for } |x| \leq \varepsilon \end{cases}$. Recall $\Gamma = \frac{-1}{(n-2)|\partial B_1|} \frac{1}{|x|^{n-2}}$.
Step 2. $\triangle \Gamma * \psi = \psi$.
Step 3. $\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1} \in C_0^\infty$

$$\int_{\mathbb{R}^n} u \triangle (\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1}) = 0 \Rightarrow \int_{\mathbb{R}^n} u \psi_{\varepsilon_2} = \int_{\mathbb{R}^n} u \psi_{\varepsilon_1}$$

- $u * \psi_\varepsilon$ is independent of ε
- $u * \psi_\varepsilon \in C^\infty$ (Review distribution theory, try it.)
- $u * \psi_\varepsilon = u$ as a distribution (Exercise).

Pointwise (viscosity) formulation.

Definition: $u \in C^0$ is a viscosity solution to $\triangle u = 0$, if for any quadratic $P \underset{(\leq)}{\geq} u$ near an interior point x_0 and "$=$" at x_0, then $\triangle P \underset{(\leq)}{\geq} 0$.

RMK. If there is no quadratic touching u from above or blow at x_0, then one checks nothing. No touching, no checking!

RMK. We can replace those quadratics by equivalent C^2/C^∞ testing functions. Certainly C^2 harmonic functions satisfy this definition. We do have C^0 but non C^2 solutions to (fully nonlinear) elliptic equations such as Monge-Ampere/Special Lagrangian equations.

RMK. Motivation and origin of viscosity solution. As we see the Dirichlet energy of harmonic function with some highly oscillating continuous boundary data could be infinite, and many other nonlinear elliptic equations such as Pucci and Isaacs equations from stochastic optimization do not have divergence structure, the variational approach is not adequate. There is a need of pointwise approach. The classic one is the Perron method for Laplace equation. The modern (since early 1980s) "twist" is viscosity solution. The origin of viscosity solution (Crandall-Lions, Evans) started from solving Hamilton-Jacobi equation $H(Du) = 0$, say in B_1, with nice C^2 boundary data on ∂B_1, no easy job. One regularizes the first order equation by adding higher order derivatives, say $\varepsilon \triangle u + H(Du) = 0$. Recall $\triangle u$ represents viscosity in fluid mechanics. One can now solve this elliptic equation for solution u_ε with boundary value φ. By the comparison principle for elliptic equations, one can show that the Lipschitz norm of the approximate solutions u_ε is uniformly bounded in terms of the C^2 norm of φ independent of ε. By Ascoli-Arzela, there is a uniform convergent limit v in, say C^0 norm. The viscosity solution is closed under C^0 uniform limit. Thus v is solution to the vanished viscosity equation—Hamilton-Jacobi equation.

We verify C^0 harmonic functions in the viscosity sense are in fact smooth and satisfy the "harmonic" equation by Poisson representation formula. Note explicitly representation for solutions to nonlinear equations are NOT available in general.

Let

$$h = \int_{\partial B_1} P(x,y)\, u(y)|_{\partial B_1}\, dA_y$$

$\cdot\ h = u$ on ∂B_1.
$\cdot\ \triangle h = 0$ in B_1.

Now if $u > h$ somewhere at $x_0 \in \overset{0}{B}_1$, say $(u-h)(x_0) = \max_{B_1}(u-h) > 0$

u,h graph $figure$

That is, $h + \max \geq u$ in B_1, " $=$" at x_0. Next for a small ($<\max$) but positive ε, $h + \max - \varepsilon|x|^2$ is still above u on the boundary ∂B_1, may not be so inside (for example at $x_0 \neq 0$). Any way, if necessary, we move up $h + \max - \varepsilon|x|^2$ until it leaves graph u at interior x_0'. Analytically, there is δ such that

$$h + \max - \varepsilon|x|^2 + \delta \underset{= \text{ at } x_0'}{\geq} u \quad \text{in } B_1.$$

By the viscosity subharmonicity of u, $0 \leq \triangle\left(h + \max - \varepsilon|x|^2 + \delta\right) = -2n\varepsilon$. This contradiction shows $u \leq h$ in B_1.

Similarly, if $u < h$ somewhere at $x_0 \in \overset{0}{B}_1$, say $(u-h)(x_0) = \min_{B_1}(u-h) < 0$

u,h graph $figure$

That is, $h + \min \leq u$ in B_1, " $=$" at x_0. Next for a small ($<|\min|$) but positive ε, $h + \min + \varepsilon|x|^2$ is still below u on the boundary ∂B_1, may not be so inside (for example at $x_0 \neq 0$). Any way, if necessary, we move down $h + \min + \varepsilon |x|^2$ until it leaves graph u at interior x_0'. Analytically, there is δ such that

$$h + \min + |x|^2 + \delta \underset{= \text{ at } x_0'}{\leq} u \quad \text{in } B_1.$$

By the viscosity superharmonicity of u, $0 \geq \triangle \left(h + \min + \varepsilon |x|^2 + \delta \right) = 2n\varepsilon$. This contradiction shows $u \geq h$ in B_1.

Thus $u \equiv h$.

Energy method

Uniqueness. $u \in C^2 \left(\bar{\Omega} \right)$ solution to $\begin{cases} \triangle u = f & \text{in } \Omega \\ u = \varphi \text{ or } u_\gamma = \psi & \text{on } \partial\Omega \quad \text{say } C^1 \text{ boundary} \end{cases}$ is unique in Dirichlet problem and unique up to a constant in Nuemann problem.

Proof. Let $w = u - v$, the difference fo two solutions. Then

$$\begin{cases} \triangle w = 0 & \text{in } \Omega \\ w = 0 \text{ or } w_\gamma = 0 & \text{on } \partial\Omega \end{cases}.$$

And

$$0 = \int_{\partial\Omega} w w_\gamma = \int_\Omega \operatorname{div} (wDw) = \int_\Omega Dw \cdot Dw + w \overbrace{\triangle w}^{0}.$$

It follows $|Dw| \equiv 0$.

Dirichlet principle. $u \in C^2 \left(\bar{\Omega} \right)$ solution to $\begin{cases} \triangle u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$ is equivalent to

$$E[u] = \min_{\substack{v \in C^2 \left(\bar{\Omega} \right) \\ v = \varphi \text{ on } \partial\Omega}} \int \frac{1}{2} |Dv|^2 + fv.$$

Proof. "$\Longleftarrow$" Let u be the minimizer ($E[v]$ is a convex functional). We have for all $\eta \in C_0^\infty (\Omega)$

$$0 = \frac{d}{d\varepsilon} E[u + \varepsilon\eta] \Big|_{\varepsilon = 0} = \int Du \cdot D\eta + f\eta = \int (- \triangle u + f) \ \eta.$$

Consequently $\triangle u = f$.

"$\Rightarrow$" Verify $\int_\Omega \frac{1}{2} |Du|^2 + fu \leq \int_\Omega \frac{1}{2} |Dv|^2 + fv$

$$\begin{aligned} RHS - LHS &= \int_\Omega \frac{1}{2} |D(v-u)|^2 + Dv \cdot Du - |Du|^2 + \underset{\triangle u}{f} \ (v - u) \\ &\geq \int_\Omega \underline{Dv \cdot Du - |Du|^2} + \int_{\partial\Omega \ 0 \text{ Neumann}} u_\gamma \ \overrightarrow{(v-u)}^0 - \int_\Omega \underline{Du \cdot D(v-u)} \\ &= 0. \end{aligned}$$

Neumann version. $u \in C^2\left(\bar{\Omega}\right)$ solution to $\begin{cases} \triangle u = f & \text{in } \Omega \\ u_\gamma = 0 & \text{on } \partial\Omega \end{cases}$ "natural" free boundary condition $\Rightarrow \int_\Omega f = 0$ is equivalent to

$$E\left[u\right] = \min_{v \in C^2\left(\bar{\Omega}\right)} \int \frac{1}{2}\left|Dv\right|^2 + fv.$$

Proof. "$\Rightarrow$" same as above.

"$\Longleftarrow$" same for equation; but one more variation for boundary condition. Let u a minimizer

$$0 = \frac{d}{d\varepsilon} E\left[u + \varepsilon\eta\right]\Big|_{\varepsilon=0} = \int_\Omega Du \cdot D\eta + f\eta = \int_{\partial\Omega} u_\gamma \eta + \int_\Omega \left(- \triangle u + f\right) \eta.$$

First $\eta \in C_0^\infty\left(\Omega\right) \Rightarrow - \triangle u + f = 0$

Second $\eta \in C^\infty\left(\Omega\right) \Rightarrow \int_{\partial\Omega} u_\gamma \eta = 0 \Longrightarrow u_\gamma = 0!$

RMK. We emphasize the necessity of the enough smoothness of the boundary data. For example, for mere continuous boundary data, the Dirichlet energy of the harmonic solution may be infinity, then no minimizer exists. On the unit circle, take rapidly oscillating continuous boundary function

$$\varphi\left(\theta\right) = \sum \frac{1}{m^2} \sin 2^m\theta.$$

The harmonic function in B_1 taking φ as boundary value is

$$u\left(r,\theta\right) = \sum \frac{1}{m^2} r^{2^m} \sin 2^m\theta.$$

However (Exercise)

$$\begin{aligned} \int_{B_1} \left|Du\right|^2 dx &= \int_0^1 \int_0^{2\pi} \left(u_r^2 + \frac{1}{r^2} u_\theta^2\right) r d\theta dr \\ &= \sum \frac{2^m}{m^4} = \infty. \end{aligned}$$