Lecture 3 Heat equations

- ◦ invariance, explicit solutions
- ◦ self-similar way
- ◦ "Green's" function for the "whole" space time and "half" space time

$$\circ \text{ mean value formulas} \begin{cases} \text{maximum principle} \\ \text{higher order derivative estimates} \\ \text{regularity-analytic in } x, \text{ smooth in t} \\ \text{Harnack} \\ \text{Liouville} \end{cases}$$

$$\circ \text{ weak formulation} \begin{cases} \text{mean value} \\ \text{weak formulation} \\ \text{distribution} \end{cases}$$

◦ uniqueness with exponential growth, Tikhonov non-uniqueness, nonexistence with super quadratic exponential growth, nonanalytic in t, backward uniqueness

<u>Invariance</u> of $u_t - \triangle u = 0$

$\cdot\ u(x + x_0, t + t_0)$
$\cdot\ u\left(e^A x, t\right)$
$\cdot\ u(\mu x, \mu^2 t)$
$\cdot\ u + v, au$

$\cdot\ D_x^\alpha D_t^l u$ still temperature/caloric
$\cdot\ Du \cdot Ax = \frac{d}{d\varepsilon} u\left(e^{\varepsilon A} x, t\right)\Big|_{\varepsilon=0}$, $A = -A^T$
$\cdot\ D_x u(x,t) \cdot x + 2D_t u(x,t) = \frac{d}{d\mu} u(\mu x, \mu^2 t)\Big|_{\mu=1}$
$\cdot\ \int u(x - y, t - s)\, \varphi(y, s)\, dyds$
$\cdot$ Kelvin $t^{-n/2} e^{-|x|^2/4t} u(x/t, -1/t)$

Examples.
$\cdot$ All harmonic functions are caloric
$x_1 x_2 x_3$, $|x|^{2-n}$, $\mathrm{Re}\,/\,\mathrm{Im}\ e^{3x_1+4x_2+i5x_3}$
$\cdot\ t + \frac{1}{2n}|x|^2$, $t^2 + t|x|^2/n + (x_1^4 + \cdots + x_n^4)/(12n)$, more later
$\cdot\ \mathrm{Re}\,/\,\mathrm{Im}\ e^{i\xi\cdot x - |\xi|^2 t}$, $e^{\xi\cdot x + |\xi|^2 t}$
RMK. $e^{\xi\cdot x+|\xi|t}$, $\mathrm{Re}\,/\,\mathrm{Im}\, e^{i(\xi\cdot x + |\xi|t)}$, $u(\xi\cdot x + |\xi|\, t)$ are wave functions; e^{x_1+t} are both caloric and wave functions.

More examples.
$\cdot$ Caloric polynomials

$$\Phi(x,t) = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$$

now

$$\Phi(x,t)\, D_x^\alpha D_t^l \Phi\left(\frac{x}{t}, \frac{-1}{t}\right)$$

[0]December 1, 2016

is a caloric polynomial of degree $|\alpha| + 2l$.

eg.

$$D_{123}\Phi = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \frac{x_1x_2x_3}{(-2t)^3}\Bigg|_{\left(\frac{x}{t},\frac{-1}{t}\right)} \to (-t)^{n/2} e^{\frac{|x|^2}{4t}} \frac{x_1x_2x_3}{8}$$
$$\overset{\Phi(x,t)\cdot}{\to} (-1)^{n/2} \frac{x_1x_2x_3}{8}.$$

$$D_{111}\Phi = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \left\{ \left(\frac{-x_1}{2t}\right)_{x_1x_1} + \left(\frac{-x_1}{2t}\right)^2_{x_1} + \left(\frac{-x_1}{2t}\right)\left[\left(\frac{-x_1}{2t}\right)_{x_1} + \left(\frac{-x_1}{2t}\right)^2\right]\right\}$$
$$= \Phi \left\{ \frac{x_1}{2t^2} + \frac{-x_1}{2t}\left[\frac{-1}{2t} + \frac{x_1^2}{4t^2}\right]\right\}$$
$$= \Phi \frac{6tx_1 - x_1^3}{8t^3} \overset{\left(\frac{x}{t},\frac{-1}{t}\right)}{\to} (-t)^{n/2} e^{\frac{|x|^2}{4t}} \frac{6tx_1 + x_1^3}{8} \overset{\Phi(x,t)\cdot}{\to} (-1)^{n/2} \frac{6tx_1 + x_1^3}{8}.$$

· Radial ones

$$u_t - \triangle u = \delta_{(0,0)}(x,t) = \delta_0(x)\,\delta_0(t)$$

Fourier transform way in space and time

$$\hat{u}(\xi,s) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{R^{n+1}} u(x,t)\, e^{-i(\xi\cdot x + st)} dxdt$$

satisfies

$$is\hat{u} + |\xi|^2 \hat{u} = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \quad \text{and } \hat{u} = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \frac{1}{is + |\xi|^2}.$$

Then

$$u(x,t) = \frac{1}{(2\pi)^{n+1}} \int_{R^{n+1}} \frac{1}{is + |\xi|^2} e^{i(x\cdot\xi + ts)} d\xi ds.$$

We see the radiality $u(x,t) = u(|x|, t)$. Question, how to invert?

Fourier transform way in space only

$$\hat{u}(\xi,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} u(x,t)\, e^{-i\xi\cdot x} dx$$

satisfies

$$\hat{u}_t + |\xi|^2 \hat{u} = \frac{1}{(2\pi)^{n/2}} \delta_0(t).$$

Then

$$\left(e^{|\xi|^2 t}\, \hat{u}\right)_t = \frac{e^{|\xi|^2 t}}{(2\pi)^{n/2}} \delta_0(t) = \begin{cases} 0 & \text{for } t > 0 \\ \frac{1}{(2\pi)^{n/2}} \delta_0(0) & \text{for } t = 0 \\ 0 & \text{for } t < 0 \end{cases}$$

and

$$e^{|\xi|^2 t}\ \hat{u} = \begin{cases} \frac{1}{(2\pi)^{n/2}} & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases} .$$

RMK. If the jump function at $t=0$ is chosen as $e^{|\xi|^2 t}\ \hat{u} = \begin{cases} 0.1\frac{1}{(2\pi)^{n/2}} & \text{for } t>0 \\ -0.9\ \frac{1}{(2\pi)^{n/2}} & \text{for } t<0 \end{cases}$, then $\hat{u} = -0.9e^{-|\xi|^2 t}$ is not invertible for $t<0$.

We proceed with the conversion.

For $t<0$, $\quad u=0$.

For $t>0$

$$\begin{aligned} u &= \frac{1}{(2\pi)^{n/2}} \int_{R^n} \frac{1}{(2\pi)^{n/2}} e^{-|\xi|^2 t + ix\cdot\xi} d\xi = \frac{1}{(2\pi)^n} \frac{1}{t^{n/2}} \int_{R^n} e^{-|\xi|^2 + i\frac{x}{\sqrt{t}}\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^n} \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \int_{R^n} e^{-\left|\xi - \frac{ix}{2\sqrt{t}}\right|^2} d\xi = \frac{1}{(2\pi)^n} \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \underbrace{\prod_k \int_{R^1} e^{-\left|\xi_k - \frac{ix_k}{2\sqrt{t}}\right|^2} d\xi}_{\pi^{n/2}} \\ &= \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} . \end{aligned}$$

RMK. In switching each of the line integrals to the real axis, we take holomorphic function $h(z) = e^{-z^2}$ and domain Ω as an "infinite" rectangle with bottom $y=c$ and top $y=0$ in the following divergence formula (FTC):
Complex version

$$\int_{\partial\Omega} h(z)\, dz = \int_{\Omega} h_{\bar{z}}(z)\, d\bar{z} \wedge dz = 0.$$

Real version

$$\begin{aligned} &\int_{\partial\Omega} (udx - vdy) + i\,(udy + vdx) \overset{(dy,-dx)=\gamma dA}{\underset{(dx,dy)=T dA}{=}} \int_{\partial\Omega} -(v,u)\cdot\gamma + i\,(u,-v)\cdot\gamma\ dA \\ &= \int_{\Omega} -(v_x + u_y) + i\,(u_x - v_y)\, dxdy = 0. \end{aligned}$$

RMK. Splitting way for Gaussian integral

$$\begin{aligned} I &= \int_{R^n} e^{-|x|^2} dx = \left(\int_{R^2} e^{-x_1^2 - x_2^2} dx_1 dx_2 \right)^{n/2} \\ &= \left(\int_0^\infty e^{-r^2} 2\pi r dr \right)^{n/2} = \left(\pi e^{-r^2} \Big|_0^\infty \right)^{n/2} = \pi^{n/2}. \end{aligned}$$

Radial way

$$I = \int_0^\infty e^{-r^2} |\partial B_1|\, r^{n-1} dr \overset{\rho=r^2}{=} \frac{|\partial B_1|}{2} \int_0^\infty e^{-\rho}\ \rho^{\frac{n}{2}-1} d\rho = \frac{|\partial B_1^n|}{2} \Gamma(n/2).$$

Consequently

$$|\partial B_1^n| = \frac{2\pi^{n/2}}{\Gamma(n/2)} = \begin{cases} 2 & n=1 \\ \frac{2\pi^{n/2}}{\left(\frac{n}{2}-1\right)!} & n \text{ even} \\ \frac{2\pi^{(n-1)/2}}{\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots\left(\frac{3}{2}\right)} & n \text{ odd} \end{cases}.$$

In chart

n	1	2	3	4	5	6	7	8		9
$\lvert\partial B_1^n\rvert$	2	2π	4π	$2\pi^2$	$\frac{4}{3}\pi^2$	π^3	$\frac{8}{15}\pi^3$	$\frac{1}{2}\pi^4$	$\searrow$	$\frac{16}{105}\pi^4$.

Self-similar way

* self similar solution $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{r}{\sqrt{t}}\right)$ to $u_t - \triangle u = \text{“}\delta(x,t)\text{''}$ or 0

$u_t = -\alpha \frac{1}{t^{\alpha+1}} v + \frac{1}{t^\alpha} \frac{-r}{2t^{3/2}} v' = \frac{1}{t^{\alpha+1}} \left(-\alpha v - \frac{r}{2\sqrt{t}} v'\right)$

$\triangle u = \frac{1}{t^\alpha} \left(\partial_{rr} + \frac{n-1}{r}\partial_r\right) v\left(r/\sqrt{t}\right)$

$\quad = \frac{1}{t^\alpha}\left(\frac{1}{t} v'' + \frac{n-1}{r}\frac{1}{\sqrt{t}} v'\right) = \frac{1}{t^{\alpha+1}}\left(v'' + \frac{n-1}{r/\sqrt{t}} v'\right)$

$v''\left(\frac{r}{\sqrt{t}}\right) + \frac{n-1}{\frac{r}{\sqrt{t}}} v'\left(\frac{r}{\sqrt{t}}\right) + \frac{1}{2}\frac{r}{\sqrt{t}} v'\left(\frac{r}{\sqrt{t}}\right) + \alpha v\left(\frac{r}{\sqrt{t}}\right) = 0$

Set $\rho = \frac{r}{\sqrt{t}}$, then

$v''(\rho) + \frac{n-1}{\rho} v'(\rho) + \frac{1}{2}\rho v'(\rho) + \alpha v(\rho) = 0$

$\left(\rho^{n-1} v'\right)' + \underbrace{\frac{1}{2}\rho^n v' + \alpha \rho^{n-1} v}_{=\left(\frac{1}{2}\rho^n v\right)' \text{ when } \alpha = n/2} = 0$

* Let $\alpha = n/2$

$\rho^{n-1} v' + \frac{1}{2}\rho^n v = c$ or $v' + \frac{\rho}{2} v = c/\rho^{n-1}$ or $\left(v e^{\frac{\rho^2}{4}}\right)' = c e^{\frac{\rho^2}{4}}/\rho^{n-1}$

Then $v(\rho) = c' e^{-\frac{\rho^2}{4}} + c\, e^{-\frac{\rho^2}{4}} \int e^{\frac{\rho^2}{4}}/\rho^{n-1} d\rho$ and

$u(x,t) = \frac{1}{t^{n/2}}\left[c' e^{-\frac{|x|^2}{4t}} + \underbrace{c e^{-\frac{|x|^2}{4t}} \left(\int e^{\frac{\rho^2}{4}}/\rho^{n-1} d\rho\right)\Big|_{\rho = r/\sqrt{t}}}\right]$

After some testing for the fundamental solution, we find $c = 0$.

Green's function for the whole space-time and "half" space-time

Whole space-time

$$\Phi(x,t,y,s) = \begin{cases} \frac{1}{(4\pi)^{n/2}} \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} & -\infty < s < t \\ 0 & s > t \end{cases}.$$

$$\begin{aligned} u(x,t) = \Phi * f &= \frac{1}{(4\pi)^{n/2}} \int_{-\infty}^t \int_{R^n} \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy ds \\ &= \frac{1}{(4\pi)^{n/2}} \int_0^\infty \int_{R^n} \frac{1}{\bar{s}^{n/2}} e^{-\frac{|\bar{y}|^2}{4\bar{s}}} f(x-\bar{y}, t-\bar{s})\, d\bar{y} d\bar{s} \end{aligned}$$

$$\text{for} \begin{cases} f \in C^0 & & \text{not enough} \\ f \in C^{\alpha}_{\alpha/2} & \text{with compact support} & \text{just fine, little involved calculus} \\ f \in C^2_1 & & \text{more than enough} \end{cases} \quad \text{to get } u \in C^2_1 \,,$$

solving

$$u_t - \triangle u = f(x,t).$$

When C^2_1 function f having compact support in R^{n+1}, we see the convolution with the heat kernel $(4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$ leads

$$\Phi * D_t f / D_x f / D_{xx} f : C^0 \to C^0.$$

Exercise $(f(y,s) = \delta_0(y))$

$$\begin{aligned} \int_0^\infty \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} dt &\overset{s=\frac{|x|^2}{4t}}{=} \frac{1}{\pi^{n/2}} \frac{1}{|x|^n} \int_0^\infty s^{n/2} e^{-s} \frac{|x|^2}{4s^2} ds \\ &= \frac{1}{4\ \pi^{n/2}} \frac{1}{|x|^{n-2}} \int_0^\infty s^{\frac{n}{2}-2} e^{-s} ds = \frac{1}{4\ \pi^{n/2}} \frac{1}{|x|^{n-2}} \frac{1}{\frac{n}{2}-1} \int_0^\infty s^{\frac{n}{2}-1} e^{-s} ds \\ &= \frac{\Gamma(n/2)}{2\ \pi^{n/2}\ (n-2)} \frac{1}{|x|^{n-2}} = \frac{1}{(n-2)\ |\partial B_1|} \frac{1}{|x|^{n-2}}! \ \text{ for } n \geq 3. \end{aligned}$$

Question. $n = 1, 2$?

As for the equation $u_t - \triangle u = f(x,t)$, when D_t hits the upper limit in $\int_{-\infty}^t$, one just substitute s with t in the integrand, we reach $f(x,t)$ or

$$\lim_{s \to t^-} \int_{R^n} \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dy = f(x,t);$$

when $D_t - \triangle$ hits the integrand, as the kernel is caloric, one gets 0,

$$(D_t - \triangle_x)\, \Phi(x,t,y,s) = 0.$$

RMK. We indicate the Schauder estimates $C^{2,\alpha}_{1,\alpha/1}$ for mere $C^{\alpha}_{\alpha/2}$ function f with compact support in the heat potential. The idea is to split off integrable factors in space or time from the bounded function $t^{-\beta} e^{-1/t}$. The order is D_{12}, D_{11}, and then by equation $D_t u = \triangle u$.

$$D_{12} u(x,t) = c_n \int_{-\infty}^t \int_{R^n} \frac{(x_1-y_1)(x_2-y_2)}{4(t-s)^2} \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dyds$$

$$= c_n \int_{-\infty}^t \int_{R^n} \frac{(x_1-y_1)(x_2-y_2)}{4(t-s)^2} \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \Big[f(y,s) - \underbrace{f(x,t)}_{\text{by cancellation}} \Big] dyds$$

$$\approx \int_{-1}^t \int_{B_1} \frac{|x-y|^2}{4(t-s)^2} \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[|x-y|^{\alpha} + (t-s)^{\alpha/2} \right] dyds$$

$$= \int_0^1 \int_{B_1} \frac{|y|^2}{4s^2} \frac{1}{s^{n/2}} e^{-\frac{|y|^2}{4s}} \left[|y|^{\alpha} + s^{\alpha/2} \right] dyds \ \text{ change variables to move singularity to } (0,0)$$

$$\approx \int_0^1 \int_{B_1} \underbrace{\frac{1}{|y|^{n-\alpha/2}} \frac{1}{s^{1-\alpha/4}}}_{\text{integrable}} \underbrace{\left(\frac{|y|^2}{s} \right)^{\frac{n+1}{2}+\frac{\alpha}{4}} e^{-\frac{|y|^2}{4s}}}_{\text{bounded}} + \underbrace{\frac{1}{|y|^{n-\alpha/2}} \frac{1}{s^{1-\alpha/4}}}_{\text{integrable}} \underbrace{\left(\frac{|y|^2}{s} \right)^{\frac{n}{2}+1-\frac{\alpha}{4}} e^{-\frac{|y|^2}{4s}}}_{\text{bounded}} dyds$$

$$D_{11}u = c_n \int_{-\infty}^{t} \int_{R^n} \left[\frac{(x_1 - y_1)^2}{4(t-s)^2} - \frac{1}{2(t-s)} \right] \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)\, dyds$$
$$= c_n \int_{-\infty}^{t} \int_{R^n} \left[\frac{(x_1 - y_1)^2}{4(t-s)^2} - \frac{1}{2(t-s)} \right] \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[f(y,s) - \underbrace{f(x,t)}_{\text{by cancellation}} \right] dyds$$
$$\approx \int_{-1}^{t} \int_{B_1} \left[\frac{|x-y|^2}{4(t-s)^2} + \frac{1}{2(t-s)} \right] \frac{1}{(t-s)^{n/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[|x-y|^\alpha + (t-s)^{\alpha/2} \right] dyds$$
$$= \int_0^1 \int_{B_1} \left(\underbrace{\frac{|y|^2}{4s^2}}_{\text{similar to } D_{12}} + \frac{1}{2s} \right) \frac{1}{s^{n/2}} e^{-\frac{|y|^2}{4s}} \left[|y|^\alpha + s^{\alpha/2} \right] dyds.$$

We finish the estimate involving $\frac{1}{s}$ term, whose singularity is still of order $-n-2+\alpha$ in space-time, then integrable.

$$\int_0^1 \int_{B_1} \frac{1}{s} \frac{1}{s^{n/2}} e^{-\frac{|y|^2}{4s}} \left[|y|^\alpha + s^{\alpha/2} \right] dyds$$
$$= \int_0^1 \int_{B_1} \underbrace{\frac{1}{s^{1-\alpha/4}} \frac{1}{|y|^{n-\alpha/2}}}_{\text{integrable}} \underbrace{\left(\frac{|y|^2}{s} \right)^{\frac{n}{2}+\frac{\alpha}{4}} e^{-\frac{|y|^2}{4s}}}_{\text{bounded}} + \underbrace{\frac{1}{s^{1-\alpha/4}} \frac{1}{|y|^{n-\alpha/2}}}_{\text{integrable}} \underbrace{\left(\frac{|y|^2}{s} \right)^{\frac{n}{2}-\frac{\alpha}{4}} e^{-\frac{|y|^2}{4s}}}_{\text{bounded}} dyds$$

Space–half time

Physical observation: temperature/density evolves forward as time goes forward.

Mathematical observation: initial "velocity" u_t already determined by initial "position" $\triangle u$. So only initial temperature can be prescribed:

$$\text{(IVP)} \quad \begin{cases} u_t - \triangle u = 0 & \text{for } t > 0 \\ u(x,0) = \varphi(x) \end{cases}.$$

It is similar to, but much better (behaved) than harmonic functions over half space

$$\begin{cases} \triangle u = 0 & \text{for } x_n > 0 \\ u(x',0) = \varphi(x') \in L^1(R^n) & \text{say } C^k(R^n) \cap L^1(R^n) \end{cases}.$$

Bonus

$$u(x,t) = \Phi(\cdot,t) * \varphi = (4\pi t)^{-n/2} \int_{R^n} \exp\left(-|x-y|^2/4t\right) \varphi(y)\, dy$$
$$= (\pi)^{-n/2} \int_{R^n} \exp\left(-|y|^2\right) \varphi\left(x - 2\sqrt{t}y\right) dy$$

solves the (IVP). Namely

i) $u(x,t) \overset{t \to 0^+}{\to} \varphi(x)$;

ii) $u(x,t)$ analytic in x&t for $t > 0$, as long as $\varphi(x) \in L^1(R^n)$;

iii) $\begin{cases} \varphi \in C^0 \Rightarrow u \in C_0^0\left(R^n \times [0,\infty)\right) \\ \varphi \in C^1 \Rightarrow u \in C_{1/2}^1\left(R^n \times [0,\infty)\right) \\ \varphi \in C^2 \Rightarrow u \in C_1^2\left(R^n \times [0,\infty)\right) \end{cases}.$

Proof.

i) $\varepsilon - \delta$ way, Lebesgue dominate convergence way.

ii) $\int_{R^n} (4\pi t)^{-n/2} \exp\left(-|x-y|^2/4t\right)\varphi(y)\,dy$ is a sum of x-t analytic function for $t>0$.

iii) We only need to show the continuity near $t=0$. We skip the C^0 & C^1 case. Now $\varphi \in C^2$ and $D_x^2 u$ takes the form

$$D_x^2 u(x,t) = (\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right) D_x^2\varphi\left(x-2\sqrt{t}y\right)dy \overset{t\to 0^+}{\to} D_x^2\varphi(x).$$

The limit can be done via ε-δ way assuming $\|D^2\varphi\|_{L^\infty(R^n)} < \infty$, or Lebesgue dominated convergence way assuming $\|D^2\varphi\|_{L^1(R^n)} < \infty$.

As for the continuity of u_t, one way is to use the continuity of $D_x^2 u$ coupled with the equation: $u_t(x,t) = \triangle u(x,t) \overset{t\to 0^+}{\to} \triangle\varphi(x)$. Another way is direct and fun. We also include here.

$$u_t(x,t) = (\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right)\frac{-y}{\sqrt{t}}\cdot\left[D_x\varphi\left(x-2\sqrt{t}y\right) - D_x\varphi(x) + \overset{\text{odd integral}^0}{\overrightarrow{D_x\varphi(x)}}\right]dy$$

$$= (\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right)\frac{-y}{\sqrt{t}}\cdot\underbrace{\left[D_x\varphi\left(x-2\sqrt{t}y\right) - D_x\varphi(x)\right]}_{\approx -2\sqrt{t}y\|D^2\varphi\|_{L^\infty}}dy$$

$$\le (\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right) 2|y|^2\,dy \quad \left\|D^2\varphi\right\|_{L^\infty}.$$

So either ε-δ way or Lebesgue dominated convergence theorem implies

$$u_t(x,t) \overset{t\to 0^+}{\to} (\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right) 2\left\langle y, D_x^2\varphi(x)\,y\right\rangle dy$$

$$= 2(\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right)\left[\varphi_{11}(x)\,y_1^2 + \cdots + \varphi_{nn}(x)\,y_n^2\right]dy$$

$$= 2(\pi)^{-n/2}\int_{R^n} \exp\left(-|y|^2\right)\left[\varphi_{11}(x)\frac{|y|^2}{n} + \cdots + \varphi_{nn}(x)\frac{|y|^2}{n}\right]dy$$

$$= \frac{2\triangle\varphi(x)}{n\pi^{n/2}}\int_{R^n} \exp\left(-|y|^2\right)|y|^2\,dy$$

$$= \frac{\triangle\varphi(x)}{n\pi^{n/2}}\int_0^\infty 2\exp\left(-r^2\right) r^2 r^{n-1}|\partial B_1|\,dr \overset{r^2=\rho}{=} \frac{\triangle\varphi(x)}{n\pi^{n/2}}|\partial B_1|\int_0^\infty \exp(-\rho)\,\rho^{n/2}d\rho$$

$$= \frac{\triangle\varphi(x)}{n\pi^{n/2}}\frac{2\pi^{n/2}}{\Gamma(n/2)}\Gamma\left(\frac{n}{2}+1\right) = \frac{\triangle\varphi(x)}{n\pi^{n/2}}\frac{2\pi^{n/2}}{\Gamma(n/2)}\Gamma\left(\frac{n}{2}\right)\frac{n}{2} = \triangle\varphi(x)!$$

RMK. In contrast, recall for harmonic function in the upper half space $\left\{ \begin{array}{l} \triangle u = 0 \quad \text{for } x_n > 0 \\ u(x', 0) = \varphi(x') \end{array} \right.$

$$\begin{aligned} \varphi \in C^1\left(R^{n-1}\right) &\nRightarrow u_{x_n} \in C^1\left(\overline{R^n_+}\right) \\ \varphi \in C^{1,\alpha}\left(R^{n-1}\right) &\nRightarrow u_{x_n} \in C^{\alpha}\left(\overline{R^n_+}\right). \end{aligned}$$

Nonhomogeneous problem

$$\left\{ \begin{array}{l} u_t - \triangle u = f(x,t) \quad \text{in } R^n \times (0,\infty), \text{ say } C_1^2 \text{ compact support} \\ u(x,0) = \varphi(x), \quad \text{say } C^0 \end{array} \right. .$$

has A solution

$$u(x,t) = \frac{1}{(4\pi)^{n/2}} \int_0^t \int_{R^n} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{n/2}} f(y,s)\, dyds + \frac{1}{(4\pi t)^{n/2}} \int_{R^n} \frac{e^{-\frac{|x-y|^2}{4t}}}{t^{n/2}} \varphi(y)\, dy,$$

where we assumed $f \equiv 0$ with $t < 0$ for the integral $\int_{-\infty}^t$, satisfying

i) $u \in C_1^2$ for $t > 0$;

ii) $u(x,t) \overset{t \to 0^+}{\to} \varphi(x)$.

We only prove ii) with

$$\begin{aligned} &\left| \frac{1}{(4\pi)^{n/2}} \int_0^t \int_{R^n} \frac{e^{-\frac{|x-y|^2}{4(t-s)}}}{(t-s)^{n/2}} f(y,s)\, dyds \right| \\ &\leq \left| \frac{1}{\pi^{n/2}} \int_0^t \int_{R^n} e^{-|y|^2} f\left(x - 2\sqrt{t-s}y, s\right) dyds \right| \\ &\leq t \|f\|_{L^\infty} \frac{1}{\pi^{n/2}} \int_{R^n} e^{-|y|^2} dy \\ &= t \|f\|_{L^\infty} \overset{t \to 0^+}{\to} 0. \end{aligned}$$

<u>Mean value equality</u>

Recall the derivation of the “solid" mean value formula for harmonic functions

$$\int_{B_R} u \triangle v - v \triangle u = \int_{\partial B_R} u v_\gamma - v u_\gamma$$

Set $v = \underbrace{\frac{-1}{(n-2)|\partial B_1|}}_{c_n} \left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right)$

$$u(0) = \int_{\partial B_R} u v_\gamma dA = \int_{\Gamma = \frac{-1}{c_n R^{n-2}} = l} u |D\Gamma|\, dA$$

where $\Gamma = \frac{-1}{c_n} \frac{1}{|x|^{n-2}}$.

As R goes from 1 to 0, the level of Γ runs from $-1/c_n$ to $-\infty$, we seek a power weight $w(l) = c(-l)^\alpha$ satisfying $1 = \int_{-\infty}^{-1/c_n} w(l)\,dl$, and then

$$\begin{aligned} u(0) &= \int_{-\infty}^{-1/c_n} w(l) \int_{\Gamma=l} u\,|D\Gamma|\,dAdl \\ &= \int_{B_1} u\ w(l)\,|D\Gamma|^2\,dx, \end{aligned}$$

where we used change of variable or co-area formula: $dx = dvol = dA\ \frac{dl}{|D\Gamma|}$

figure gradient length$=dl/|D\Gamma|$

Now we choose w so that the weight $w(l)\,|D\Gamma|^2 = 1/|B_1|$, then

$$w(l) = \frac{1}{|B_1|}\left[\frac{c_n}{(n-2)}R^{n-1}\right]^2.$$

At $R=1$ and $l = -1/c_n$, we have

$$c\,(1/c_n)^\alpha = \frac{1}{|B_1|}\left[\frac{c_n}{(n-2)}\right]^2.$$

The integral for the weight implies

$$-c\frac{1}{1+\alpha}\left(\frac{1}{c_n}\right)^{1+\alpha} = 1.$$

Thus $-(1+\alpha) = \frac{c_n}{|B_1|(n-2)^2} = n/(n-2)$, then

$$w(l) = \frac{n}{n-2}(c_n)^{-n/(n-2)}(-l)^{-\frac{2(n-1)}{n-2}}.$$

RMK. Old weight way, $1 = \int_0^1 nr^{n-1}dr \overset{r=(-c_n l)^{-1/(n-2)}}{=} \frac{n}{n-2}c_n^{\frac{-n}{n-2}} \int_{-\infty}^{-1/c_n} (-l)^{-\frac{2(n-1)}{n-2}}\,dl.$ And pleasantly $w(l)\,|D\Gamma|^2 = 1/|B_1|$!

Mean value equality for caloric functions (sphere version)

$$u(0,0) = \frac{1}{(4\pi R^2)^{n/2}} \int_{\Phi=(4\pi R^2)^{-n/2}} u(x,t)\frac{|x|^2\,\Phi(0,0,x,t)}{\sqrt{4t^2\,|x|^2 + \left(2nt + |x|^2\right)^2}}dA.$$

figure: heat sphere $\Phi(0,0,x,t) = [4\pi(-t)]^{-n/2}\exp\frac{|x|^2}{4t} = (4\pi)^{-n/2}$

One "solid" version

$$u(0,0) = \frac{1}{(4\pi R^2)^{n/2}} \int_{\Phi\geq(4\pi R^2)^{-n/2}} u(x,t)\frac{|x|^2}{4t^2}dA,$$

where

$$\Phi\left(x_0, t_0; x, t\right)=\frac{1}{\left[4\pi\left(t_0-t\right)\right]^{n/2}} e^{-\frac{\left|x_0-x\right|^2}{4\left(t_0-t\right)}}$$

figure: backward heat kernel graphs

Derivation of the hollow version.

$$\begin{aligned}
&\int_{U_T} Du\cdot Dv+u\triangle v=\int_{U_T} div\left(uDv\right)=\int_{U_T}\left(div, D_t\right)\left(uDv, 0\right)=\int_{\partial U_T}\left(uDv, 0\right)\cdot\left(\gamma_x, \gamma_t\right) dA\\
-)&\int_{U_T} Dv\cdot Du+v\triangle u=\int_{U_T} div\left(vDu\right)=\int_{U_T}\left(div, D_t\right)\left(uvDu, 0\right)=\int_{\partial U_T}\left(vDu, 0\right)\cdot\left(\gamma_x, \gamma_t\right) dA\\
+)&\int_{U_T} uD_tv+vD_tu=\int_{U_T} D_t\left(uv\right)=\int_{U_T}\left(div, D_t\right)\left(0, uv\right)=\int_{\partial U_T}\left(0, uv\right)\cdot\left(\gamma_x, \gamma_t\right) dA
\end{aligned}$$

$\Rightarrow$

$$\int_{U_T} u\Big(\underbrace{D_tv+\triangle v}_{\text{“}\delta(0,0)''}\Big)+v\Big(\underbrace{D_tu-\triangle u}_{0}\Big)=\int_{\partial U_T} uv_{\gamma_x}-\underbrace{v}_{0}u_{\gamma_x}+uv\gamma_t\ dA.$$

Take $v=\frac{1}{[4\pi(-t)]^{n/2}}e^{\frac{|x|^2}{4t}}-\frac{1}{[4\pi]^{n/2}}$, then

$$\begin{aligned}
E&=\left\{\frac{1}{\left[4\pi\left(-t\right)\right]^{n/2}}e^{\frac{|x|^2}{4t}}\geq\frac{1}{\left[4\pi\right]^{n/2}}\right\}\Leftrightarrow\left\{|x|^2\leq 2nt\ln\left(-t\right)\right\}\\
U_T&=E\cap\left\{t\leq s\right\}.
\end{aligned}$$

figure U_T

We have from the Green's identity

$$0=\int_{\partial E\cap\{t\leq s\}} uv_{\gamma_x}+\underbrace{\int_{I_s} uvdA}_{\to u(0,0)}$$

as $s\ \to 0^-$, say for $u\in C^0$

$$\begin{aligned}
v_{\gamma_x}&=D\Phi\cdot\gamma_x=D\Phi\cdot\frac{-D\Phi}{\left|\left(D\Phi, D_t\Phi\right)\right|}=\frac{-\left|D\Phi\right|^2}{\left|\left(D\Phi, D_t\Phi\right)\right|}\\
&=\frac{-\Phi^2\left|\frac{x}{2t}\right|^2}{\sqrt{\Phi^2\left|\frac{x}{2t}\right|^2+\Phi^2\left(\frac{n}{2}\frac{1}{-t}-\frac{|x|^2}{4t^2}\right)^2}}=-\Phi\frac{|x|^2}{\sqrt{4t^2\left|x\right|^2+\left(2nt+\left|x\right|^2\right)^2}}.
\end{aligned}$$

Therefore

$$u(0,0)=\frac{1}{(4\pi)^{n/2}}\int_{\Phi=(4\pi)^{-n/2}}u(x,t)\,\frac{|x|^2\ \ \Phi(0,0,x,t)}{\sqrt{4t^2|x|^2+\left(2nt+|x|^2\right)^2}}dA$$

$$u(0,0)=\frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi=(4\pi R^2)^{-n/2}}u(x,t)\,\frac{|x|^2\ \ \Phi(0,0,x,t)}{\sqrt{4t^2|x|^2+\left(2nt+|x|^2\right)^2}}dA$$

$$=\int_{\Phi=(4\pi R^2)^{-n/2}=l}u(x,t)\,\frac{|D\Phi|^2}{|(D\Phi,D_t\Phi)|}dA$$

Having this sphere version, let us get a "solid" mean value formula, by integrating level by level of the heat kernel.

Set $w(l)$ s.t. $1=\int_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty}w(l)\,dl$,

$$\begin{aligned}u(0,0)&=\int_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty}w(l)\int_{\Phi=l}u\frac{|D\Phi|^2}{|(D\Phi,D_t\Phi)|}dA\ dl\\&=\int_{\Phi\geq\left(\frac{1}{4\pi}\right)^{n/2}}uw(l)\,|D\Phi|^2\,dxdt\qquad\text{recall } dA\ \frac{dl}{|\nabla\Phi|}=dvol\\&=\int_{\Phi\geq\left(\frac{1}{4\pi}\right)^{n/2}}u\ \underbrace{w(l)\,\Phi^2}_{\text{constant}}\ \left|\frac{x}{2t}\right|^2\ dxdt\end{aligned}$$

Let $w(l)=\left(\frac{1}{4\pi}\right)^{n/2}\frac{1}{l^2}$, then $\int_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty}w(l)\,dl=\left(\frac{1}{4\pi}\right)^{n/2}\frac{-1}{l}\Big|_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty}=1$. Thus

$$u(0,0)=\frac{1}{(4\pi)^{n/2}}\int_{\Phi\geq\left(\frac{1}{4\pi}\right)^{n/2}}u\ \ \frac{|x|^2}{4t^2}dxdt$$

or

$$u(x_0,t_0)=\frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi(x_0,t_0,x,t)\geq\left(\frac{1}{4\pi R^2}\right)^{n/2}}u\ \ \frac{|x|^2}{4t^2}dxdt$$

figure heat ball

rescale the unit heat ball by R^2 in time and R in space

In particular, for $u\equiv 1$

$$1=\frac{1}{(4\pi R^2)^{n/2}}\int_{[4\pi(-t)]^{-n/2}e^{\frac{|x|^2}{4t}}\geq\frac{1}{(4\pi R^2)^{n/2}}}\frac{|x|^2}{4t^2}dxdt=\int_{\underbrace{\Phi\geq 1}_{\text{say }4\pi R^2=1}}\frac{|x|^2}{4t^2}dxdt.$$

RMK. Other choices of weights. For example $w(l)=c/l^{2-\alpha}$, $0<\alpha<1$

$$u(0,0)=c\int_{\Phi\geq 1}u\ \ \Phi^{\alpha}\frac{|x|^2}{4t^2}dxdt,$$

the kernel is still singular, though the kernel/weight has integrable singularity of order $-n\alpha - 2$ in space-time.

Applications of mean value formulas

App1. Strong max principle:

Let $u \in C_1^2$ solution to $u_t - \triangle u = 0$ in U_T. THEN

$\cdot$ $\max u$ only attains at the parabolic boundary of U_T;

$\cdot$ otherwise, if $\max u = u(x_0, t_0)$, where (x_0, t_0) is an interior or non-parabolic boundary of U_T, then we have $u(x,t) \equiv u(x_0, t_0)$ for all (x,t) in the closure of the connected set of $U_T \cap \{t \leq t_0\}$ by chain of downward heat balls.

Def: Parabolic boundary points cannot center any heat ball inside the domain U_T.

Examples of U_T

figure parabolic bdry

Proof of the strong max principle.

Suppose $u(x_0, t_0) = \max_{U_T} u$ and (x_0, t_0) is not a parabolic boundary point, that is, (x_0, t_0) centers a heat ball in U_T. By the mean value formula in this ball

$$
\begin{aligned}
u(x_0, t_0) &= \frac{1}{(4\pi R^2)^{n/2}} \int_{\Phi(x_0,t_0,x,t) \geq (4\pi R^2)^{-n/2}} u(x,t) \quad \frac{|x|^2}{4t^2} dxdt \\
&\overset{\text{kernel } \geq 0}{\leq} \frac{1}{(4\pi R^2)^{n/2}} \int_{\Phi(x_0,t_0,x,t) \geq (4\pi R^2)^{-n/2}} u(x_0,t_0) \quad \frac{|x|^2}{4t^2} dxdt \\
&\overset{\int \text{kernel}=1}{=} u(x_0, t_0).
\end{aligned}
$$

Thus $u(x,t) \equiv u(x_0, t_0)$ in

$$
\left\{ (x,t) | \quad \frac{1}{(t_0 - t)^{n/2}} e^{-\frac{|x_0 - x|^2}{4(t_0 - t)}} \geq \frac{1}{R^n} \right\}.
$$

RMK. The closure includes the points at horizontal level $\{t = t_0\}$, this is because such a point (y, t_0) is the limit of (y, s) as $s \uparrow t_0$, and the segment (x_0, t_0) -(y, s) can be covered a chain of heat balls.

figure downward segment

Then $u(y,t) = \lim u(y,s) = \lim u(x_0, t_0)$.

Uniqueness of caloric function on bounded domains

Let u, v be two $C_1^2(U_T) \cap C_0^0(\bar{U}_T)$ solutions to $w_t - \triangle w = 0$ in U_T, and $u = v$ on the parabolic boundary of U_T. THEN $u \equiv v$.

RMK. U_T including U_∞ domains like $\{t > \text{convex}(x)\}$, say

$$\text{figure } t \geq |x|^4 .$$

Question. What happens to $R^n \times [0,T]$ or $R_+^1 \times [0,T]$?

App2. Regularity

Faking space dimension $R^n \to R^{n+m}$ will lead us to a C_1^2 (even better ones for larger m) kernel in the mean value formula:

$$u_t(x,t) - (\triangle_x + \triangle_y) u(x,t) = 0$$

$$\begin{aligned} u(x_0,t_0) &= \int_{\Phi(x_0,t_0,x,t) \geq (4\pi R^2)^{-n/2}} u(x,t)\ K(x_0 - x, t_0 - t)\, dxdt \\ &= \int_{R^n \times R^1} u(x,t)\ K(x_0 - x, t_0 - t)\, dxdt \end{aligned}$$

where K is the kernel of Kuptsov (c.f. Neil A. Waston 2002). Then starting from L^1 function, satisfying the parabolic mean value formula, we immediately have C_1^2 solution to the heat equation (no need existence). We can also get interior estimates.

RMK. One way to verify those C_1^2 functions (out of L^1, enjoying solid, then hollow mean value formulas) satisfy the heat equation, comes from the derivation of the heat sphere mean value derivation

$$\int_{\Phi \geq c} v(D_t u - \triangle u)\, dxdt = \int_{\Phi = c} u v_{\gamma_x} dA + u(x_0,t_0) = 0.$$

Next, by using a different argument via Green's identity over a cylinder $U_T = B_R \times [0,T]$, we show the C_1^2 solutions are C^∞ in x, t and C^ω in x. Recall the fundamental solution is not C^ω in t.

Green's identity

$$\int_{U_T} u\Big(\underbrace{D_t v + \triangle v}_{\text{“}\delta(0,0)\text{''}}\Big) + v\Big(\underbrace{D_t u - \triangle u}_{0}\Big) = \int_{\partial U_T} u v_{\gamma_x} - v u_{\gamma_x} + u v \gamma_t \ dA.$$

$$v = \Phi(x,t;y,s) = e^{\frac{|x-y|^2}{4(t-s)}} / \left[4\pi(t-s)\right]^{n/2}$$

$$u(x,t) = -\int_{\partial U\times[0,t]} u(y,s) \underbrace{\Phi_{\gamma_y}(x,t;y,s)}_{C^\omega \text{ in } x \text{ not in } t,\ C^\infty \text{ in } t} dA$$
$$+\int_{\partial U\times[0,t]} u_{\gamma_y}(y,s) \underbrace{\Phi(x,t;y,s)}_{C^\omega \text{ in } x \text{ not in } t,\ C^\infty \text{ in } t} dA$$
$$+\int_U u(y,0) \underbrace{\Phi(x,t;y,0)}_{C^\omega \text{ in } x,t \text{ for } t\geq\delta_0} dy.$$

So we conclude u is $C^\omega(x)$ and $C^\infty(t)$ in $U_T' \subset U_T$.

Interior estimates

$$\max_{C_1} \left|D_x^k D_t^l u\right| \leq C(k,l,K) \max_{C_2} \left|D_x^2 u\right| + |D_t u| \leq C(k,l,K) \max_{C_3} |u|$$

or

$$\max_{C_R} \left|D_x^k D_t^l u\right| \leq \frac{C(k,l,K)}{R^{k+2l}} \max_{C_{3R}} |u|$$

via scaling $v(x,t) = u(Rx, R^2 t)$, $D_x^k D_t^l v(x,t) = R^{k+2l} \ D_x^k D_t^l u|_{(Rx,R^2t)}$. Here $C_R = B_R(x_0) \times (t_0 - R^2, t_0)$.

Liouville Theorem: Global (eternal) solution, say C_1^2 to $u_t - \triangle u = 0$ in $R^n \times (-\infty, +\infty)$ satisfying

$$|u(x,t)| \leq A\left(|x|^k + |t|^l\right) \quad \text{for large } |x| + |t|$$

must be a caloric polynomial of degree less than $k+2l$.

Proof.
$$\left|D_x^k D_t^l u(0,0)\right| \leq \frac{C(k',l',K)}{R^{k'+2l\prime}} A \ \left(R^k + R^{2l}\right) \overset{R\to\infty}{\to} 0$$

for $k' + 2l\prime > k + 2l$. Note $(0,0)$ could be anywhere, so $u\ (x,t)$ is a caloric polynomial of degree $\leq k+2l$.

App3. Harnack inequality
$u \geq 0$ solution to $u_t - \triangle u = 0$, then

$$\max_{C_r^-} u \leq C(n) \min_{C_r^+} u.$$

figure Harnack.

One proof is via "fake" dimension mean value formula, it is little "involved" in calculating the positive weight. However, everything is at calculus level.

Uniqueness revisited

- If a caloric function is analytic in terms of x and t, like

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{-\frac{|x-y|^2}{4t}} \varphi(y)\, dy \quad \text{for } t > 0,$$

then $u(x, 0.9)$ determines all the temperature for $t > 0 : t > 0.9$ or $0 < t < 0.9$, also like

$$u(x,t) = e^{x+t} \quad \text{or } e^{-t} \sin x,$$

$u(x, 0)$ determines all temperature for $-\infty < t < \infty$.

- In general, caloric functions are not analytic in t, like

$$u(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{x^2}{4t}} & \text{for } x \neq 0 \text{ and } t > 0 \\ 0 & \text{for } x \neq 0 \text{ and } t \leq 0 \end{cases},$$

then $u(x, 0)$ cannot determine $u(x,t)$ for $t > 0$.

Recall we have from the maximum principle, that two caloric functions agree on the parabolic boundary of a bounded space-time domain, they agree everywhere.

figure cylinder domain and $\left\{t > |x|^2\right\}$ intersecting $t \leq T$

Energy proof of the uniqueness: $w \in C_1^2\left(\bar{U}_T\right)$ solution to $w_t - \triangle w = 0$, vanishing on the parabolic boundary, then $w \equiv 0$.

C^1 way. We calculate/estimate the energy

$$0 \leq \int\int_{U_T} |Dw|^2\, dxdt = \int_0^T \int_{\Omega_t} \operatorname{div}(wDw) - w \triangle w dxdt = \int_0^T \left(\int_{\partial\Omega_t} \vec{w}^0 w_\gamma dA - \int_{\Omega_t} w \triangle w dx \right) dt$$

$$= -\int_0^T \int_{\Omega_t} w w_t dxdt = -\int_{\Omega_T} \int_{t(x)}^T \frac{1}{2}\left(w^2\right)_t dtdx = -\int_{\Omega_T} \frac{1}{2} w^2(x,T)\, dx \leq 0.$$

So $|Dw| = 0$ and then $w \equiv 0$.

To prepare for a proof of backward uniqueness, we present another

C^0 way. We calculate/estimate the L^2 norm of w. Set $E(t) = \int_{\Omega_t} w(x,t)^2\, dx$, say $\Omega_t = \Omega$.

$$\frac{d}{dt} E(t) = \dot{E}(t) = \int_\Omega 2 w w_t dx = \int_\Omega 2w \triangle w dx$$
$$= 2\int_{\partial\Omega} \vec{w}^0 w_\gamma dA - 2\int_\Omega |Dw|^2\, dx = -2\int_\Omega |Dw|^2\, dx \leq 0.$$

Then $\int_\Omega w^2(x,t)\, dx = E(t) \leq E(0) = 0$. So $w(x,t) \equiv 0$.

Backward uniqueness: Let u and v be two C_1^2 solutions to $\begin{cases} w_t - \triangle w = 0 & \text{in } \Omega \times [0,T] \\ u = v & \text{on } \partial\Omega \times [0,T] \\ u = v & \text{on } \Omega \times T \end{cases}$,

then $u \equiv v$ on $\Omega \times [0,T]$.

Proof. By linearity, we only need to show from $w_t - \triangle w = 0 \quad \text{in } \Omega \times [0,T]$ and $w = 0$ on $\Omega \times T$ and $\partial\Omega \times [0,T]$, we have $w \equiv 0$ or $w(x,0) = 0$.Set

$$E(t) = \int_\Omega w(x,t)^2\, dx.$$

The proof is based on the observation that $\ln E(t)$ is convex in terms of t.

Step1. As in the C^0 way in the above, we have

$$\dot{E}(t) = -2\int_\Omega |Dw|^2\, dx \le 0.$$

We will also need $\int_\Omega |Dw|^2\, dx \overset{w=0 \text{ on } \partial\Omega}{=} \int_\Omega -w\triangle w dx \le \left(\int_\Omega w^2 dx\right)^{1/2} \left(\int_\Omega (\triangle w)^2\, dx\right)^{1/2}$.
Take one more derivative

$$\begin{aligned}\ddot{E}(t) &= -4\int_\Omega Dw \cdot Dw_t dx = -4\int_{\partial\Omega} w_\gamma \cdot \overrightarrow{w_t}^0 dA + 4\int_\Omega \triangle w\ w_t dx \\ &= 4\int_\Omega (\triangle w)^2\ dx.\end{aligned}$$

Step2. Suppose $E(t) > 0$ for $0 \le t < T' \le T$ and $E(T') = 0$. Then

$$\frac{d}{dt}\ln E = \frac{\dot{E}}{E}$$

and

$$\frac{d^2}{dt^2}\ln E = \frac{\ddot{E}E - \dot{E}^2}{E^2} \ge 0,$$

since $\dot{E}^2 = 4\left(\int_\Omega |Dw|^2\, dx\right)^2 \le 4\int_\Omega w^2 dx \int_\Omega (\triangle w)^2\, dx = \ddot{E}E$.

Now the convex function $\ln E(t)$ cannot go to $-\infty$ as t goes to T', for it should stay above the tangent line at $(0, \ln E(0))$. This contradiction shows $E(t) \equiv 0$.

Uniqueness for Cauchy problem with constraints in $R^n \times R^+$.

Max Principle: Let u be $C_1^2(R^n \times (0,T)) \cap C(R^n \times [0,T])$ solution to

$$\begin{cases} u_t - \triangle u = 0 \\ u(x,0) = g(x) \end{cases}.$$

Suppose $|u(x,t)| \le Ae^{a|x|^2}$ in $R^n \times [0,T]$. Then

$$|u(x,t)| \le \sup_{R^n} g(x) \quad \text{in } R^n \times [0,T].$$

Proof. We only need to prove $u(x,t) \le \sup_{R^n} g(x) \triangleq M$ for subcaloric solution $u_t - \triangle u \le 0$ with sub quadratic-exponential growth $u(x,t) \le Ae^{a|x|^2}$.

Step1.

figure t-direction thin domain.

For any $\mu > 0$, set

$$\begin{aligned} v &= M + \mu \frac{1}{\left(\frac{1}{8a} - t\right)^{n/2}} e^{\frac{|x|^2}{4\left(\frac{1}{8a} - t\right)}} \\ v_t - \triangle v = 0 &\geq u_t - \triangle u \\ v &\geq u \text{ on } \partial B_{R_\mu} \times \left[0, \frac{1}{16a}\right] \text{ for } R_\mu \text{ large} \\ v &\geq u \text{ on } R^n \times \{0\}. \end{aligned}$$

RMK. Invariance way to construct this barrier:

* $\frac{1}{\sqrt{t}} e^{\frac{|x|^2}{-4t}} \overset{\sqrt{-1}}{\dashrightarrow} \frac{1}{\sqrt{-t}} e^{\frac{|x|^2}{-4t}}$ still sol.

* the time shift sol v is quadratic-exponential growth at $t = 0$, it goes to ∞ as $t \nearrow \frac{1}{8a}$ for every x.

It follows from the maximum principle on the bounded domain,

$$u(x,t) \leq M + \mu \frac{1}{\left(\frac{1}{8a} - t\right)^{n/2}} e^{\frac{|x|^2}{4\left(\frac{1}{8a} - t\right)}} \text{ in } B_{R_\mu} \times \left[0, \frac{1}{16a}\right].$$

So for any fixed (x_0, t_0) with $t_0 \leq \frac{1}{16a}$, we have

$$u(x_0, t_0) \leq M + \mu \frac{1}{\left(\frac{1}{8a} - t_0\right)^{n/2}} e^{\frac{|x_0|^2}{4\left(\frac{1}{8a} - t_0\right)}} \overset{\mu \to 0}{\to} M.$$

Step2. The above argument works equally well on $\left[\frac{1}{16a}, \frac{2}{16a}\right], \left[\frac{2}{16a}, \frac{3}{16a}\right], \cdots$, still $[0, T]$.

Corollary. The Cauchy problem with growth constraint

$$\begin{cases} u_t - \triangle u = f \\ u(x, 0) = g(x) \end{cases} \text{ in } R^n \times (0, T)$$

with $|u(x,t)| \leq Ae^{a|x|^2}$ in $R^n \times [0, T]$, has at most one solution.

Proof. The difference of any two solutions satisfies the condition in the max principle with $g = 0$ and difference is less $2Ae^{a|x|^2}$, so the difference is 0.

eg. The caloric function, or a solution to $\begin{cases} u_t - \triangle u = 0 \\ u(x, 0) = e^{a|x|^2} \end{cases}$

Integral way to construct the barrier:

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{\frac{|x-y|^2}{-4t}} e^{a|y|^2} dy \\ &= \frac{1}{\pi^{n/2}} \int_{R^n} e^{-|y|^2 + a\left|2\sqrt{t}y - x\right|^2} dy \\ &= \frac{1}{(1 - 4at)^{n/2}} e^{\frac{\alpha}{1-4at}|x|^2} \end{aligned}$$

is

* $a>0$ unique in $R^n \times [0,\frac{1}{4a})$ with constraint $|u(x,t)| \le \frac{1}{(1-4at)^{n/2}} e^{\frac{\alpha}{1-4at}|x|^2}$ and grows faster than $e^{a|x|^2}$ for $t>0$;

* $a<0$ uniqueness in $R^n \times [0,\infty)$ with constraint $|u(x,t)| \le e^{100|x|^2}$ and grows faster than $e^{a|x|^2}$ for $t>0$.

The message: the growth/decay rate is not preserved precisely.

Nonuniqueness of Cauchy problem $\begin{cases} u_t - \triangle u = 0 \text{ in } R^n \times [0,\infty) \\ u(x,0) = 0 \end{cases}$

Tikhonov's counterexample.

Idea of construction:

* along $t=0$, position $u(x,0)$ alone determines all the derivatives (if analytic).

* along $x=0$, position $u(0,t)$ and velocity $u_x(0,t)$ determines all the derivatives in x, (if analytic in x).

Now we solve a "real" Cauchy problem along the t-axis

$$\begin{cases} u(0,t) = g(t) \\ u_x(0,t) = 0 \end{cases}$$

$u_x(0,t) = 0$, $u_{xxx}(0,t) = u_{xt}(0,t) = 0, \cdots, D_x^{2k+1}u(0,t) = D_t^k u_x(0,t) = 0$

$u_{xx}(0,t) = u_t(0,t) = g'(t), u_{xxxx}(0,t) = u_{tt}(0,t) = g''(t), \cdots, D_x^{2k}u(0,t) = D_t^k u(0,t) = D_t^k g(t)$.

Assuming u is C^ω in terms of x, then

$$u(x,t) = g(t) + \sum_{k=1}^{\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k}.$$

Technical realization:

$$\text{graph for } g(t) = \begin{cases} e^{-\frac{1}{t^\alpha}} & t>0 \\ 0 & t \le 0 \end{cases} \quad \text{need } \alpha > 1.$$

How to control the derivatives

1st Direct try.

$g(t) = e^{-t^{-\alpha}}$

$g' = e^{-t^{-\alpha}} \alpha t^{-\alpha-1}$

$g'' = e^{-t^{-\alpha}} \left[(\alpha t^{-\alpha-1})^2 - \alpha(\alpha+1) t^{-\alpha-2} \right]$

$g''' = e^{-t^{-\alpha}} \left[(\alpha t^{-\alpha-1})^3 + \cdots + \alpha(\alpha+1)(\alpha+2) t^{-\alpha-3} \right]$

$\cdots$

$g^{(k)} = e^{-t^{-\alpha}} \left[(\alpha t^{-\alpha-1})^k + \cdots \pm \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1) t^{-\alpha-k} \right]$

$\quad \approx e^{-t^{-\alpha}} k! (\alpha t^{-\alpha-1})^k$

$\frac{g^{(k)}}{(2k)!} \approx e^{-t^{-\alpha}} \frac{k!}{(2k)!} \left(\alpha t^{-\alpha-1}\right)^k$, and $\frac{k!}{(2k)!} = \frac{1}{2^k \ 1\cdot 3\cdot 5\cdots(2k-1)} \approx \frac{1}{2^{2k} \ k!}$

$|u(x,t)| \leq g(t) + \sum_{k=1}^{\infty} \frac{e^{-t^{-\alpha}}}{k! \ 2^{2k}} \left(\alpha t^{-\alpha-1}\right)^k \ x^{2k} = e^{-t^{-\alpha}} e^{\frac{\alpha t^{-\alpha-1}}{4} x^2} = e^{-\frac{1}{t^\alpha} + \frac{\alpha x^2}{4} \frac{1}{t^{\alpha+1}}} \overset{t\to 0+}{\longrightarrow} \infty$.

2nd complex try

Observe $e^{-t^{-\alpha}}$ is analytic when $t > 0$

figure complex plan $z = t + is$

$g(t) = \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{z-t} dz$

$g^{(k)}(t) = \frac{k!}{2\pi i} \int_\gamma \frac{g(z)}{(z-t)^{k+1}} dz$

Now by continuity of $z^{-\alpha}$ at 1, $\operatorname{Re} z^{-\alpha} \geq \frac{1}{2}$ for $|z-1| \leq \mu$, where $\mu = \mu(1/2) < 1$, then $\operatorname{Re}(tz)^{-\alpha} \geq \frac{1}{2} t^{-\alpha}$ for $|tz - t| \leq \mu t$

or $\operatorname{Re} z^{-\alpha} \geq \frac{1}{2} t^{-\alpha}$ for $|z - t| \leq \mu t$

and

$$\left| e^{-z^{-\alpha}} \right| = e^{\operatorname{Re} - z^{-\alpha}} \leq e^{-\frac{1}{2} t^{-\alpha}} \quad \text{for } |z-t| \leq \mu t.$$

So

$$\left| g^{(k)}(t) \right| \leq \frac{k!}{2\pi} \left| \int_{|z-t|=\mu t} \frac{g(z)}{(z-t)^{k+1}} dz \right| \leq \frac{k!}{2\pi} \frac{e^{-\frac{1}{2}t^{-\alpha}}}{(\mu t)^{k+1}} 2\pi \mu t = \frac{k! \ e^{-\frac{1}{2}t^{-\alpha}}}{(\mu t)^k}$$

and

$$\begin{aligned} |u(x,t)| &\leq g(t) + \sum_{k=1}^{\infty} \frac{1}{(2k)!} \frac{k! \ e^{-\frac{1}{2}t^{-\alpha}}}{(\mu t)^k} \ x^{2k} \\ &\leq e^{-t^{-\alpha}} + e^{-\frac{1}{2}t^{-\alpha}} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{x^2}{\mu t} \right)^k \\ &\leq e^{-\frac{1}{2}t^{-\alpha} + \frac{x^2}{\mu t}} \begin{cases} < \infty & \text{for all } (x,t) \text{ with } t > 0, \\ \to 0 & \text{as } t \to 0+ \text{ for each fixed } x, \end{cases} \quad \text{provided } \alpha > 1. \end{aligned}$$

Then the Tikhonov's series converges, we've constructed a "super" quadratic-exponential caloric function such that $u_t - u_{xx} = 0$ and $u(x,0) = 0$, $u(x,t)$ is not identically 0 for $t > 0$.

RMK. Choosing

$$g(t) = \begin{cases} e^{-\frac{1}{t^\alpha} - \frac{1}{(1-t)^\alpha}} & t > 0 \\ 0 & t \leq 0 \text{ or } t \geq 1 \end{cases},$$

figure complex plan $z = t + is$ for this g

we have another Tikhonov solution/caloric function vanishes before 0 or after 1.

figure Tikhonov double sided vanishing

RMK. Let $v = u^2$, where u is the above Tikhonov caloric function, then $v_t - \triangle v = 2uu_t - 2u \triangle u - 2|Du|^2 = -2|Du|^2 \leq 0$. This v is non-negative sub-caloric function vanishing at $t = 0$ and $t = 1$, yet doesn't vanish identically between $(0, 1)$.

Nonanalytic, yet smooth solution in $R^n \times [0, \infty)$
eg.

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x, 0) = e^{-x^4} \end{cases}$$

has the bounded solution/quadratic-exponential growth solution

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{1/2}} \int_{R^1} e^{-\frac{|x-y|^2}{4t}} e^{-|y|^4} dy \\ &= \frac{1}{\pi^{1/2}} \int_{R^1} e^{-y^2} e^{-\left|x-2\sqrt{t}y\right|^4} dy \end{aligned}$$

which is $C^\infty(R^1 \times [0, \infty))$, but NOT C^ω at $t = 0$.

In fact, if $u(0, t)$ is analytic in terms of t near $t = 0$, then

$$u(0,t) = \sum_{k=0}^{\infty} a_k t^k,$$

where

$$\begin{aligned} a_k &= \frac{1}{k!} D_t^k u(0,0) = \frac{1}{k!} D_x^{2k} u(0,0) = \frac{1}{k!} D_x^{2k} \sum_{m=0}^{\infty} \frac{(-x^4)^m}{m!} \Bigg|_{x=0} \\ &\overset{2k=4m}{=} \frac{1}{(2m)!} \frac{(4m)!}{m!} > m!. \end{aligned}$$

So

$$\left|a_{2m} t^{2m}\right| > m! t^{2m} \overset{m\to\infty}{\longrightarrow} \infty \quad \text{for any fixed } t > 0.$$

Then the series diverges, $u(x, t)$ cannot be analytic in t at $(0, 0)$.

Nonexistence of nonnegative solution to Cauchy problem

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x, 0) = e^{x^4} \end{cases}$$

First note the representation

$$\frac{1}{(4\pi t)^{1/2}} \int_{R^1} e^{-\frac{|x-y|^2}{4t}} e^{y^4} dy = \infty.$$

Overheated, the nonnegative solution blows up once time starts.

Now the proof.

figure for e^{x^2} and g_k

$g_k(x) \in C_0^\infty(B_{k+1})$ and $g_k(x) = e^{x^4}$ on B_k. The bounded C_1^2 solution to

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = g_k \end{cases}$$

is

$$u_k(x,t) = \frac{1}{\pi^{1/2}} \int_{R^1} e^{-y^2} g_k\left(x - 2\sqrt{t}y\right) dy.$$

figure for e^{-y^2} and $g_k\left(R - 2\sqrt{t}y\right)$

For each fixed k and say, 0.9, there exists $R_k = R(k, 0.9)$ large so that

$$0 \le u_k(\pm R, t) \le 0.9 \quad \text{for } 0 \le t \le 1.$$

This is because

$$u_k(\pm R_k, t) \le \frac{1}{\pi^{1/2}} e^{-\left(\frac{R-k-1}{2\sqrt{t}}\right)^2} e^{k^4} \frac{k+1}{\sqrt{t}}.$$

Then as u is nonnegative, $u_k \le 0.9 + u$ on the parabolic boundary of the cylinder $B_{R_k} \times [0,1]$. The maximum principle implies $u_k \le 0.9 + u$ in $B_{R_k} \times [0,1]$. In particular

$$u_k(0,1) \le 0.9 + u(0,1) \quad \text{for all } k.$$

But $u_k(0,1)$ goes to $+\infty$, as k goes to $+\infty$. A contradiction!

In fact, $u(0,l)$ is forced to be ∞ for all small $l > 0$.

Question: Existence of sign-changing solutions? Answer: YES, F. B. Jr. Jones 1977.