Lecture 4 Wave equations

∘ invariance, explicit solutions
∘ radial way, self-similar way, and Fourier way
∘ Cauchy problem in 1d, 3d, & 2d (spherical means)
∘ energy method, uniqueness, domain of dependence

<u>Invariance</u> of $u_{tt}-\triangle u=0$ in R^{n+1}
· $u(x+x_0,t+t_0)$
· $u\left(e^{A}x,t\right)$ with $A=-A^T$
· $u\left(e^{L}(x,t)\right)$ with $L\begin{pmatrix} I & \\ & -1\end{pmatrix}+\begin{pmatrix} I & \\ & -1\end{pmatrix}L^T=0$, $e^L\in O(n,1)$ and $L\in o(n,1)$, eg. $n=1$

$$e^L=\begin{pmatrix}\cosh\varepsilon & \sinh\varepsilon\\ \sinh\varepsilon & \cosh\varepsilon\end{pmatrix}\quad\text{and } L=\varepsilon\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

· $u(\mu x,\mu t)$
· $u+v, au$

· $D_x^\alpha D_t^l u$ still wave
· $Du\cdot Ax=\frac{d}{d\varepsilon}u\left(e^{\varepsilon A}x,t\right)\Big|_{\varepsilon=0}$ with $A=-A^T$
· $(Du,D_tu)\cdot L(x,t)=\frac{d}{d\varepsilon}u\left(e^{\varepsilon L}(x,t)\right)\Big|_{\varepsilon=0}$ with $L\in o(n,1)$, eg. $n=1$

$$L=\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\text{and } tu_x+xu_t\ \text{ is still wave}$$

· $D_xu(x,t)\cdot x+D_tu(x,t)=\frac{d}{d\mu}u(\mu x,\mu t)\Big|_{\mu=1}$
· $\int u(x-y,t-s)\,\varphi(y,s)\,dyds$
· Kelvin like $\left(|x|^2-t^2\right)^{(1-n)/2}u\left(\frac{\left(x,\sqrt{-1}t\right)}{|x|^2-t^2}\right)$ by the twist harmonic generation

Examples.
· Harmonic generation
* all harmonic functions in R^n are stationary/time independent wave $x_1x_2x_3$, $|x|^{2-n}$, $\operatorname{Re}/\operatorname{Im} e^{3x_1+4x_2+i5x_3}$
* all harmonic functions $h(x,t)$ in $R^n\times R$ with imaginary twist on last variable $h\left(x,\sqrt{-1}t\right)$ are wave
$nt^2+|x|^2$, $nt^4+6t^2|x|^2+x_1^4+\cdots+x_n^4$, ..., all those harmonic polynomials in $R^n\times R^1$ should produce all those wave polynomials, as the "twist" operation is reversible.
$\left(|x|^2-t^2\right)^{(1-n)/2}$ for $n\geq 2$, $\ln\left(|x|^2-t^2\right)$ for $n=1$

[0]November 28, 2016

$\cdot\ e^{\xi\cdot x+|\xi|t}$, $\operatorname{Re}/\operatorname{Im} e^{i(\xi\cdot x+|\xi|t)}$, $u(\xi\cdot x+|\xi|\,t)$ (already seeing no smoothness improvement for, say $u\in C^2$)

RMK. e^{x_1+t} are both caloric and wave functions.

$\cdot$ Radial ones: $u(x,t)=u(r,t)$

$u_{tt}-u_{rr}-\frac{n-1}{r}u_r=0$ (and $u_{rtt}=u_{rrr}+\frac{n-1}{r}u_{rr}-\frac{n-1}{r^2}u_r$)

$n=1$: $u_{xx}=u_{tt}$, $u(x,t)=F(x+t)+G(x-t)$

$n=2$: $(ru)_{tt}=(ru)_{rr}-u_r$?

$n=3$: $(ru)_{tt}=r\left(u_{rr}+\frac{2}{r}u_r\right)=(ru)_{rr}$, from 1-d, $u(r,t)=r^{-1}F(r+t)+r^{-1}G(r-t)$

$n=4$: $(ru)_{tt}=(ru)_{rr}+u_r$

$\quad(r^2u_r)_{tt}=(r^2u_r)_{rr}+(n-5)\,ru_{rr}-(n+1)\,u_r=(r^2u_r)_{rr}+ru_{rr}-5u_r$

$n=5$: $(ru)_{tt}=(ru)_{rr}+2u_r$

$\quad(r^2u_r)_{tt}=(r^2u_r)_{rr}+(n-5)\,ru_{rr}-(n+1)\,u_r=(r^2u_r)_{rr}+ru_{rr}-6u_r$

then $(r^2u_r+3ru)_{rr}=(r^2u_r+3ru)_{tt}$, or

$$\partial_r^2\left[\frac{1}{r}\partial_r\left(r^3u\right)\right]\overset{\text{nontrivial}}{=}\frac{1}{r}\partial_r\left[r^3\left(\partial_r^2+\frac{4}{r}\partial_r\right)u\right]=\frac{1}{r}\partial_r\left[r^3\partial_{tt}u\right]=\partial_{tt}\left[\frac{1}{r}\partial_r\left(r^3u\right)\right]$$

RMK. The nontrivial fact in general dimension is

$$\partial_r^2\left[\left(\frac{1}{r}\partial_r\right)^{\frac{n-3}{2}}\left(r^{n-2}u\right)\right]=\left(\frac{1}{r}\partial_r\right)^{\frac{n-3}{2}}\left[r^{n-2}\left(\partial_r^2+\frac{n-1}{r}\partial_r\right)u\right]=\left(\frac{1}{r}\partial_r\right)^{\frac{n-3}{2}}\left[r^{n-2}\partial_{tt}u\right]$$
$$=\partial_{tt}\left[\left(\frac{1}{r}\partial_r\right)^{\frac{n-3}{2}}\left(r^{n-2}u\right)\right].$$

Yes, $\left(\frac{1}{r}\partial_r\right)^{\frac{n-3}{2}}$ is fractional derivative in even dimensions. Also radial solutions to other evolution equations like heat equation $\triangle h(r,t)=\partial_t h(r,t)$ satisfy a similar equation with ∂_{tt} replaced by ∂_t. For example, $\partial_{rr}[rh]=\partial_t[rh]$ in 3-d and $\partial_{rr}[r^2h_r+3rh]=\partial_t[r^2h_r+3rh]$. See the method of spherical means in [Courant-Hilbert, vol 2] for a complete account.

$\cdot$ Self-similar ones: $u(x,t)=t^\alpha v(r/t)$

$$\triangle u=t^\alpha\left[t^{-2}v''(r/t)+\frac{n-1}{r}t^{-1}v'(r/t)\right]=t^{\alpha-2}\left[v''(r/t)+\frac{n-1}{r/t}v'(r/t)\right]$$

$$u_{tt}=\left(\alpha t^{\alpha-1}v-t^{\alpha-2}rv'\right)_t=\alpha(\alpha-1)\,t^{\alpha-2}v-\alpha t^{\alpha-3}rv'-(\alpha-2)\,t^{\alpha-3}rv'+t^{\alpha-4}r^2v''$$
$$=t^{\alpha-2}\left[\frac{r^2}{t^2}v''(r/t)-2(\alpha-1)\,\frac{r}{t}v'(r/t)+\alpha(\alpha-1)\,v(r/t)\right]$$

Set $\rho=r/t$, we have equation for $v(\rho)$

$$\left(1-\rho^2\right)v_{\rho\rho}+\left[\frac{n-1}{\rho}+2(\alpha-1)\,\rho\right]v_\rho-\alpha(\alpha-1)\,v=0.$$

To eliminate v term, try $\alpha = 1$ or 0.

$\alpha = 1:$

$$v_{\rho\rho} + \frac{n-1}{\rho\left(1-\rho^2\right)} v_\rho = 0$$

$$\exp\left(\int \frac{n-1}{\rho\left(1-\rho^2\right)} d\rho\right) v_\rho = c \text{ or } \exp\left[(n-1)\ln\frac{\rho}{\left(1-\rho^2\right)^{1/2}}\right] v_\rho = c$$

then

$$u(x,t) = tv(\rho) = t\left\{c\int \frac{\left(1-\rho^2\right)^{n-1/2}}{\rho^{n-1}} d\rho + c'\right\} \quad \text{with } \rho = r/t.$$

$\alpha = 0:$

$$v_{\rho\rho} + \left[\frac{n-1}{\rho\left(1-\rho^2\right)} - \frac{2\rho}{1-\rho^2}\right] v_\rho = 0$$

$$u(x,t) = v(\rho) = c\int \frac{\left(1-\rho^2\right)^{n-1/2}}{\rho^{n-1}} + \frac{1}{1-\rho^2} d\rho + c' \quad \text{with } \rho = r/t.$$

Fourier transform for $u_{tt} - \triangle u = 0$

In space and time

$$\hat{u}(\xi, s) = \frac{1}{(2\pi)^{(n+1)/2}} \int_{R^{n+1}} u(x,t)\, e^{-i(\xi\cdot x + st)} dxdt.$$

Now transform both sides of $u_{tt} - \triangle u = \delta_0(x)\,\delta_0(t)$, we have

$$-s^2\hat{u} + |\xi|^2\,\hat{u} = \frac{1}{(2\pi)^{(n+1)/2}}, \text{ then } \hat{u} = \frac{c_n}{|\xi|^2 - s^2}.$$

How to convert?

Space only

$$\hat{u}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \int_{R^{n+1}} u(x,t)\, e^{-i\xi\cdot x} dx.$$

Space transform both sides of $u_{tt} - \triangle u = \delta_0(x)\,\delta_0(t)$, we have

$$\hat{u}(\xi, t)_{tt} + |\xi|^2\,\hat{u}(\xi, t) = \frac{1}{(2\pi)^{n/2}} \delta_0(t).$$

Let try to use variation of coefficient method to solve this equation. The general solution to the homogeneous equation is $e^{\pm i|\xi|t}$, let us just go with $e^{i|\xi|t}$ in the variation $a(t)\,e^{i|\xi|t}$. We solve

$$\left[a(t)\,e^{i|\xi|t}\right]_{tt} + |\xi|^2\, a(t)\, e^{i|\xi|t} = c_n\delta_0(t)$$

then

$$\left[a_{tt} + 2a_t i\,|\xi|\right] e^{i|\xi|t} = c_n\delta_0(t) \quad \text{or} \quad a_{tt} + 2a_t i\,|\xi| = c_n\delta_0(t)\, e^{-i|\xi|t} = c_n\delta_0(t)$$

and

$$\left[e^{i2|\xi|t} a_t\right]_t = c_n\delta_0(t)\, e^{i2|\xi|t} = c_n\delta_0(t).$$

So

$$e^{i2|\xi|t}a_t = \begin{cases} c_n & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases} \quad \text{or say} \begin{cases} 0.5c_n & \text{for } t>0 \\ -0.5c_n & \text{for } t<0 \end{cases}$$

and

$$a(t) = \begin{cases} c_n \frac{e^{-i2|\xi|t}}{-i2|\xi|} & \text{for } t>0 \\ c_n \frac{1}{-i2|\xi|} & \text{for } t<0 \end{cases} \quad \text{or} \quad \begin{cases} 0.5c_n \frac{e^{-i2|\xi|t}}{-i2|\xi|} & \text{for } t>0 \\ -0.5c_n \frac{e^{-i2|\xi|t}}{-i2|\xi|} + c_n \frac{1}{-i2|\xi|} & \text{for } t<0 \end{cases}.$$

Finally

$$\hat{u}(\xi,t) = a(t)e^{i|\xi|t} = \begin{cases} c_n \frac{e^{-i|\xi|t}}{-i2|\xi|} & \text{for } t>0 \\ c_n \frac{e^{i|\xi|t}}{-i2|\xi|} & \text{for } t<0 \end{cases} \quad \text{or} \quad \begin{cases} 0.5c_n \frac{e^{-i|\xi|t}}{-i2|\xi|} & \text{for } t>0 \\ -0.5c_n \frac{e^{-i|\xi|t}}{-i2|\xi|} + c_n \frac{e^{i|\xi|t}}{-i2|\xi|} & \text{for } t<0 \end{cases}.$$

How to convert? ...

Let try to solve the Cauchy problem for wave equation in the whole space time, by directly space Fourier transform the initial position and initial velocity in

$$\begin{cases} u_{tt} - \triangle u = 0 & \text{in } R^n \times R^1 \\ u(x,0) = g(x) & \text{in } R^n \\ u_t(x,0) = h(x) & \text{in } R^n \end{cases}.$$

We have

$$\hat{u}(\xi,t)_{tt} + |\xi|^2 \hat{u}(\xi,t) = 0$$
$$\hat{u}(\xi,0) = \hat{g}(\xi)$$
$$\hat{u}_t(x,0) = \hat{h}(\xi)$$

and

$$\hat{u}(\xi,t) = \hat{g}(\xi)\cos|\xi|t + \hat{h}(\xi)\frac{\sin|\xi|t}{|\xi|}.$$

Thus

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{R^n}\left[\hat{g}(\xi)\cos|\xi|t + \hat{h}(\xi)\frac{\sin|\xi|t}{|\xi|}\right]e^{-x\cdot\xi}d\xi.$$

Hard to convert.

RMK. If we repeat the same procedure to the Cauchy problem for heat equation

$$\begin{cases} u_t - \triangle u = 0 & \text{in } R^n \times [0,T) \\ u(x,0) = g(x) & \text{in } R^n \end{cases}.$$

We quickly get an explicit representation the solution. From

$$\hat{u}(\xi,t)_t + |\xi|^2 \hat{u}(\xi,t) = 0$$
$$\hat{u}(\xi,0) = \hat{g}(\xi)$$

we have

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-|\xi|^2 t}$$

and

$$
\begin{aligned}
u(x,t) &= \frac{1}{(2\pi)^n}\int_{R^n}\hat{g}(\xi)\,e^{-|\xi|^2 t}e^{ix\cdot\xi}d\xi = \frac{1}{(2\pi)^n}\int_{R^n}\int_{R^n} g(y)\,e^{-i\xi\cdot y}dy\ e^{-|\xi|^2 t}e^{ix\cdot\xi}d\xi \\
&= \frac{1}{(2\pi)^n}\int_{R^n} g(y)\int_{R^n} e^{-i\xi\cdot y+ix\cdot\xi-|\xi|^2 t}\ d\xi\ \ dy \\
&= \frac{1}{(2\pi)^n}\int_{R^n} g(y)\,\frac{\pi^{n/2}}{t^{n/2}}e^{-\frac{|x-y|^2}{4t}}\ dy = \frac{1}{(4\pi t)^{n/2}}\int_{R^n} g(y)\,e^{-\frac{|x-y|^2}{4t}}\ dy!
\end{aligned}
$$

<u>Cauchy problem</u> in 1d

$$
\begin{cases}
u_{tt}-u_{xx}=0 \ \text{ in } \ R\times(-\infty,\infty) \\
u(x,0)=g(x) \quad \text{on } R \\
u_t(x,0)=h(x) \quad \text{on } R
\end{cases}
$$

First when g and h are analytic, we can get (an) analytic local solution in terms of x and t, say near $(0,0)$, by finding all space and time derivatives at $(0,0)$, then justifying the convergence (general version is Cauchy-Kowalevskaya). This works for all dimensions $R^n\times(-\infty,\infty)$.

When g and h are smooth, in fact $g\in C^2$ and $h\in C^1$, we can directly solve the 1-d wave equation, by factoring the operator $D_t^2-D_x^2=(D_t+D_x)(D_t-D_x)$ and integrating along the characteristic lines. We reduce the order of the equations as follows

$$u_{tt}-u_{xx}=(D_t+D_x)(D_t-D_x)\,u=0.$$

Let $v(x,t)=(D_t-D_x)\,u(x,t)$, we immediately solve (the transport equation) $v_t+v_x=(D_t+D_x)\,v=D_{(1,1)}v=0$. That is, along the characteristic curve $\gamma(s)=(x(s),t(s))$ with $\gamma_s=(1,1)$, $v(\gamma(s))$ is constant. If we come back from (x,t) to initial time 0, then (x,t) and $(x-t,0)$ support the same value for v, namely $v(x,t)=v(x-t,0)\overset{\text{define}}{=}a(x-t)$.

Next we solve for $(D_t-D_x)\,u=a(x-t)$. This time the characteristic curve is $\gamma(s)=(x(s),t(s))$ with $\gamma_s=(-1,1)$. (Solutions to the homogenous equation $(D_t-D_x)\,u=0$ are $b(x+t)$.) We need to integrate a along $\gamma(s)$ to get solutions to the nonhomogeneous equation. We want to start from $t=0$, and go to (x,t). Then

$$\gamma(s)=(x+t,0)+s(-1,1)\quad s\in(0,t).$$

$$
\begin{aligned}
u(x,t) &= u(x+t,0)+\int_0^t \frac{d}{ds}u(\gamma(s))\,ds = g(x+t)+\int_0^t\ (D_t-D_x)\,u(\gamma(s))\,ds \\
&= g(x+t)+\int_0^t\ a(x+t-s-s)\,ds \\
&\overset{x+t-2s=\tau}{=}\ g(x+t)+\frac{1}{2}\int_{x-t}^{x+t}\ a(\tau)\,d\tau.
\end{aligned}
$$

Finally we go back to the initial velocity to figure out a. We have

$$h(x) = u_t(x,0) = g'(x+0) + a(x).$$

Then $a(\tau) = h(\tau) - g'(\tau)$. In turn

$$\begin{aligned} u(x,t) &= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(\tau) - g'(\tau)\ d\tau \\ &= \frac{1}{2}\left[g(x+t) + g(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} h(\tau)\ d\tau. \end{aligned}$$

figure: the cone with vertex (x,t)
$|y-x| \le |\tau - t|$

Certainly one verifies the above representation satisfies the Cauchy problem for C^2 g and C^1 h. For nonhomogeneous equation $u_{tt} - u_{xx} = f(x,t)$, by Duhamel principle (for derivation, see the end of 3d Cauchy problem), we only need to add

$$\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)\, dyds.$$

We will handle uniqueness (energy method, no maximal principle anymore) later on.

Cauchy problem in 3d

$$\begin{cases} u_{tt} - \triangle u = f(x,t) & \text{in} \ \ R^3 \times (-\infty,\infty) \\ u(x,0) = g(x) & \text{on } R^3 \\ u_t(x,0) = h(x) & \text{on } R^3 \end{cases}$$

The solution is

$$\begin{aligned} u(x,t) = {} & \frac{1}{4\pi}\int_{S^2} g(x+t\omega) + t\omega \cdot Dg(x+t\omega)\, d\omega + \frac{t}{4\pi}\int_{S^2} h(x+t\omega)\, d\omega \\ & + \frac{1}{4\pi}\int_{B_t(0)} \frac{f(x+y, t-|y-x|)}{|y|} dy. \end{aligned}$$

We have seen the "fundamental" solutions for the wave equations:

$$\left(|x|^2 - t^2\right)^{(1-n)/2} \ \text{for } n \ge 2,\ \ln\left(|x|^2 - t^2\right) \ \text{ for } \ n = 1.$$

But as the singularity is along the whole cone $|x| = |t|$, it is tough (still possible) to use the fundamental theorem of calculus–Green's formula–to work out the representation formula, as for Laplace and heat equation. Instead we go with usual spherical mean way.

We first work out case $f = g = 0$, then $f = 0$, finally general nonhomogeneous case. We only deal with $t > 0$ case. The negative $t < 0$ case can be done with change of time $\tau = -t$ and $v(x,\tau) = u(x,-\tau)$ with $v_{\tau\tau} - \triangle u = f(x,-\tau)$.

Case $f = g = 0$

Let $\rho = |x - y|$, recall from general 3-d radial solutions, $\rho^{-1} F(\rho - t)$ is a 3-d wave function in terms t and x for all functions F. Adding those with weight $h(y)$, we get another wave function

$$\int_{R^3} h(y) \frac{F(|x-y| - t)}{|x-y|} dy.$$

Now take F as a bump function such that $F(s) \in C_0^\infty(R^1)$ and $\int_{R^1} F(s)\, ds = 1$. Then the family of wave functions (the integration is over an ε-thin shell around sphere $\partial B_t(x)$, use spherical coordinates around x)

$$\int_{R^3} h(y) \frac{\frac{1}{\varepsilon} F\left(\frac{|x-y|-t}{\varepsilon}\right)}{|x-y|} dy \overset{\varepsilon \to 0^+}{\to} \int_{|x-y|=t} h(y) \frac{1}{t} dA(y)$$
$$\overset{y=x+t\omega}{=} t \int_{S^2} h(x + t\omega)\, d\omega.$$

We quickly check

$$u(x,t) = \frac{t}{4\pi} \int_{S^2} h(x+t\omega)\, d\omega = \frac{1}{4\pi t} \int_{\partial B_t(0)} h(x+y)\, dA(y)$$

satisfies the above wave equation (for all $t \in R$) with $u(x,0) = 0$ and $u_t(x,0) = h(x)$. As for the equation, the above procedure shows the validation. Another direct way is as follows. The trick is to use divergence theorem in rewriting $u_t(x,t)$, and not to create full Hessian terms $\langle D^2 h(x+t\omega)\omega, \omega\rangle$ and any higher order derivatives.

$$\begin{aligned} u_t(x,t) &= \frac{1}{4\pi} \int_{S^2} h(x+t\omega)\, d\omega + \frac{t}{4\pi} \int_{S^2} \omega \cdot Dh(x+t\omega)\, d\omega \\ &= \frac{1}{4\pi} \int_{S^2} h(x+t\omega)\, d\omega + \frac{1}{4\pi t} \int_{B_t(0)} \triangle h(x+y)\, dy \end{aligned}$$

$$\begin{aligned} u_{tt}(x,t) &= \frac{1}{4\pi} \int_{S^2} \omega \cdot Dh(x+t\omega)\, d\omega \\ &\quad - \frac{1}{4\pi t^2} \int_{B_t(0)} \triangle h(x+y)\, dy + \frac{1}{4\pi t} \int_{\partial B_t(0)} \triangle h(x+y)\, dA(y) \\ &= \frac{1}{4\pi t} \int_{\partial B_t(0)} \triangle h(x+y)\, dA(y) = \triangle \left[\frac{t}{4\pi} \int_{S^2} h(x+t\omega)\, d\omega\right] = \triangle u(x,t). \end{aligned}$$

Case $f = h = 0$

Note that the t-derivative of the above solution $u_t(x,t)$ satisfies

$$\begin{gathered} u_{ttt}(x,t) - \triangle u_t(x,t) = 0 \\ u_t(x,0) = h(x) \\ u_{tt}(x,0) = \lim_{t \to 0} \frac{1}{4\pi t} \int_{\partial B_t(0)} \triangle h(x+y)\, dA(y) = 0 \quad \text{or} \quad \triangle\, u(x,0) = 0 \end{gathered}$$

Replacing h by g, we see that

$$\frac{1}{4\pi}\int_{S^2} g(x+t\omega)+t\omega\cdot Dg(x+t\omega)\,d\omega$$

is a wave function with initial position $g(x)$ and velocity 0. Thus the solution to

$$\begin{cases} u_{tt}-\triangle u=0 \quad \text{in} \quad R^3\times(-\infty,\infty) \\ u(x,0)=g(x) \quad \text{on } R^3 \\ u_t(x,0)=h(x) \quad \text{on } R^3 \end{cases}$$

is

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\int_{S^2} g(x+t\omega)+t\omega\cdot Dg(x+t\omega)\,d\omega+\frac{t}{4\pi}\int_{S^2} h(x+t\omega)\,d\omega \\ &= \frac{1}{4\pi t^2}\int_{\partial B_t(0)} g(x+y)+y\cdot Dg(x+y)+th(x+y)\;\; dA(y). \end{aligned}$$

RMK. If g and h have compact support, say in $B_{100}(0)\subset R^3$, then $u(x,t)$ decays in the order of $\frac{1}{t}$ uniformly for large time t. This is because all the integrals happen in finite region $B_{100}(0)$:

$$|u(x,t)|\le\frac{C(100)}{t}\left[\|g\|_{C^1(B_{100}(0))}+\|h\|_{L^\infty(B_{100}(0))}\right].$$

Case full nonhomogeneous.

By Duhamel's principle, we just add/superpose

$$\begin{aligned} &\int_0^t\frac{t-s}{4\pi}\int_{S^2} f(x+(t-s)\omega,s)\,d\omega ds \\ &=\int_0^t\frac{s}{4\pi}\int_{S^2} f(x+s\omega,t-s)\,d\omega ds\overset{s\omega=y}{=}\frac{1}{4\pi}\int_{B_t(0)}\frac{f(x+y,t-|y-x|)}{|y|}dy \end{aligned}$$

RMK. Duhamel principle: solution to $\begin{cases} u_{tt}-\triangle u=f(x,t) \\ u(x,0)=u_t(x,0)=0 \end{cases}$ can be integrated from the solution

$$\frac{t-s}{4\pi}\int_{S^2} f(x+(t-s)\omega,s)\,d\omega$$

to

$$\begin{cases} v(x,t;s)_{tt}-\triangle v(x,t;s)=0 & \text{for } t>s \\ v(x,s;s)=0 & t=s \\ v_t(x,s;s)=f(x,s) & t=s \end{cases}.$$

Let us verify $u(x,t)=\int_0^t v(x,t;s)\,ds$ satisfies the above non-homogeneous equation. The initial conditions are obvious

$$\begin{aligned} &u(x,0)=0 \\ &u_t(x,t)=\underset{0}{v(x,t;t)}+\int_0^t v_t(x,t;s)\,ds\overset{t=0}{=}0 \end{aligned}$$

As for the equation

$$
\begin{aligned}
u_{tt}(x,t) &= v_t(x,t;t) + \int_0^t v_{tt}(x,t;s)\,ds = f(x,t) + \int_0^t \triangle v(x,t;s)\,ds \\
&= f(x,t) + \triangle u(x,t).
\end{aligned}
$$

Duhamel heat equation version: solution to $\begin{cases} u_{tt} - \triangle u = f(x,t) \\ u(x,0) = 0 \end{cases}$ can be integrated from the solution to

$$
\begin{cases} v(x,t;s)_t - \triangle v(x,t;s) = 0 & \text{for } t > s \\ v(x,s;s) = f(x,s) & t = s \end{cases}.
$$

Now $u(x,t) = \int_0^t v(x,t;s)\,ds$, then $u(x,0) = 0$ and

$$
\begin{aligned}
u_t(x,t) &= v(x,t;t) + \int_0^t v_t(x,t;s)\,ds = f(x,t) + \int_0^t \triangle v(x,t;s)\,ds \\
&= f(x,t) + \triangle u(x,t).
\end{aligned}
$$

Duhamel cubic version, ode version... similar. For example solution to

$$
\begin{cases} u_{ttt} - \triangle u = f(x,t) \\ u(x,0) = u_t(x,0) = u_{tt}(x,0) = 0 \end{cases}
$$

can be integrated from the solution to

$$
\begin{cases} v(x,t;s)_{ttt} - \triangle v(x,t;s) = 0 \quad \text{for } t > s \\ v(x,s;s) = v_t(x,s;s) = 0, \quad v_{tt}(x,s;s) = f(x,s) & t = s \end{cases}.
$$

That is $u(x,t) = \int_0^t v(x,t;s)\,ds$.

Cauchy problem in 2d

$$
\begin{cases} u_{tt} - \triangle u = f(x,t) \quad \text{in } R^2 \times (-\infty,\infty) \\ u(x,0) = g(x) \quad \text{on } R^2 \\ u_t(x,0) = h(x) \quad \text{on } R^2 \end{cases}
$$

Again we only handle $t > 0$ case. The negative t case can be done by "symmetry". As the radial solution in $R^2 \times R^1$ is hard to find, a direct way might (should) be hard. But any solution to the 2-d wave equation is automatically a solution to 3-d (higher dimensional) wave equation with the initial data independent of the "fake" variables. Setting $x = (x', x_3) = (x_1, x_2, x_3)$ and $\omega = \left(y_1, y_2, \pm\sqrt{1 - y_1^2 - y_2^2}\right)$ with

$$
\begin{aligned}
d\omega &= \sqrt{1 + \left|D\sqrt{1 - |y'|^2}\right|^2}\, dy_1 dy_2 \\
&= \frac{1}{\sqrt{1 - |y'|^2}} dy_1 dy_2,
\end{aligned}
$$

the 2-d solution is

$$
\begin{aligned}
u(x_1,x_2,t) &= \frac{1}{4\pi}\int_{S^2} g(x+t\omega)+t\omega\cdot Dg(x+t\omega)\,d\omega+\frac{t}{4\pi}\int_{S^2} h(x+t\omega)\,d\omega \\
&+\int_0^t \frac{s}{4\pi}\int_{S^2} f(x+s\omega,t-s)\,d\omega ds. \\
&= \frac{1}{2\pi}\int_{B_1(0)} g(x'+ty)+ty\cdot Dg(x'+ty)+t\ h(x'+ty)\ \frac{1}{\sqrt{1-|y'|^2}}dy_1dy_2 \\
&+\frac{1}{2\pi}\int_0^t s\int_{B_1(0)} f(x'+sy,t-s)\ \frac{1}{\sqrt{1-|y'|^2}}dy_1dy_2ds.
\end{aligned}
$$

The reason $1/4\pi$ becomes $1/2\pi$ is because the sphere S^2 has upper and lower parts seeing from the 2-d disk $B_1(0)$.

RMK. Descend further down to 1-d, or if we start with 1-d Cauchy problem of the wave equation, integrate out

$$
\int_{[-1,1]}\int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} \frac{1}{\sqrt{1-y_1^2-y_2^2}}dy_2dy_1=\int_{[-1,1]}\pi dy_1,
$$

then the solution is

$$
\begin{aligned}
u(x,t)=u(x_1,t) &= \frac{1}{2}\int_{[-1,1]} g(x_1+ty_1)+ty_1\cdot D_1g(x_1+ty_1)+t\ h(x_1+ty_1)\,dy_1 \\
&+\frac{1}{2}\int_0^t s\int_{[-1,1]} f(x_1+sy_1,t-s)\,dy_1ds \\
&= \frac{1}{2}\left[g(x+t)+g(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy \\
&+\frac{1}{2}\int_0^t s\int_{[-1,1]} f(x+sy,t-s)\,dyds \\
&= \frac{1}{2}\left[g(x+t)+g(x-t)\right]+\frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy+\frac{1}{2}\int_0^t\int_{x-t+s}^{x+t-s} f(y,s)\,dyds\ !
\end{aligned}
$$

RMK. Huygens' principle. For the homogeneous $f=0$ case, comparing the "sphere" representaion of wave with the "ball" representations of wave in 1-d and 2-d, we see the 3d wave/sound (and all higher odd dimensions) effect at (x,t) only depends on the disturbance on the **sphere** centered at x with radius t, while 1-d string wave and 2-d surface wave (and all even dimensions) both depend on the whole **ball** centered at x with radius t. For example, effect of the disturbance from a stone thrown in a lake is forever on the water surface.

Apprarently Shakespeare first studied this "forever-to-nought" effect, as noted in F. John's PDE book:

> Glory is like a circle in the water,
> which never ceaseth to enlarge itself,
> Till by broad spreading it diperse to nought.

Cauchy problem in higher dimensional case can be handled by spherical means. The spherical mean of the wave function (harmonic, caloric, ...) over spheres with arbitrary radius ρ centered at x ($|x| \neq \rho$) satisfies the wave equation in terms ρ and t. The odd radial wave equation can be reduced to 1-d wave equation, as we saw for 3d case, then solvable. Note that the spherical mean takes exactly the (wave) function value at (x,t), when the radius $\rho = 0$, an explicit representation of the solution is then reached. Descending to the even dimensional from, say the nearest odd dimensional space, one completes the general case.

For any function $f(x)$, set

$$M(x,\rho) = \frac{1}{|\partial B_\rho|} \int_{\partial B_\rho(x)} f(y)\, dA(y) = \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} f(x+\rho y)\, dA(y)$$

Claim:

$$\triangle_\rho M = \left(\partial_\rho^2 + \frac{n-1}{\rho} \partial_\rho \right) M = \triangle_x M.$$

In fact

$$\begin{aligned} M_\rho &= \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} y \cdot Df(x+\rho y)\, dA(y) = \frac{1}{|\partial B_1|} \int_{B_1(0)} \operatorname{div}\left[Df(x+\rho y) \right] dy \\ &= \frac{1}{|\partial B_1|} \int_{B_1(0)} \rho \triangle f(x+\rho y)\, dy \\ &= \frac{\rho^{1-n}}{|\partial B_1|} \int_{B_\rho(0)} \triangle f(x+y)\, dy \qquad \text{(to avoid further derivative of } f \text{ in } M_{\rho\rho}\text{)} \end{aligned}$$

and

$$M_{\rho\rho} = -\frac{(n-1)\rho^{-n}}{|\partial B_1|} \int_{B_\rho(0)} \triangle f(x+y)\, dy + \frac{\rho^{1-n}}{|\partial B_1|} \int_{\partial B_\rho(0)} \triangle f(x+y)\, dA(y).$$

Thus

$$\left(\partial_\rho^2 + \frac{n-1}{\rho} \partial_\rho \right) M(x,\rho) = \triangle_x M(x,\rho).$$

Next, when $f(x)$ also depends on time t, we average $u(x,t)$ for each fixed t

$$M(x,\rho,t) = \frac{1}{|\partial B_\rho|} \int_{\partial B_\rho(x)} u(y,t)\, dA(y) = \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} u(x+\rho y, t)\, dA(y).$$

And

$$\begin{aligned} \left(\partial_\rho^2 + \frac{n-1}{\rho} \partial_\rho \right) M(x,\rho,t) &= \triangle_x M(x,\rho,t) = \frac{1}{|\partial B_1|} \int_{\partial B_1(0)} \triangle_x u(x+\rho y, t)\, dA(y) \\ &= \partial_t^2 M(x,\rho,t), \end{aligned}$$

provided $\triangle u = u_{tt}$.

One nontrivial fact needed is, as we remarked in the beginning,

$$\partial_\rho^2\left[\left(\rho^{-1}\partial_\rho\right)^{k-1}\left(\rho^{2k-1}w\left(\rho\right)\right)\right]=\left(\rho^{-1}\partial_\rho\right)^{k-1}\left[\rho^{2k-1}\left(\partial_\rho^2+\frac{n-1}{\rho}\partial_\rho\right)w\left(\rho\right)\right],$$

when $n=2k+1$

$$\left(\rho^{-1}\partial_\rho\right)^{k-1}\left[\rho^{2k-1}\frac{1}{\left|\partial B_\rho\right|}\int_{\partial B_\rho(x)}u\left(y,t\right)dA\left(y\right)\right]$$

satisfies the 1-d wave equation in terms of ρ and t, $w_{tt}-w_{\rho\rho}=0$.

RMK. If u satisfies heat equation $\triangle u=u_t$, or harmonic equation $\triangle u=0$, etc, one has $\triangle_\rho M=\partial_t M$, or certainly $\triangle_\rho M=0$, etc

Energy method

Physically, the sum of potential and kinetic energy of the wave function

$$\int_{R^n}\left|Du\right|^2+\left|D_t u\right|^2dx$$

should be preserved, as suggested by one 1-d string oscillation such as $\sin\left(x+t\right)$. Analytically, multiply the wave equation by u_t (others u, u_x?) and integrate by parts (divergence theorem) in space (assuming compact support of the wave), we confirm

$$\begin{aligned}0&=\int_{R^n}u_t\left(u_{tt}-\triangle u\right)dx=\int_{R^n}\frac{1}{2}D_t\left[\left(u_t\right)^2+\left|Du\right|^2\right]-\operatorname{div}\left(u_tDu\right)dx\\&=\frac{1}{2}D_t\int_{R^n}\left[\left(u_t\right)^2+\left|Du\right|^2\right]dx.\end{aligned}$$

From the spherically mean representation of the solutions to the Cauchy problem, we see $u\left(x_0,t_0\right)$ only depends on the wave in the backward cone

$$C=\left\{\left(x,t\right):0\le t\le t_0,\quad\left|x-x_0\right|\le t_0-t\right\}.$$

This suggests a cone dependence or uniqueness, quite different from caloric functions.

Uniqueness-Cone dependence-Finite time propagation: For wave function u, $u_{tt}-\triangle u=0$, if $u=u_t=0$ in the ball centered at x_0 and with radius t_0 initially $B_{t_0}\left(x_0\right)$, then $u\equiv 0$ in the cone C.

Proof. For $t\in\left(0,t_0\right)$, we estimate the above energy changing rate inside the cone

$$\begin{aligned}&\frac{d}{dt}\int_{B_{t_0-t}(x_0)}\left|Du\right|^2+\left|D_tu\right|^2dx\\&=\int_{B_{t_0-t}(x_0)}2\operatorname{div}\left(u_tDu\right)dx-\int_{\partial B_{t_0-t}(x_0)}\left|Du\right|^2+\left|D_tu\right|^2dA\quad\text{(boundary is changing)}\\&=\int_{\partial B_{t_0-t}(x_0)}2u_tu_\gamma-\left(\left|Du\right|^2+\left|D_tu\right|^2\right)dA\\&\le\int_{\partial B_{t_0-t}(x_0)}\left(u_t\right)^2+\left(u_\gamma\right)^2-\left(\left|Du\right|^2+\left|D_tu\right|^2\right)dA\le 0.\end{aligned}$$

Thus the energy $\int_{B_{t_0-t}(x_0)}\left|Du\right|^2+\left|D_tu\right|^2dx=0$ for $t\in\left[0,t_0\right]$. And consequently $u\equiv 0$ in C.