Lecture 5 Minimal surface equation–Bernstein problem

- ∘ explicit examples
- ∘ Bernstein
- ∘ Jörgens

Minimal surface equation

Consider the variational problem for area functional $A[f]=\int\sqrt{1+|Df|^2}dx$

$$\inf_{f=\varphi \text{ on } \partial U} A[f],$$

any critical f satisfies for all $\eta\in C_0^\infty(U)$

$$0=\frac{d}{d\varepsilon}A[f+\varepsilon\eta]\Big|_{\varepsilon=0}=\int_U\frac{Df\cdot D\eta}{\sqrt{1+|Df|^2}}dx$$
$$=\int_U -div\left(\frac{Df}{\sqrt{1+|Df|^2}}\right)\eta dx.$$

So

$$\text{mean curvature } H\triangleq div\left(\frac{Df}{\sqrt{1+|Df|^2}}\right)=0.$$

Note also

$$H=\frac{\triangle f}{\sqrt{1+|Df|^2}}-\frac{\langle Df,D^2f\ Df\rangle}{\left(\sqrt{1+|Df|^2}\right)^3}$$
$$=\frac{1}{\left(\sqrt{1+|Df|^2}\right)^3}\left[\left(1+|Df|^2\right)\triangle f-\langle Df,D^2f\ Df\rangle\right]$$
$$\overset{2d}{=}\frac{1}{\left(\sqrt{1+|Df|^2}\right)^3}\left[\left(1+f_2^2\right)f_{11}-2f_1f_2f_{12}+\left(1+f_1^2\right)f_{22}\right]$$
$$=\frac{1}{\left(\sqrt{1+|Df|^2}\right)^3}Lf,$$

where $L=(1+f_2^2)\,\partial_{11}-2f_1f_2\partial_{12}+(1+f_1^2)\,\partial_{22}$.

Explicit examples of minimal surfaces

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RMK. Solutions, in particular explicit ones are hard to come by for nonlinear equations.)

catenoid: $|(x, y)| = \cosh z$
helicoid: $z = \arctan \frac{y}{x}$
Sherk's surface: $z = \ln \frac{\cos y}{\cos x}$

figure minimal surfaces

Exercise: invariance for minimal surface equation?

Bernstein. Let smooth f satisfies $div \left(\frac{Df}{\sqrt{1+|Df|^2}} \right) = 0$ in R^2. Then f is linear.
Bernstein's proof 1910s–40s
Strange obs. $L \arctan f_1 = 0$! Only in 2d.
Stunning Theorem. Bounded global saddle surface is flat, really horizontal.
That is $\arctan f_1 = const$. Similarly $\arctan f_2 = const$. Thus f is linear.

Lewy 1930s
In studying the Monge-Ampere equations $\det D^2 u = 1$, really Darboux equation

$$\det_g \nabla^2 u = K_g \left(1 - |\nabla_g u|^2\right)$$

for the isometric embedding problem, introduced the/his transformation $\eta(\xi)$ =Lewy$[u(x)]$ with

$$\begin{cases} \xi_1 = 1 + u_1 \\ \xi_2 = 1 + u_2 \end{cases},$$

and $u - \eta$ satisfying the contact transformation

$$0 = u + \frac{1}{2}\left(x_1^2 + x_2^2\right) + \eta - x_1 \xi_1 - x_2 \xi_2.$$

In fact, Lewy rotation is just the usual Legendre transformation of function $u + \frac{1}{2}\left(x_1^2 + x_2^2\right)$. Lewy was trying to get a priori estimates (in order to solve the equations).

Jörgens 1954

$$\det D^2 u = 1 \quad \text{in } R^2 \;\Rightarrow\; u \text{ is quadratic.}$$

Jörgens used Heinz's "hard" estimates on the 3rd order derivatives.
Exercise: Verify $v = x_1^3 + \frac{x_2^2}{12 x_1}$ satisfies the 2d M-A equation.
Contrasting example in hyperbolic case.

$$v = x_1 x_2 + \arctan x_1$$
$$D^2 v = \begin{bmatrix} \frac{1}{1+x_1^2} & 1 \\ 1 & 0 \end{bmatrix}$$
$$\det D^2 v = -1.$$

Heinz 1952 observed, there exists a scalar function u such that

$$\frac{1}{\sqrt{1+|Df|^2}}\left[I+(Df)(Df)^T\right]=\frac{1}{\sqrt{1+|Df|^2}}\begin{bmatrix}1+f_1^2 & f_1f_2\\ f_1f_2 & 1+f_2^2\end{bmatrix}=D^2u$$

and

$$\det D^2u=1.$$

The second equation is easy, just note

$$\frac{1}{\sqrt{1+|Df|^2}}g\sim\frac{1}{\sqrt{1+|Df|^2}}\begin{bmatrix}1+|Df|^2 & \\ & 1\end{bmatrix}.$$

The first super potential part is a little hard.

Geometrically, we know $\triangle_g(x_1,x_2,f)=\vec{H}=0$, in fact

$$\triangle_g f=0\Leftrightarrow\triangle_g x_1=0\Leftrightarrow\triangle_g x_2=0,$$

where

$$\triangle_g=\frac{1}{\sqrt{g}}div\left(\sqrt{g}g^{-1}D\right)=\frac{1}{\sqrt{1+|Df|^2}}div\left(\frac{1}{\sqrt{1+|Df|^2}}\begin{bmatrix}1+f_2^2 & -f_1f_2\\ -f_1f_2 & 1+f_1^2\end{bmatrix}D\right)$$

figure mean curvature

Then $\triangle_g x_1=0$ implies

$$\partial_1\left(\frac{1+f_2^2}{\sqrt{1+|Df|^2}}\right)-\partial_2\left(\frac{f_1f_2}{\sqrt{1+|Df|^2}}\right)\overset{*}{=}0$$

and $\triangle_g x_2=0$ implies

$$-\partial_1\left(\frac{f_1f_2}{\sqrt{1+|Df|^2}}\right)+\partial_2\left(\frac{1+f_1^2}{\sqrt{1+|Df|^2}}\right)\overset{*}{=}0.$$

We'll also verify these two identifies directly. It follows that

$$\begin{bmatrix}\frac{1+f_2^2}{\sqrt{1+|Df|^2}} & \frac{-f_1f_2}{\sqrt{1+|Df|^2}}\\ \frac{-f_1f_2}{\sqrt{1+|Df|^2}} & \frac{1+f_1^2}{\sqrt{1+|Df|^2}}\end{bmatrix}=\begin{bmatrix}DF\\ DG\end{bmatrix}.$$

As $F_2=G_1$, there exists u such that $(F,G)=Du$. Thus the existence of the double potential u.

Direct verification of $*$.

Let $V = \sqrt{1+|Df|^2}$,

$$
\begin{aligned}
LHS &= \partial_1\left(\frac{V^2 - f_1^2}{V}\right) - \partial_2\left(f_1 \frac{f_2}{V}\right) \\
&= \partial_1\left[V - f_1\left(\frac{f_1}{V}\right)\right] - \partial_2\left[f_1\left(\frac{f_2}{V}\right)\right] \\
&= \partial_1 V - f_{11}\left(\frac{f_1}{V}\right) - \underbrace{f_1 \partial_1\left(\frac{f_1}{V}\right)} \\
&\quad - f_{12}\left(\frac{f_2}{V}\right) - f_1 \underbrace{\partial_2\left(\frac{f_2}{V}\right)} \\
&= \frac{f_1 f_{11} + f_2 f_{21}}{V} - \frac{f_1 f_{11} + f_2 f_{21}}{V} - f_1 div\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) \\
&= 0.
\end{aligned}
$$

In summary:
Heinz

$$
div\left(\frac{Df}{\sqrt{1+|Df|^2}}\right) = 0 \text{ in } R^2 \Rightarrow \text{there exists } u \text{ such that in } R^2
$$

$$
\det D^2 u = 1 \text{ and } \frac{1}{\sqrt{1+|Df|^2}}\begin{bmatrix} 1+f_1^2 & f_1 f_2 \\ f_1 f_2 & 1+f_2^2 \end{bmatrix} = D^2 u.
$$

Jörgens result implies Bernstein theorem in 2d.

RMK. A by product divergence $\triangle_g = \frac{1}{\sqrt{g}} div\left(\sqrt{g} g^{-1} D\right) = \sum g_{ij}\partial_{ij}$ nondivergence on minimal graphs.

Nitsche's proof of Jorgens' Theorem via Lewy rotation (1956).

Step1. Let $\tilde{u}(y)$ be the Legendre-Lewy rotation of u, namely Legendre transform of $v(x) = u(x) + \frac{1}{2}|x|^2$. The following distance increasing argument shows the map from x to $Du(x) + \frac{1}{2}x$ is 1-1 and onto.

Step2. By the property of Legendre transform

$$
D^2 \tilde{v}(y) = \left[D^2 u(x) + I\right]^{-1} \sim \begin{bmatrix} \frac{1}{1+\lambda_1} & \\ & \frac{1}{1+\lambda_2} \end{bmatrix}.
$$

Further we see

$$
0 < D^2 \tilde{v}(y) < I
$$

and

$$
\triangle \tilde{v}(y) = \tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{2+\lambda_1+\lambda_2}{1+\lambda_1+\lambda_2+\lambda_1\lambda_2} = \frac{2+\lambda_1+\lambda_2}{2+\lambda_1+\lambda_2} = 1.
$$

Step3. Liouville shows $D^2\tilde{v}$ is constant, in turn D^2u is constant.

Now new interpretation of Nitsche's proof. (2001)
Geometric way.
Step 1. Set-up

$$\underset{\text{assume } \lambda_i>0}{\lambda_1\lambda_2 = 1} \Leftrightarrow \arctan\lambda_1 + \arctan\lambda_2 = \frac{\pi}{2} \text{ or } \theta_1 + \theta_2 = \frac{\pi}{2}.$$

figure $\pi/4$ rotation

Step 2. $0 < \theta_i < \frac{\pi}{2}$ graph over x-R^2 plane
Make a $U(2)$ rotation

$$\begin{bmatrix} e^{-\sqrt{-2}\pi/4} & \\ & e^{-\sqrt{-2}\pi/4} \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \begin{cases} z_1 = x_1 + \sqrt{-1}y_1 \\ z_2 = x_2 + \sqrt{-1}y_2 \end{cases}$$

Obs. $U(2)$ rotation preserves the Lagrangian structure i.e. J Tangent space= Normal Space or $iT = N$. This is because $iUT = UiT = UN$. Locally Lagrangian means the graph has a "gradient" structure.

Obs. This $U(2)$ rotation decreases the angles

$$-\frac{\pi}{4} < \bar{\theta}_i = \theta_i - \frac{\pi}{4} < \frac{\pi}{4} \longleftrightarrow -1 < \tan\bar{\theta}_i = \bar{\lambda}_i < 1$$

Then $(x, Du(x))$ still a graph over $\bar{x}$-R^2 plane. In fact a Lagrangian graph $(\bar{x}, D\bar{u}(\bar{x}))$ with bounded Hessian $D^2\bar{u}$.

Step 3. $\bar{\theta}_1 + \bar{\theta}_2 = 0 \Leftrightarrow \triangle\bar{u} = 0$. Also $-I < D^2\bar{u} < I$. By Liouville, $\bar{u}$ is quadratic. Then $(\bar{x}, D\bar{u})$ is a plane, finally u is quadratic in terms of x.

RMK. In justifying the rotation $e^{-\sqrt{-1}\pi/4}$, we assumed D^2u is diagonal, this can be achieved by another $U(2)$ rotation induced from the $O(2)$ rotation on x-R^2 plane

$$Rx + \sqrt{-1}Ry \quad \text{or } [R]_{2\times2}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Analytic way.
Step 1. Set up $\lambda_1\lambda_2 = 1$, say $\lambda_i > 0$.

figure $\pi/4$ rotation

Step 2. Change of variables
Now $(x, Du(x))$ represented by $(\Phi(x), \Psi(x))$ in $\bar{x} - \bar{y}$ coordinate system

$$\bar{x} = \Phi(x) = \frac{1}{\sqrt{2}}(x + Du(x))$$
$$\bar{y} = \Psi(x) = \frac{1}{\sqrt{2}}(-x + Du(x))$$

Note $\frac{\partial\Phi}{\partial x}=\frac{1}{\sqrt{2}}\left(I+D^2u\right)\geq\frac{1}{\sqrt{2}}I$. Then Φ is $\frac{1}{\sqrt{2}}$ distance expanding and an open map. It follows that

* Φ is globally 1-1 and onto from x-R^2 to $\bar{x}-R^2$

* $(x, Du(x))$ is still a graph over $\bar{x}$-plane.

Instead of this infinitesimal argument, we argue without derivative.

$$\begin{aligned}\left|\bar{x}^P-\bar{x}^Q\right|^2&=\frac{1}{2}\left|x^P-x^Q+Du\left(x^P\right)-Du\left(x^Q\right)\right|^2\\&=\frac{1}{2}\left|x^P-x^Q\right|^2+\frac{1}{2}\left|Du\left(x^P\right)-Du\left(x^Q\right)\right|^2+\underbrace{\left\langle x^P-x^Q, Du\left(x^P\right)-Du\left(x^Q\right)\right\rangle}_{\geq 0,\text{ since } u \text{ is convex}}\\&\geq\frac{1}{2}\left|x^P-x^Q\right|^2+\frac{1}{2}\left|Du\left(x^P\right)-Du\left(x^Q\right)\right|^2=\frac{1}{2}\left|P-Q\right|^2.\end{aligned}$$

So different points P and Q have different projections on $\bar{x}$-plane. So $(x, Du(x))$ is still a graph over $\bar{x}$-plane.

Checking the Lagrangian structure

$$curl_{\bar{x}}\Psi=\frac{\partial\Psi^2}{\partial\bar{x}_1}-\frac{\partial\Psi^1}{\partial\bar{x}_2}=0\Leftrightarrow 0=\underbrace{d\bar{x}_1\wedge d\Psi^1+d\bar{x}_2\wedge d\Psi^2}_{\bar{x}\text{ parametrization}}=\underbrace{d\Phi^1\wedge d\Psi^1+d\Phi^2\wedge d\Psi^2}_{x\text{ parametrization}}.$$

Now

$$\begin{aligned}&d\left(x_1+u_1\right)\wedge d\left(-x_1+u_1\right)+d\left(x_2+u_2\right)\wedge d\left(-x_2+u_2\right)\\&=dx_1\wedge du_1-du_1\wedge dx_1+dx_2\wedge du_2-du_2\wedge dx_2\\&=2\left(dx_1\wedge du_1+dx_2\wedge du_2\right)\\&=2\left(u_{12}dx_1\wedge dx_2+u_{21}dx_2\wedge dx_1\right)=0.\end{aligned}$$

Calculating the new Hessian $D^2\bar{u}$, and another way of checking "gradient" structure.

$$\begin{aligned}\begin{bmatrix}\frac{\partial\Psi^1}{\partial\bar{x}_1}&\frac{\partial\Psi^2}{\partial\bar{x}_1}\\\frac{\partial\Psi^1}{\partial\bar{x}_2}&\frac{\partial\Psi^2}{\partial\bar{x}_2}\end{bmatrix}&=\frac{\partial\Psi}{\partial\bar{x}}=\frac{\partial\Psi}{\partial x}\frac{\partial x}{\partial\bar{x}}=\frac{\partial\Psi}{\partial x}\left(\frac{\partial\bar{x}}{\partial x}\right)^{-1}=\frac{\partial\Psi}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)^{-1}\\&=\underbrace{\left(-I+D^2u\right)\left(I+D^2u\right)^{-1}}_{\text{symmetric, thus }\frac{\partial\Psi^1}{\partial\bar{x}_2}=\frac{\partial\Psi^2}{\partial\bar{x}_1}}\sim\begin{bmatrix}\frac{-1+\lambda_1}{1+\lambda_1}&\\&\frac{-1+\lambda_2}{1+\lambda_2}\end{bmatrix}.\end{aligned}$$

figure graph $\dfrac{-1+\lambda}{1+\lambda}$

So

$$-I<\left(D^2\bar{u}\right)=\frac{\partial\Psi}{\partial\bar{x}}<I.$$

Step 3. Equation

$$\begin{aligned}\triangle\bar{u}=\bar{\lambda}_1+\bar{\lambda}_2&=\frac{-1+\lambda_1}{1+\lambda_1}+\frac{-1+\lambda_2}{1+\lambda_2}=\\&=\frac{2\lambda_1\lambda_2-2}{\left(1+\lambda_1\right)\left(1+\lambda_2\right)}=0!\end{aligned}$$

We have a harmonic function with bounded Hessian on R^2. Liouville theorem implies that $\bar{u}$ is quadratic, then so is u.