Lecture 10 De Giorgi–Nash

- Statement
- motivation
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- Liouville

Theorem 1 *Let u be a weak solution to*

$$\sum_{i,j=1}^{n} D_i\left(a_{ij}(x) D_j u\right) = 0 \quad \text{in } B_1 \subset \mathbb{R}^n$$

with

$$\begin{aligned} &\mu I \le (a_{ij}) \le \mu^{-1} I && \text{when } (a_{ij}) \overset{def}{=} A = A^T, \\ &\mu I \le (a_{ij}) \quad \text{and } |a_{ij}| < \mu^{-1} && \text{when } A \ne A^T. \end{aligned} \tag{*}$$

Namely for all $v \in H_0^1(B_1)$

$$\int a_{ij}(x) D_i v D_j u = 0.$$

Then u is Hölder continuos in $B_{1/2}$ and

$$\|u\|_{C^\alpha\left(B_{1/2}\right)} \le C(\mu, n) \|u\|_{L^2(B_1)} \quad \text{with } \alpha = \alpha(\mu, n) > 0.$$

RMK. The general equations

$$\sum_{i,j=1}^{n} D_i\left(a_{ij}(x) D_j u\right) + b_i D_i u + cu = 0$$

with $|b| \le \mu^{-1}$ and $|c| \le \mu^{-1}$ can be reduced to the above model case. To write the equation in full divergence form, let $\bar{c} = x_1 c$, then

$$\sum D_i\left(a_{ij}(x) D_j u\right) + \sum b_i D_i u - \bar{c} D_1 u + D_1(\bar{c} u) = 0.$$

Set

$$u^0\left(x^0, x\right) = x^0 u(x)$$

and

$$\left(a_{ij}^0\right) = \begin{bmatrix} a_{00} & \frac{x^0}{2}(b_1 - \bar{c}) & \frac{x^0}{2} b_2 & \cdots & \frac{x^0}{2} b_n \\ x^0 \bar{c} & & & & \\ 0 & & (a_{ij}) & & \\ \cdots & & & & \\ 0 & & & & \end{bmatrix}.$$

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By choosing a_{00} large for, say $1 \le x^0 \le 3$, the nonsymmetric $\left(a^0_{ij}\right)$ satisfies the uniform ellipticity condition (*).

Exercise: Verify $u^0\left(x^0, x\right)$ satisfies in both pointwise and integral senses the equation

$$\sum_{i,j=\underline{\mathbf{0}}}^{n} D_i\left(a^0_{ij}\left(x\right) D_j u\right) = 0.$$

Exercise: Verify the unbounded function $f\left(x\right) = \left|x\right|^{-\alpha}$ and $g\left(x\right) = \sum_{k=1}^{\infty} 2^{-k}\left|x - Q_k\right|^{-\alpha}$ with $0 < \alpha < \left(n-2\right)/2$ and Q_k rational points in a unit ball B_1 are in $W^{1,2}\left(B_1\right)$. What is an analogous example in dimension two?

Motivation/Application 1. Minimal surface

$$\inf \int F\left(Du\right)$$

Euler-Lagrangian

$$\sum_{i,j=1}^{n} D_i\left(F_{p_i}\left(Du\right)\right) = 0$$

or

$$\sum_{i,j=1}^{n} F_{p_i p_j}\left(Du\right) D_{ij} u = 0.$$

In order to apply Schauder, or Calderon-Zygmund, need $F_{p_i p_j}\left(Du\right) \in C^\alpha$ or C^0. Let $e \in \mathbb{R}^n$

$$D_e \sum_{i,j=1}^{n} D_i\left(F_{p_i}\left(Du\right)\right) = \sum_{i,j=1}^{n} D_i\left(F_{p_i p_j}\left(Du\right) D_j u_e\right) = 0.$$

Let us first assume

$$\mu I \le \left(F_{p_i p_j}\left(Du\right)\right) \le \mu^{-1} I.$$

De Giorgi–Nash then implies $u_e \in C^\alpha$. (A long blow-up argument came up later.)

M/A 2. Homogenization

$$\triangle_g u = \frac{1}{\sqrt{g}} D_i\left(\sqrt{g} g^{ij} D_j u\right) = 0$$

with the Riemannian metric g periodic.

periodic figure   salty water, potato soup

Look at u from far away, what happens? We have solution u to

$$\sum_{i,j=1}^{n} D_i\left(a_{ij}\left(x\right) D_j u\left(x\right)\right) = 0.$$

What happens to $u_\varepsilon(x)$ as ε goes to 0?

$$\sum_{i,j=1}^{n} D_i\left(a_{ij}\left(\frac{x}{\varepsilon}\right) D_j u_\varepsilon(x)\right) = 0$$

De Giorgi–Nash says $u_\varepsilon(x) \to u \in C^a$ in $C^{\alpha-\delta}$ norm.

Eg. 1-d

$$D_x\left(\frac{1}{2+\cos\frac{x}{\varepsilon}} D_x\left(2x + \varepsilon\sin\frac{x}{\varepsilon}\right)\right) = 0$$

$u_\varepsilon(x) = 2x + \varepsilon\sin\frac{x}{\varepsilon} \overset{\varepsilon\to 0}{\longrightarrow} 2x$, which satisfies

$$D_x\left(\frac{1}{2} D_x(2x)\right) = 0.$$

Equations for limit solutions are very interesting to know.

Proof.

Step 1. Cacioppoli, ($\|Du\|_2 \le \|u\|_2$): As for any convex function, its gradient is bounded by its oscillation, any subharmonic function's Dirichlet energy is bounded by its potential energy.

Step 2. L^∞ bound: Dyadically and simultaneously, going up to level 1 of the subsol and going down to the half ball from the unit ball, by Holder, Sobolev, and Caccioppoli, each L^2 integral (potential energy) in dyadic ball the above dyadic level is bounded by the product of the reciprocals of the gaps of subsol levels and dyadic balls, and multiplied by a **super linear** power of the previous L^2 integral. If the initial above zero L^2 integral in the unit but is small enough, then the dyadic L^2 integrals decays geometrically to zero. This means the subsol is less than 1 in the half ball.

Step 3. Oscillation: The product of upper and lower function level set measures is bounded above by the L^1 Dirichlet energy, then by the L^2 one multiplied by the in between level set measure, and in turn, by Caccioppoli applied to the subsol function, solely by the in between measure.

Dyadically going up to level 1, in finite steps, the upper level set measure would be less than a fixed small amount. Otherwise, those dyadic in between level set measures add up to impossible infinity in the unit ball. It follows that the integral of the subsol above that finite dyadic level is less than a fixed small number as in Step 2. Repeating the iteration in Step 2, the less than that finite dyadic level set has measure zero in the half ball, or the oscillation in the half ball shrinks.

Step 1. Let $v = \eta^2 u$, (technical, trial-error picking)

$$\int \underbrace{D\left(\eta^2 u\right)}_{\eta D(\eta u)+\eta u D\eta} ADu = 0.$$

Let us move one η to the other u, in this the non-commuative situation

$$0 = \int \left[D(\eta u) + uD\eta\right] A \left[D(\eta u) - uD\eta\right].$$

Then

$$\mu\int|D(\eta u)|^2\le\int D(\eta u)AD(\eta u)\overset{\#}{=}\int u^2D\eta AD\eta+uD(\eta u)AD\eta-uD\eta AD(\eta u)$$
$$=\int u^2D\eta AD\eta+uD(\eta u)\left(\frac{A-A^T}{2}\right)D\eta-uD\eta\left(\frac{A-A^T}{2}\right)D(\eta u)$$
$$\le\begin{cases}\mu^{-1}\int u^2|D\eta|^2 & \text{when } A \text{ is symmetric}\\ C(n,\mu)\int u^2|D\eta|^2+\underbrace{\bar{C}(n,\mu)\int u|D\eta||D(\eta u)|}_{\le\frac{[\bar{C}(n,\mu)]^2}{2\mu}\int u^2|D\eta|^2+\frac{\mu}{2}\int|D(\eta u)|^2} & \text{otherwise}\end{cases}.$$

Thus

$$\int|D(\eta u)|^2\le C(n,\mu)\int u^2|D\eta|^2.$$

RMK 1. By design, the "intrinsic" case is "equally" symmetric as for flat Laplacian

$$0=\int\eta^2u\,\triangle_g\,udv_g=-\int\sum D_i\left(\eta^2u\right)g^{ij}D_judv_g$$
$$=-\int\left\langle\nabla_g\left(\eta^2u\right),\nabla_gu\right\rangle_gdv_g=-\int\left\langle\nabla_g(\eta u)+u\nabla_g\eta,\nabla_g(\eta u)-u\nabla_g\eta\right\rangle_gdv_g$$
$$=-\int\left(|\nabla_g(\eta u)|_g^2-u^2|\nabla_g\eta|_g^2\right)dv_g.$$

RMK 2.

$$\int_{B_{1/2}}|Du|^2\le C(n,\mu)\frac{1}{(1/2)^2}\int_{B_1\setminus B_{1/2}}u^2$$
$$\int_{B_{1/2}}|Du|^2\le C(n,\mu)\frac{1}{\varepsilon^2}\int_{B_{\frac{1}{2}+\varepsilon}\setminus B_{\frac{1}{2}}}u^2$$

figure

RMK 3. Let $v=\eta^2(u-a)^+$ (be a nonnegative test function), we have for (sub) solution u

$$0\ge\int D\left[\eta^2(u-a)^+\right]ADu=\int D\left[\eta^2(u-a)^+\right]AD(u-a)^+.$$

Hence by repeating the above argument, we get

$$\int\left|D\left[\eta(u-a)^+\right]\right|^2\le C(n,\mu)\int\left[(u-a)^+\right]^2|D\eta|^2.$$

Recall Sobolev

$$\left[\int |\eta u|^{\frac{2n}{n-2}}\right]^{\frac{n-2}{2n}} \leq C(n)\left[\int |D(\eta u)|^2\right]^{\frac{1}{2}} \quad \text{when } n>2$$

$$\left[\int_{B_1} |\eta u|^p\right]^{\frac{1}{p}} \leq \underset{\text{scaling }\mathbf{variant}}{C(p,B_1)}\left[\int_{B_1} |D(\eta u)|^2\right]^{\frac{1}{2}} \quad \text{for all } p<\infty \quad \text{when } n=2.$$

Step 2. Claim: There exists $\varepsilon_0(n,\mu)>0$ small such that (for sub solution u)

$$\int_{B_1} \left(u^+\right)^2 \leq \varepsilon_0 \quad \Rightarrow \quad \sup_{B_{\frac{1}{2}}} u \leq 1.$$

With the claim in hand, $w=\varepsilon_0^{1/2}u/\|u\|_{L^2}$ still a solution satisfies

$$\int_{B_1} w^2 \leq \varepsilon_0$$

$$\sup_{B_{\frac{1}{2}}} w \leq 1 \quad \text{and} \quad \sup_{B_{\frac{1}{2}}} -w \leq 1.$$

So

$$\|u\|_{L^\infty\left(B_{1/2}\right)} \leq \frac{1}{\sqrt{\varepsilon_0}}\|u\|_{L^2(B_1)} = C(n,\mu)\|u\|_{L^2(B_1)}.$$

Question: A mean value inequality approach as for subharmonic function $\triangle u \geq 0$ on R^n or $\triangle_g u \geq 0$ on minimal surfaces

$$u(0) \leq \frac{1}{|B_r|}\int_{B_r} u dv \ ?$$

Now to the claim.

Domain goes from 1 to $\frac{1}{2}$, range goes from $\frac{1}{2}$ to 1.

domain range figure

Set

$$\eta_k(x) = \begin{cases} 1 \ \text{ for } \ |x| < \frac{1}{2}+\frac{1}{2^{k+1}} \\ \text{linear interpolation in between} \\ 0 \ \text{ for } \ |x| > \frac{1}{2}+\frac{1}{2^k} \end{cases}$$

$$u_k = \left[u-\left(1-\frac{1}{2^k}\right)\right]^+.$$

Observe $u_k \le u_{k-1}$, and $u_{k-1} > \frac{1}{2^k}$ when $u_k > 0$. Set

$$A_k \overset{\text{def}}{=} \int_{B_1} (\eta_k u_k)^2 \ge \int_{B_{1/2}} \left[(u-1)^+\right]^2 .$$

We prove

$$\left|\{u > 1\} \cap B_{1/2}\right| = 0$$

via iteration

$$A_k \le [b(n,\mu)]^{S(k)} A_1^{S(k)} \to 0 \ \text{ as } k \to \infty.$$

Case $n > 2$.

$$\begin{aligned}
A_k &\le \left[\int_{B_1} (\eta_k u_k)^{2\frac{n}{n-2}}\right]^{\frac{n-2}{n}} \left(\int_{\eta_k u_k > 0} 1\right)^{\frac{2}{n}} \\
&\overset{\text{Sobolev}}{\le} C(n) \int_{B_1} |D(\eta_k u_k)|^2 \left(\int_{\eta_k u_k > 0} 1\right)^{\frac{2}{n}} \\
&\overset{\text{Step 1}}{\le} C(n,\mu) \int_{B_1} |D\eta_k|^2 u_k^2 \left(\int_{\eta_k u_k > 0} 1\right)^{\frac{2}{n}} \\
&\le C(n,\mu) \left(2^{k+1}\right)^2 \int_{B_1} \eta_{k-1}^2 u_k^2 \left(\int_{\eta_{k-1} u_{k-1} > \frac{1}{2^k}} 1\right)^{\frac{2}{n}} \\
&\le C(n,\mu) \left(2^{k+1}\right)^2 \int_{B_1} \eta_{k-1}^2 u_{k-1}^2 \left(\int_{B_1} \left(\frac{\eta_{k-1} u_{k-1}}{\frac{1}{2^k}}\right)^2\right)^{\frac{2}{n}} \\
&\le C(n,\mu)\ 4\ 16^k \left(\int_{B_1} \eta_{k-1}^2 u_{k-1}^2\right)^{1+\frac{2}{n}} .
\end{aligned}$$

Case $n = 2$. Say $p = 4$ in the Sobolev, then

$$\begin{aligned}
C(n,\mu) &\dashrightarrow C\left(n, \mu, \overset{=4}{p}\right) \\
\frac{2}{n} &\dashrightarrow \frac{1}{2}.
\end{aligned}$$

Any way both $1 + \frac{2}{n} > 1$ and $1 + \frac{1}{2} > 1$. At this point we have

$$\begin{aligned}
&A_k \le b^k A_{k-1}^{\beta} \ \text{ with} \\
&b = C(n,\mu) \ \text{ and } \beta = 1 + \frac{2}{n} \ \text{ or } 1 + \frac{1}{2}.
\end{aligned}$$

To skip a direct but "tedious" iteration, let us go with a short cut. We need

$$\begin{aligned}
d^k A_k &\le \left(d^{k-1} A_{k-1}\right)^{\beta} \ \text{ and really} \\
b^k &\le \frac{\left(d^{k-1}\right)^{\beta}}{d^k} = d^{\beta(k-1)-\beta}
\end{aligned}$$

or

$$\frac{k\ln b}{(\beta-1)\,k-\beta}\le\ln d.$$

This can be achieved for $k\ge k_0\,(n,\mu)$ with $\ln d>0$, $d>1$. Thus we have

$$d^kA_k\le\left(d^{\kappa-1}A_{k-1}\right)^{\beta}\le\left(d^{\kappa-2}A_{k-2}\right)^{\beta^2}\le\cdots\le\left(d^{k_0}A_{k_0}\right)^{\beta^{k-k_0}}.$$

It follows that

$$\begin{aligned}A_k&\le\frac{1}{d^k}\left(d^{k_0}A_{k_0}\right)^{\beta^{k-k_0}}\\&\le\frac{1}{d^k}\left[d^{k_0}\int_{B\ 1}\left(u^+\right)^2\right]^{\beta^{k-k_0}}\\&\to 0\ \text{ as } k\to\infty\end{aligned}$$

provided

$$\int_{B\ 1}\left(u^+\right)^2\le\varepsilon_0\,(n,\mu)\quad\text{small enough.}$$

Therefore

$$\int_{B_{1/2}}\left[(u-1)^+\right]^2=0$$

and

$$\sup_{B_{1/2}}u^+\le 1.$$

Step 3. Drop claim: Assume (sub) solution u in B_2 satisfying

$$u\le 1\ \text{ in } B_2$$
$$\frac{|\{u\le 0\}\cap B_1|}{|B_1|}\ge\delta_0>0.$$

Then

$$u\le 1-\varepsilon\,(\delta_0,n,\mu)\quad\text{in } B_{1/2}.$$

Consequence: "Full" solution u

$$\operatorname*{osc}_{B_1}u\le 2\ \Rightarrow\ \operatorname*{osc}_{B_{1/2}}u\le 2\theta\,(n,\mu)\ <2.$$

In fact, by linearity of the equation, suppose $-1\le u\le 1$.

Case $\frac{|\{u\le 0\}\cap B_1|}{|B_1|}\ge 1/2 \overset{u\text{ sub sol}}{\Rightarrow} u\le 1-\varepsilon\left(\frac{1}{2},n,\mu\right)$ in $B_{1/2}$ $\Rightarrow$

$$\operatorname*{osc}_{B_{1/2}}u\le 2-\varepsilon\left(\frac{1}{2},n,\mu\right)=2\cdot\frac{2-\varepsilon\left(\frac{1}{2},n,\mu\right)}{2}=2\theta;$$

Case $\frac{|\{u\le 0\}\cap B_1|}{|B_1|}<1/2$, then sub sol $-u$ satisfies $\frac{|\{-u\le 0\}\cap B_1|}{|B_1|}\ge 1/2$. From Case "$\ge 1/2$"

$$2\theta\ge\operatorname*{osc}_{B_{1/2}}(-u)=\operatorname*{osc}_{B_{1/2}}u.$$

Therefore, for all $x_0 \in B_{1/2}(0)$

$$\operatorname*{osc}_{B_{2^{-k}}(x_0)} u \le \theta^{k-2} \operatorname*{osc}_{B_{1/4}(x_0)} u,$$

and continuously for $2^{-k-1} < \rho \le 2^{-k}$

$$\operatorname*{osc}_{B_\rho(x_0)} u \le \theta^{k+1}\ \theta^{-3} \operatorname*{osc}_{B_{1/4}(x_0)} u \le \left(2^{-k-1}\right)^{\log_2 \theta^{-1}}\ \ \theta^{-3} \operatorname*{osc}_{B_{1/4}(x_0)} u$$
$$\le \rho^\alpha\ C(n,\mu)\,\|u\|_{L^2(B_1(0))}$$

with $\alpha(n,\mu) = \log_2 \theta^{-1} > 0$. Our theorem is proved up to the Drop claim.

Now we prove the drop claim. We proceed with the following preparation of the "almost case".

Case almost: $\frac{|\{u\le 0\}\cap B_1|}{|B_1|} \ge 1 - h_0$ with small enough $h_0 = \frac{\varepsilon_0}{4|B_1|}$. We have

$$\int_{B\ \ 1} \left(u^+\right)^2 \le 1^2 h_0\,|B_1| \le \frac{1}{4}\varepsilon_0.$$

Apply Step 2 to $2u$, we get $2u \le 1$ in $B_{1/2}$ or $u \le 1/2$ in $B_{1/2}$.

Drop claim-Case δ_0 : $\frac{|\{u\le 0\}\cap B_1|}{|B_1|} \ge \delta_0 > 0 \quad (\delta_0 < 1 - h_0)$. The proof is through next

Energy claim: $v \in H^1(B_1)\quad 0 \le v \le 1,\ \ \Sigma_0 = \{v = 0\}\quad \Sigma_1 = \{v = 1\}$. Then

$$|\Sigma_0|\ \ |\Sigma_1| \le C(n)\,\|Dv\|_{L^2(B_1)}\quad |\{0 < v < 1\}|^{\frac{1}{2}}.$$

Assuming Energy claim, let us proceed.

$$u \text{ from } 1 - \frac{1}{2^{k-1}} \quad \text{to } 1 - \frac{1}{2^k}\ \mathit{figure}$$

Apply Step 1 Cacioppoli to test function $\eta^2 u_k$ with

$$u_k = \min\left\{2^k\left[u - \left(1 - \frac{1}{2^{k-1}}\right)\right]^+,\ \ 1\right\} \in W^{1,2},$$

we get

$$\int_{B_1} |Du_k|^2 \le C(n,\mu)\int_{B_2\setminus B_1} u_k^2 \le C(n,\mu)\int_{B_2\setminus B_1} 1.$$

By Energy claim applying to $v = u_k$,

$$\underbrace{\left|\left\{u \le 1 - \frac{1}{2^{k-1}}\right\}\cap B_1\right|}_{\ge \delta_0|B_1|}\ \left|\left\{1 \ge u \ge 1 - \frac{1}{2^k}\right\}\cap B_1\right|$$
$$\le C(n)\,\|Du_k\|_{L^2}\ \left|\left\{1 - \frac{1}{2^{k-1}} < u < 1 - \frac{1}{2^k}\right\}\right|^{1/2}.$$

If

$$\left|\left\{1 \geq u \geq 1-\frac{1}{2^k}\right\} \cap B_1\right| \geq h_0 \left|B_1\right| \quad \text{for all } k=1,2,3,\cdots,$$

then

$$\delta_0 \left|B_1\right| \; h_0 \left|B_1\right| \; C\left(n,\mu\right) \leq \; \left|\left\{1-\frac{1}{2^{k-1}} < u < 1-\frac{1}{2^k}\right\} \cap B_1\right|^{1/2}.$$

It implies

$$\left|\left\{0 < u \leq 1\right\} \cap B_1\right| > \sum_{k=1}^{\infty} \left|\left\{1-\frac{1}{2^{k-1}} < u < 1-\frac{1}{2^k}\right\} \cap B_1\right|^{1/2} = \infty.$$

This contradiction shows there exists (large) $k_1 = k_1\left(n,\mu\right)$ such that

$$\frac{\left|\left\{1 \geq u \geq 1-\frac{1}{2^{k_1}}\right\} \cap B_1\right|}{\left|B_1\right|} < h_0.$$

Now (sub) solution $w = \left[u - \left(1-\frac{1}{2^{k_1}}\right)\right] \; 2^{k_1}$ satisfies

$$\int_{B_1} \left(w^+\right)^2 \leq 1 \; \left|B_1\right| \; h_0 \leq \frac{\varepsilon_0}{4}.$$

Applying Step 2 to $2w$ we get

$$\sup_{B_{1/2}} 2w \leq 1$$

or

$$\left[u - \left(1-\frac{1}{2^{k_1}}\right)\right] \; \leq \frac{1}{2}\frac{1}{2^{k_1}} \quad \text{in } B_{1/2}.$$

That is

$$u \leq 1 - \frac{1}{2}\frac{1}{2^{k_1}} \;\; \text{in } B_{1/2} = 1 - \varepsilon\left(n,\mu\right).$$

Finally we prove Energy claim.

"solid angle" from x to reach all Σ_1

$S_x\left(\Sigma_1\right)$ "solid" angle from $x \in \Sigma_0$ to reach all $y \in \Sigma_1$. First

$$1 = v\left(y\right) - v\left(x\right) = \int_0^{|y-x|} D_\rho v\left(x+\rho w\right) d\rho \leq \int_0^{|y-x|} \left|Dv\left(x+\rho w\right)\right| d\rho.$$

Next fix x and integrate over $\int_{S_x(\Sigma_1)} d\omega$

$$|S_x(\Sigma_1)| = \int_{S_x(\Sigma_1)} 1\, d\omega \le \int_{S_x(\Sigma_1)} \int_0^{|y-x|} \frac{\left| Dv\left(\overbrace{x+\rho w}^{z}\right)\right|}{\rho^{n-1}} \rho^{n-1}\, d\rho d\omega$$
$$\le \int_{B_1} \frac{|Dv(z)|}{|z-x|^{n-1}} dz.$$

Then integrate over $\int_{\Sigma_0} dx$

$$|\Sigma_0| \min_{x\in\Sigma_0} |S_x(\Sigma_1)| = \int_{\Sigma_0} |S_x(\Sigma_1)|\, dx \le \int_{B_1}\int_{B_1} \frac{|Dv(y)|}{|z-x|^{n-1}} dz dx$$
$$\le C(n) \int_{B_1} |Dv(z)|\, dz$$
$$\le C(n) \left(\int_{B_1} |Dv(z)|^2\, dz\right)^{1/2} \left(\int_{|Dv(z)|\ne 0} 1 dz\right)^{1/2}$$
$$= C(n) \|Dv\|_{L^2(B_1)} |\{0<v<1\}|^{1/2}.$$

Lastly we solve the puzzle $\min_{x\in\Sigma_0} |S_x(\Sigma_1)| \ge c(n)|\Sigma_1|$. This is because

$$|\Sigma_1| = \int_{\Sigma_1} dy \le \int_{\rho_1(y)}^{\rho_2(y)} \int_{S_x(\Sigma_1)} \rho^{n-1} d\omega d\rho$$
$$\le \int_0^2 \int_{S_x(\Sigma_1)} \rho^{n-1} d\omega d\rho = \frac{2^n}{n} |S_x(\Sigma_1)|.$$

Thus $\min_{x\in\Sigma_0} |S_x(\Sigma_1)| \ge \frac{n}{2^n}|\Sigma_1|$.

RMK. For $u \in C_0^\infty(\Omega)$ we have

$$u(y) = -\int_0^\infty D_\rho u(y+\rho\omega)\, d\rho$$

then

$$u(y)|\partial B_1| = -\int_{\partial B_1}\int_0^\infty D_\rho u(y+\rho\omega)\, d\rho d\omega = -\int_\Omega \frac{\langle z-y, Du(z)\rangle}{|z-y|^n} dz$$
$$= \frac{1}{(n-2)} \int_\Omega \left\langle D_z \frac{1}{|z-y|^{n-2}}, Du(z)\right\rangle dz = \frac{-1}{n-2} \int_\Omega \frac{1}{|z-y|^{n-2}} \triangle u(z)\, dz,$$

that is

$$u(y) = \int_\Omega \frac{-1}{(n-2)|\partial B_1|\ |z-y|^{n-2}} \triangle u(z)\, dz.$$

Also we have

$$
\begin{aligned}
u(y) &= \frac{z}{-|\partial B_1|\,|z|^n} * Du \quad \text{and} \\
\|u\|_{L^r(\Omega)} &\leq \left\| \frac{z}{-|\partial B_1|\,|z|^n} \right\|_{L^p(\Omega)} \|Du\|_{L^q(\Omega)} \quad \text{with say} \\
& r = p < \frac{n}{n-1} \quad \text{and } q = 1 \quad \text{in the condition } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
\end{aligned}
$$

By the way last Young's inequality is proved as follows: decompose

$$
fg = f^{\frac{p}{r}} g^{\frac{q}{r}} f^{p\left(\frac{1}{p}-\frac{1}{r}\right)} g^{q\left(\frac{1}{q}-\frac{1}{r}\right)} \quad \text{with } 1 = \frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r}\right) + \left(\frac{1}{q} - \frac{1}{r}\right);
$$

apply Hölder

$$
\int f(x-y)\,g(y)\,dy \leq \left[\int f^p(x-y)\,g^q(y)\,dy\right]^{\frac{1}{r}} \left[\int f^p(x-y)\,dy\right]^{\left(\frac{1}{p}-\frac{1}{r}\right)} \left[\int g(y)\,dy\right]^{\left(\frac{1}{q}-\frac{1}{r}\right)};
$$

integrate

$$
\begin{aligned}
\int \left[\int f(x-y)\,g(y)\,dy\right]^r dx &\leq \int\int f^p(x-y)\,g^q(y)\,dydx \left[\int f^p\right]^{r\left(\frac{1}{p}-\frac{1}{r}\right)} \left[\int g\right]^{r\left(\frac{1}{q}-\frac{1}{r}\right)} \\
&= \left[\int f^p\right]^{1+r\left(\frac{1}{p}-\frac{1}{r}\right)} \left[\int g\right]^{1+r\left(\frac{1}{q}-\frac{1}{r}\right)};
\end{aligned}
$$

simplify

$$
\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.
$$

Immediate application: Assume

$$
\begin{aligned}
u \in W^{1,2}_{loc}(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n), \ \text{say } |Du| \leq \text{a google} \\
\sum_i D_{x_i}(F_{p_i}(Du)) = 0, \quad \text{say } F(p) = \sqrt{1 + |p|^2}.
\end{aligned}
$$

Then u is a linear function.

In fact

$$
\begin{aligned}
\frac{1}{\varepsilon} \sum_i D_{x_i}\left[F_{p_i}(Du(x+\varepsilon e)) - F_{p_i}(Du(x))\right] = 0 \\
\sum_i D_{x_i}\left[F_{p_i p_j}(*)\,D_{x_j}\left(\frac{Du(x+\varepsilon e) - Du(x)}{\varepsilon}\right)\right] = 0.
\end{aligned}
$$

Now De Giorgi-Nash implies

$$
\begin{aligned}
\operatorname*{osc}_{B_1} \frac{Du(x+\varepsilon e) - Du(x)}{\varepsilon} &\leq \theta^k \operatorname*{osc}_{B_{2^k}} \frac{Du(x+\varepsilon e) - Du(x)}{\varepsilon} \\
&\leq \theta^k \|Du\|_{L^\infty(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty,
\end{aligned}
$$

where positive $\theta = \theta\left(n, \|Du\|_{L^\infty(\mathbb{R}^n)}\right) < 1$. Thus

$$\operatorname*{osc}_{B_1}\left[Du\left(x+\varepsilon e\right) - Du\left(x\right)\right] = 0$$

for all $\varepsilon, e, B_1\left(x_0\right)$.

Exercise: Relying on this zero oscillation, show that u is a linear function. (Try not to use derivative.)

RMK. We use the Euclidean structure (only) in deriving the measure control of Energy claim in Step 3. This part is messy on minimal surfaces. In fact it is not true in general. Otherwise (since Step 1 and Step 2 (Sobolev) generalize to minimal surfaces), one would have Hölder estimate for harmonic functions. One consequence is that Hölder growth of non constant harmonic functions

$$\operatorname*{osc}_{B_{2^{k+1}}} h \geq \frac{1}{\theta^k} \operatorname*{osc}_{B_1} h = \left(2^k\right)^{\overbrace{\log_2 \theta^{-1}}^{\alpha}} \operatorname*{osc}_{B_1} h.$$

But the height of catenoid $z = ch^{-1}|x|$ satisfies

$$\begin{aligned} \triangle_g z &= 0 \\ z &\backsim \ln|x| \ll \rho^\alpha. \end{aligned}$$

This contradiction indicates that Energy claim is not true on Catenoid.