Lecture 11 Moser

- ∘ Statement
- ∘ strong maximal principle

Theorem 1 (Moser) *Let C^{α} u be a $W^{1,2}$ weak solution to*

$$\sum_{i,j=1}^{n} D_i\left(a_{ij}\left(x\right)D_j u\right)=0 \quad \text{in } B_1\subset\mathbb{R}^n$$

with

$$\begin{aligned} &\mu I\leq\left(a_{ij}\right)\leq\mu^{-1}I && \text{when } \left(a_{ij}\right)\overset{def}{=}A\ =A^T,\\ &\mu I\leq\left(a_{ij}\right)\quad \text{and } \left|a_{ij}\right|<\mu^{-1} && \text{when } \ A\ \neq A^T. \end{aligned} \tag{*}$$

Suppose u satisfies

$$u\geq 0 \quad \text{in } B_1\left(0\right),$$
$$u\left(0\right)\leq 1.$$

Then

$$\sup_{B_{1/2}} u\leq C\left(n,\mu\right).$$

Recall the examples r^{2-n} and $x_1 r^{1-n}$.
Proof.
Step 1. Distribution estimate of solution
Step 2. Divergent sequence

Step 1. Claim: Suppose (**super**) solution $v\geq 0$ in Q_2 cube and $v\left(0\right)\leq 1$. Then

$$\frac{\left|\{v>t\}\cap Q_1\right|}{\left|Q_1\right|}\leq\frac{2}{t^{\gamma}},$$

where $\gamma=\gamma\left(n,\mu\right)>0$, could be small.

RMK. Norm $\|v\|_{L^2(Q_1)}$ is not available. One cannot normalize so that $\|v\|_{L^2(Q_1)}=1$ and $v\left(0\right)\leq 1$ simultaneously. Otherwise the claim the trivial with $\gamma=2$.

RMK. The assumption $v\left(0\right)\leq 1$ is a conflicting condition for positive solution v, hence the reverse control of the large distribution of the positive solution v.

Let $\Sigma_k=\left\{v\geq N^k\right\}\cap Q_1$ with $N=N\left(n,\mu\right)$ to be chosen in the inductive step.

Recall Step 3 in the proof of De Giorgi: Suppose

$$w \ \text{ sub solution in } B_2$$
$$w\leq 1 \ \text{ in } B_2$$
$$\frac{\left|\{w\leq 0\}\cap B_1\right|}{\left|B_1\right|}\geq\delta_0$$

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Then $w < 1 - \varepsilon(\delta_0, n, \mu)$ in $B_{1/2}$.

Now a "variant" claim: Suppose

$$w \ \text{ sub solution in } Q_2$$
$$w \leq 1 \ \text{ in } Q_2$$
$$\frac{|\{w \leq 0\} \cap Q_1|}{|Q_1|} \geq \delta_0 \quad \text{or} \quad \frac{|\{w \leq 0\} \cap Q_2|}{|Q_2|} \geq \frac{\delta_0}{2^n}.$$

Then $w < 1 - \varepsilon(\delta_0 2^{-n}, n, \mu)$ in Q_1, where δ_0 is chosen in the inductive step as 2^{-1-n}.

Initial Step. $|\Sigma_1| < \frac{1}{2}$ with $N = \frac{2}{\varepsilon}$.

Otherwise if $|\Sigma_1| \geq \frac{1}{2}$, we seek a contradiction. Now the (sub) solution

$$w = 1 - \frac{v}{N} = 1 - \frac{\varepsilon}{2} v$$

satisfies

$$w \leq 1 \ \text{ in } Q_2$$
$$\frac{|\{w \leq 0\} \cap Q_1|}{|Q_1|} \geq \frac{1}{2}.$$

By the "variant" claim, $w \leq 1 - \varepsilon$ in Q_1 or $v > 2$ in Q_1, which contradicts $v(0) \leq 1$.

Second Step. $|\Sigma_2| \leq \frac{1}{2}|\Sigma_1|$.

RMK. The strategy is to prove $|\Sigma_2| \leq \frac{1}{2}|\Sigma_1|$ at every small scale, namely $|\Sigma_2 \cap Q| \leq \frac{1}{2}|\Sigma_1 \cap Q|$ for all Qs. Only density points of Σ_2 make contributions toward its measure. We (Calderon-Zygmund) decompose Q_1 forever.

cube Q_1

Case splitting: $\frac{|\Sigma_2 \cap Q|}{|Q|} < \frac{1}{2}$, continue splitting.

Case keeping: $\frac{|\Sigma_2 \cap Q|}{|Q|} \geq \frac{1}{2}$, keep Q. And in this case the predecessor Q^* of Q satisfies $2^{-1-n} \leq \frac{|\Sigma_2 \cap Q^*|}{|Q^*|} < \frac{1}{2}$ and Q propels Q^* inside Σ_1, that is $Q^* \subset \Sigma_1$.

Indeed consider sub solution

$$w = 1 - \frac{v/N^1}{N} < 1 \text{ in } 2Q^*$$
$$\frac{|\{w \leq 0\} \cap 2Q^*|}{|2Q^*|} \geq \frac{\frac{1}{2}|Q|}{|2Q^*|} = 2^{-2n-1}.$$

By the "variant" claim with ε chosen now, $w \leq 1 - \varepsilon(2^{-2n-1}, n, \mu) = 1 - \frac{2}{N}$ in Q^* or $v > 2N^1$ in Q^*, which implies $Q^* \subset \Sigma_1$.

Before we proceed, observe that any of those kept Qs cannot be any of the half subdivision $Q_{1/2}$ of $Q_1 = \left(Q_{1/2}\right)^*$. Otherwise, by what we have just proved, $v > 2N > 1 = v(0)$ in Q_1. Impossible.

Now let the (disjoint) collection of Q be $\{Q^j\}$, we have

$$|\Sigma_2| \overset{\text{Lebesgue}}{\leq} \sum_j \left|Q^j \cap \Sigma_2\right| \leq \sum_{l \quad \text{exclude repeated ones}} \left|\left(Q^l\right)^* \cap \Sigma_2\right|$$
$$\overset{\text{case splitting}}{<} \frac{1}{2} \sum_l \left|\left(Q^l\right)^*\right| \overset{Q^* \subset \Sigma_1}{\leq} \frac{1}{2} |\Sigma_1|.$$

Inductive step. $|\Sigma_{k+1}| \leq \frac{1}{2} |\Sigma_k|$.

Case splitting: $\frac{|\Sigma_{k+1} \cap Q|}{|Q|} < \frac{1}{2}$, continue splitting.

Case keeping: $\frac{|\Sigma_{k+1} \cap Q|}{|Q|} \geq \frac{1}{2}$, keep Q. And in this case the predecessor Q^* of Q satisfies $2^{-1-n} \leq \frac{|\Sigma_{k+1} \cap Q^*|}{|Q^*|} < \frac{1}{2}$ and Q propels Q^* inside Σ_k, that is $Q^* \subset \Sigma_k$.

Indeed consider sub solution

$$w = 1 - \frac{v/N^k}{N} < 1 \text{ in } 2Q^*$$
$$\frac{|\{w \leq 0\} \cap 2Q^*|}{|2Q^*|} \geq \frac{\frac{1}{2}|Q|}{|2Q^*|} = 2^{-2n-1}.$$

By the "variant" claim, $w \leq 1 - \varepsilon = 1 - \frac{2}{N}$ in Q^* or $v > 2N^k$ in Q^*, which implies $Q^* \subset \Sigma_k$.

Now let the (disjoint) collection of Q be $\{Q^j\}$, we have

$$|\Sigma_{k+1}| \overset{\text{Lebesgue}}{\leq} \sum_j \left|Q^j \cap \Sigma_{k+1}\right| \leq \sum_{l \quad \text{not all predecessor}} \left|\left(Q^l\right)^* \cap \Sigma_{k+1}\right|$$
$$\overset{\text{case splitting}}{<} \frac{1}{2} \sum_l \left|\left(Q^l\right)^*\right| \overset{Q^* \subset \Sigma_k}{\leq} \frac{1}{2} |\Sigma_k|.$$

So we have the claim

$$|\{v \geq t\}| \leq \left|\left\{v \geq N^k\right\}\right| \leq \frac{1}{2^k} = \frac{2}{2^{k+1}} = \frac{2}{\left(N^{\log_N 2}\right)^{k+1}} = \frac{2}{\left(N^{k+1}\right)^{\log_N 2}} < \frac{2}{t^{\log_N 2}} = \frac{2}{t^\gamma},$$

where

$$N^k \leq t < N^{k+1}$$
$$\gamma = \log_{N(n,\mu)} 2 > 0.$$

Step 2. Claim: The positive solution u in the theorem satisfies

$$\sup_{Q_{1/2}} u \leq M(n, \mu), \quad \text{large enough to be chosen in the end.}$$

Otherwise, there exist $\{x_k\} \subset Q_1$ such that

$$u(x_k) \geq l^{k-1} M \to \infty \text{ with } 1 < l = l(n, \mu) \text{ to be chosen shortly.}$$

blow up sequence figure

This contradiction proves the claim. Now let us find a blow-up sequence.

Step x_1. There exists $x_1 \in Q_{1/2}$ such that $u(x_1) \geq M$.

Step x_2. From Step 1.

$$\left| \left\{ u > \frac{M}{2} \right\} \cap Q_1 \right| \leq \frac{2}{\left(\frac{M}{2}\right)^{\gamma}} = \frac{1}{2} \left(\frac{h_1}{2} \right)^n \text{ with } h_1 = 2 \left[\frac{4}{\left(\frac{M}{2}\right)^{\gamma}} \right]^{1/n}.$$

Then

$$\frac{\left| \left\{ u \leq \frac{M}{2} \right\} \cap Q_{h_1/2}(x_1) \right|}{\left| Q_{h_1/2}(x_1) \right|} > \frac{1}{2}. \tag{*2}$$

From this we show that there exists $x_2 \in Q_{h_1}(x_1)$ such that $u(x_2) \geq lM$. Suppose otherwise, then $u(x) < lM$ in $Q_{h_1}(x_1)$.

(The heuristic idea of the following argument is, to look down u from lM with $l = 1 + \frac{1}{100000000}$, then **relatively** $u(x_1) \geq M$ is near lM, but $M/2$ is far away from lM. By Step 1, the $M/2$ far away distribution of the "flipped" solution is small. The competition of distributions from two ends then leads to a collision.)

flip figure

We have (sub) solution

$$w = \frac{lM - u}{lM - M} \geq 0 \text{ in } Q_{h_1}(x_1)$$
$$w(x_1) \leq 1.$$

By "scaled" Step 1,

$$\frac{\left| \left\{ w \geq \frac{lM - \frac{M}{2}}{lM - M} \right\} \cap Q_{h_1/2}(x_1) \right|}{\left| Q_{h_1/2}(x_1) \right|} \leq \frac{2}{\left(\frac{l - \frac{1}{2}}{l - 1} \right)^{\gamma}} < \frac{1}{2}$$

if $l = l(n, \mu) > 1$ and close to 1. In terms of u

$$\frac{\left| \left\{ u \leq \frac{M}{2} \right\} \cap Q_{h_1/2}(x_1) \right|}{\left| Q_{h_1/2}(x_1) \right|} < \frac{1}{2}.$$

This contradicts (*2).

Step x$_3$. Given $u(x_2) \geq lM$, repeat Step 2 with M replaced by lM: Again from Step 1

$$\left|\left\{u > \frac{lM}{2}\right\} \cap Q_1\right| \leq \frac{2}{\left(\frac{lM}{2}\right)^{\gamma}} = \frac{1}{2}\left(\frac{h_2}{2}\right)^n \quad \text{with } h_2 = 2\left[\frac{4}{\left(\frac{M}{2}\right)^{\gamma}}\right]^{1/n} \frac{1}{\left(l^{1/n}\right)^{\gamma}}.$$

Then

$$\frac{\left|\left\{u \leq \frac{lM}{2}\right\} \cap Q_{h_2/2}(x_2)\right|}{\left|Q_{h_2/2}(x_2)\right|} > \frac{1}{2}. \tag{*3}$$

From this we show that there exists $x_3 \in Q_{h_2}(x_2)$ such that $u(x_3) \geq l^2 M$. Suppose otherwise, then $u(x) < l^2 M$ in $Q_{h_2}(x_2)$. We have (sub) solution

$$w = \frac{l^2 M - u}{l^2 M - lM} \geq 0 \text{ in } Q_{h_2}(x_2)$$
$$w(x_2) \leq 1.$$

By "scaled" Step 1, we have

$$\frac{\left|\left\{w \geq \frac{l^2 M - \frac{lM}{2}}{l^2 M - lM}\right\} \cap Q_{h_2/2}(x_2)\right|}{\left|Q_{h_2/2}(x_2)\right|} \leq \frac{2}{\left(\frac{l - \frac{1}{2}}{l-1}\right)^{\gamma}} < \frac{1}{2}.$$

In terms of u, it is

$$\frac{\left|\left\{u \leq \frac{lM}{2}\right\} \cap Q_{h_2/2}(x_2)\right|}{\left|Q_{h_2/2}(x_2)\right|} < \frac{1}{2}.$$

It contradicts (*3).

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In particular

$$\begin{aligned} & h_1 + h_2 + h_3 + \cdots \\ &= h_1\left(1 + \frac{1}{\left(l^{\frac{\gamma}{n}}\right)} + \frac{1}{\left(l^{\frac{\gamma}{n}}\right)^2} + \cdots\right) \\ &= 2\left[\frac{4}{\left(\frac{M}{2}\right)^{\gamma}}\right]^{1/n} \frac{1}{1 - \frac{1}{\left(l^{\frac{\gamma}{n}}\right)}} < 1 \end{aligned}$$

provided we choose $M = M(\gamma, n, \mu)$ large enough. The proof of Moser is complete.

Strong Maximum Principle. Suppose $W^{1,2}$

$$u \text{ is a weak solution to } \sum D_i\left(a_{ij} D_j u\right) = 0$$
$$u \geq 0 \text{ in } B_1$$
$$u(0) = 0.$$

Then $u \equiv 0$.

Proof. For arbitrarily large K, $Ku \geq 0$ in B_1, $Ku(0) = 0$. By Moser

$$Ku \leq C(n, \mu) \quad \text{in } B_{1/2} \quad \text{or}$$
$$0 \leq \sup_{B_{1/2}} u \leq \frac{C(n,\mu)}{K} \to 0 \quad \text{as } K \to \infty.$$

Similarly $u \equiv 0$ in $B_{\frac{1}{2}+\frac{1}{4}}, B_{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \cdots, B_1$.

RMK. Once $u(0) > 0$, we can scale and upgrade Th'm 1 to

$$\sup_{B_{1/2}} u \leq C(n,\mu)\, u(0).$$