Lecture 14 Alexandrov

- $\circ$ $C^{1,1}/W^{2,n}$ version
- $\circ$ viscosity version statement

Alexandrov-Bakelman-Pucci maximum principle:
Let $u \in C^{1,1}$ be a solution to

$$\sum a_{ij}(x) D_{ij} u = f$$
$$u \geq 0 \quad \text{on } \partial B_1,$$

where

$$\mu I \leq (a_{ij}) \leq \mu^{-1} I.$$

Then

$$\max_{B_1} u^- \leq C(n,\mu) \left[\int_{B_1} (f^+)^n \right]^{1/n}.$$

Proof.

cone figure

Consider the cone with base ∂B_1 and vertex $(x_0, u(x_0))$, where $u(x_0) = \inf_{B_1} u$. Then

$$\max_{B_1} u^- \leq \overbrace{2}^{\text{diameter}} \cdot \overbrace{\inf |D\text{cone}|}^{m}.$$

We next estimate m from above. Consider the $\widetilde{\text{cone}}$ with vertex $(x_0, u(x_0))$ and a uniform slope m. For each and every tangent plane to the $\widetilde{\text{cone}}$ (along its generator), we move down parallelly the plane until it touched u (and leaves u after further down). At the touching points, which we denote by Σ,

$$Du = D\text{plane} = D\text{cone}$$
$$u < 0$$
$$D^2 u \geq 0.$$

It follows that

$$B_m \subset Du(\Sigma).$$

RMK. To better describe the contact set Σ, we let Γ be the convex envelope of $\min(u,0)$ in B_1 (mainly for viscosity version). We have

$$\Sigma = \{u = \Gamma\}$$
$$L(x) \leq \Gamma \leq u$$
$$L(x^*) = u(x^*) \quad x^* \text{ touching point.}$$

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We infer

$$
\begin{aligned}
|B_1|\ m^n \le |Du(\Sigma)| &= \int_{Du(\Sigma)} dy \\
\overset{u\in C^{1,1}}{\underset{D^2u\ge 0}{=}} \int_\Sigma \det D^2 u &= \int_\Sigma \frac{\det(AD^2u)}{\det A} \\
&\le \int_\Sigma \frac{1}{\det A}\left[\frac{tr\,(AD^2u)}{n}\right]^n \\
= \int_\Sigma \frac{1}{\det A}\left(f^+\right)^n &\le C(n,\mu)\int_\Sigma \left(f^+\right)^n.
\end{aligned}
$$

Thus we get

$$
\sup_{B_1} u^- \le C(n,\mu)\left[\int_\Sigma \left(f^+\right)^n\right]^{1/n} = C(n,\mu)\left[\int_{\Gamma=u} \left(f^+\right)^n\right]^{1/n}.
$$

In order to have a short notation, mainly to avoid linearization of fully nonlinear equations, we introduce the viscosity sub and super classes: for $\mu>0$ and $f\in C^0$

$$
\begin{aligned}
\underline{S}(\mu,f) &= \left\{u:\ M^+\left(D^2u\right)\ge f \text{ in viscosity sense}\right\}\\
\bar{S}(\mu,f) &= \left\{u:\ M^-\left(D^2u\right)\le f \text{ in viscosity sense}\right\}\\
S &= \underline{S}\cap\bar{S}.
\end{aligned}
$$

Here the subviscosity sense means whenever a quadratic Q (touching u from above, "$\triangle Q$"$\ge f(x^0)$) satisfying

$$
\left\{\begin{array}{c} Q\ge u \ \text{ near } x^0 \\ = \ \text{at } x_0 \end{array}\right\}
$$

then

$$
M^+\left(D^2Q\right) = \sum_{\lambda_i\ge 0}\mu^{-1}\lambda_i + \sum_{\lambda_i<0}\mu\lambda_i \ge f\left(x^0\right);
$$

and the superviscosity sense means whenever a quadratic Q (touching u from below, "$\triangle Q$"$\le f(x^0)$) satisfying

$$
\left\{\begin{array}{c} Q\le u \ \text{ near } x^0 \\ = \ \text{at } x_0 \end{array}\right\}
$$

then

$$
M^-\left(D^2Q\right) = \sum_{\lambda_i\ge 0}\mu\lambda_i + \sum_{\lambda_i<0}\mu^{-1}\lambda_i \le f\left(x^0\right).
$$

Theorem 1 (Caffarelli) *Let $u\in C^0(B_1)$ with $u\ge 0$ on ∂B_1 and $u\in\bar{S}(\mu,f)$. Then*

$$
\sup_{B_1} u^- \le C(n,\mu)\left[\int_{\Gamma=u}\left(f^+\right)^n\right]^{1/n}.
$$

Proof. Step 1. Γ is $C^{1,1}$.

Γ is $C^{1,1}$ at contact points $\Gamma = u$, say 0

$$\begin{cases} \Gamma \le u \\ \Gamma(0) = u(0) \end{cases} \overset{?}{\Longrightarrow} L(x) \le \Gamma \le L(x) + C|x|^2.$$

By subtracting a linear support function from Γ and u, we assume $L \equiv 0$. We want

$$0 \le \Gamma \le C|x|^2,$$

that is to avoid $u = |x|$.

Let $M = \sup_{B_\rho} \Gamma$, say $M = \Gamma(\rho e_n)$. Note convex function Γ is above its linear support function at $x = \rho e_n$

$$\Gamma \ge a + b x_n.$$

Then $\Gamma(x', \rho) \ge a + b\rho = M$. Recall $\Gamma \ge 0$ on the rest of boundary of $R = B_{K\rho}^{n-1} \times$ $\times (-\rho, \rho)$. Set

$$Q = M\left[\frac{(x_n+\rho)^2}{4\rho^2} - \frac{|x'|^2}{K^2\rho^2}\right], \quad K \text{ to be determined.}$$

Then

$$Q(0) > 0 = \Gamma(0) = (u - L)(0)$$
$$Q \le \Gamma \le u - L \text{ on } \partial R.$$

It allows one to move Q down parallelly until it touches u from above at x^* inside R. From $u - L \in \bar{S}(\mu, f)$, we have

$$M^-\left(D^2 Q\right) \le f(x^*) \le \left\|f^+\right\|_{L^\infty\left(B_{K\rho}\right)}.$$

$$M^-\left(D^2 Q\right) = \frac{M}{\rho^2}\left[\mu\frac{1}{2} - \mu^{-1}(n-1)\frac{2}{K^2}\right] = \frac{M}{\rho^2}\frac{1}{4}\mu$$

with K chosen as $K^2 = \frac{(n-1)}{8}\mu^{-2}$.We get

$$M \le 4\mu^{-1}\left\|f^+\right\|_{L^\infty\left(B_{K\rho}\right)} \rho^2.$$

So for all $\rho > 0$

$$0 \le \Gamma \le \sup_{B_\rho} \Gamma \le 4\mu^{-1}\left\|f^+\right\|_{L^\infty\left(B_{K\rho}\right)} \rho^2.$$

This implies that

$$0 \le D^2\Gamma(0) \le 4\mu^{-1} f^+(0)\ I.$$

Γ is $C^{1,1}$ elsewhere. This follows from the fact that any non-contact point, is a convex combination of $(n+1)$ but one contact points [Caffarelli-Cabre, Lem 3.5].

Step 2. Consider cone with base ∂B_1, vertex $(x_0, u(x_0))$ with

$$|u(x_0)| = \sup_{B_1} |u(x)| \le \frac{2}{\text{diam}} \cdot \inf_m |D\text{cone}|.$$

Modify the initial cone to $\widetilde{cone}$ with slope m and the same vertex $(x_0, u(x_0))$. Parallelly move down each generator, really tangent plane along each generator of $\widetilde{cone}$, until it touches u from above. Moreover, it touches Γ from below where it touches u. Thus

$$D\Gamma(\{\Gamma = u\}) \supseteq B_m.$$

Then

$$|B_1| m^n \leq |D\Gamma(\{\Gamma = u\})| = \int_{D\Gamma(\{\Gamma=u\})} dy$$

$$\overset{\Gamma \text{ is } C^{1,1}}{=} \int_{\{\Gamma=u\}} \det D^2\Gamma \leq C(\mu) \int_{\{\Gamma=u\}} \left(f^+\right)^n.$$

Therefore

$$\sup_{B_1} u^- \leq C(n, \mu) \left[\int_{\{\Gamma=u\}} \left(f^+\right)^n\right]^{1/n}.$$