Lecture 15 Krylov-Safonov

- ∘ decay estimate
- ∘ weak Harnack $\Rightarrow$ C^{α} $\Rightarrow$ Liouville
- ∘ Harnack
- ∘ L^{∞} bound in terms of L^{ε}/L^{p}

Theorem 1 (Krylov-Safonov) *Let $u \in C^0$ be a viscosity solution of $S(\mu, 0) = 0$. Then u is Hölder continuous and*

$$\|u\|_{C^{\alpha}\left(B_{1/2}\right)} \leq C(n, \mu)\|u\|_{L^{\infty}(B_1)} \text{ with (small) } \alpha = \alpha(n, \mu) > 0.$$

RMK. In this nondivergence case, the proof is relatively "easier". It only took 20 years to achieve it after the divergence results in the 1950s. The viscosity version was adapted by Caffarelli in the late 1980s.

Proof. Outline Step 1. Let

$$\begin{gathered} u \in \bar{S}(\mu, 0), \text{ i.e. } M^{-}\left(D^2 u\right) \leq 0 \\ u \geq 0 \text{ in } Q_{4\sqrt{n}} \\ \inf_{Q_3} u \leq 1. \end{gathered}$$

Then there exist large $M(\mu, n)$ and small $\eta(\mu, n) > 0$ such that

$$|\{u < M\} \cap Q_1| > \eta \text{ or } |\{u \geq M\} \cap Q_1| \leq 1 - \eta.$$

Step 2. Iterate

$$\left|\left\{u \geq M^k\right\} \cap Q_1\right| \leq (1 - \eta)^k.$$

RMK. The right formulation/consequence is: if

$$\begin{gathered} u \in \bar{S}(\mu, 0), \text{ i.e. } M^{-}\left(D^2 u\right) \leq 0 \\ u \geq 0 \text{ in } Q_{4\sqrt{n}} \\ \min_{Q_3} u \leq 1, \end{gathered}$$

then there exists $\varepsilon = \varepsilon(\mu, n)$ such that

$$\begin{aligned} \int_{Q_1} u^{\varepsilon} &\leq M^{\varepsilon}\left|\{u < M\}\right| + M^{2\varepsilon}\left|\left\{M \leq u < M^2\right\}\right| + M^{3\varepsilon}\left|\left\{M^2 \leq u < M^3\right\}\right| + \cdots \\ &\leq M^{\varepsilon}\ 1 + M^{2\varepsilon}(1 - \eta) + M^{3\varepsilon}(1 - \eta)^2 + \cdots \\ &= \frac{M^{\varepsilon}}{1 - M^{\varepsilon}(1 - \eta)} \overset{\text{def}}{=} C(\mu, n). \end{aligned}$$

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Without the assumption $\min_{Q_3} u \le 1$ for $0 \le u \in \bar{S}(\mu, 0)$ in $Q_{4\sqrt{n}}$, we just apply the above to $v = u / \min_{Q_3} u$, then

$$\left(\int_{Q_1} u^{\varepsilon} \right)^{1/\varepsilon} \le C^{1/\varepsilon} (\mu, n) \min_{Q_3} u.$$

Step 3. Oscillation

Step 1. First heuristic: If

$$\begin{aligned} \sum a_{ij} D_{ij} u &\le 0 \quad \text{in } B_1 \\ u(0) &\le 1 \\ u &\ge 0, \end{aligned}$$

then

$$|\{u < M\} \cap B_1| > \eta.$$

That is, positive super solution small at one point implies it is not too large in a nontrivial portion.

envelope figure

Let $w = u + 2\left(|x|^2 - 1\right)$, then $\sum a_{ij} D_{ij} w \le 4 \sum a_{ii}$. It follows that

$$1 \le \max_{B_1} w^- \le C(n, \mu) \left[\int_{w = \Gamma(w)} \left(4 \sum a_{ii} \right)^n \right]^{1/n} \le C(n, \mu) \left| \{ w = \Gamma(w) \} \right|^{1/n}.$$

Now

$$\{w = \Gamma(w)\} \subset \{w < 0\} \subset \{u < 2\}.$$

Thus

$$|\{u < 2\} \cap B_1| > \left[\frac{1}{C(n, \mu)} \right]^n - 0.0000001 \overset{\text{def}}{=} \eta.$$

Second realization: construct $h = A - B/r^{\alpha}$ such that

$$\begin{aligned} M^+\left(D^2 h\right) &\le 0 \quad \text{outside } Q_1, \text{ to pick out } Q_1, \\ h &\le -2 \quad \text{in } (Q_3 \subset) B_{2\sqrt{n}}, \quad \text{to have inf over } Q_3, \\ h &\ge 0 \quad \text{outside } (Q_4 \subset) B_{3\sqrt{n}}, \text{ to have nonnegative boundary data for Alexandrov}, \\ \left|D^2 h\right| &\le C, \quad \text{to bound determinant from above.} \end{aligned}$$

set inclusion figure

Let

$$h = \begin{cases} 2 - 2\frac{(3\sqrt{n})^{\alpha}}{r^{\alpha}} & \text{outside } B_{1/2} \subset Q_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]^n \\ \text{smooth inside } B_{1/2} \end{cases}.$$

Then

$$D^2h = \frac{2\left(3\sqrt{n}\right)^{\alpha}}{r^{\alpha+2}} \begin{bmatrix} -\alpha\left(\alpha+1\right) & & & \\ & \alpha & & \\ & & \cdots & \\ & & & \alpha \end{bmatrix}$$

$$M^{+}\left(D^2h\right) = \frac{2\left(3\sqrt{n}\right)^{\alpha}}{r^{\alpha+2}}\left[-\alpha\left(\alpha+1\right)\mu + \left(n-1\right)\alpha\mu^{-1}\right] \leq 0 \text{ if } \alpha\left(n,\mu\right) \text{ large enough.}$$

Set $w = u + h$, apply Alexandrov-B-P to $\min\left(u+h,0\right)$ in $B_{3\sqrt{n}} \subset Q_{4\sqrt{n},}$ we have

$$\left|\min_{B_{3\sqrt{n}}} w\right| \leq C\left(n,\mu\right)\operatorname{diam}\left(B_{3\sqrt{n}}\right)\left[\int_{w=\Gamma(w)}\left[\left[M^{-}\left(D^2w\right)\right]^{+}\right]^{n}\right]^{1/n} \text{ (minimal arithmetic mean).}$$

Observe

$$\inf_{B_{3\sqrt{n}}} w \leq \inf_{Q_3} w \leq 1-2 = -1.$$

Near contact points

$$\begin{aligned} & w = u + h \geq \Gamma \geq P \\ & u \geq P - h \\ & u \geq P - T_2h \quad \text{“}=\text{''} \text{ at contact point.} \end{aligned}$$

Recall $u \in \bar{S}\left(\mu,0\right)$ means $\mu\lambda^{+} - \mu^{-1}\lambda^{-} \leq \triangle u \leq 0$

$$\begin{aligned} 0 &\geq M^{-}\left(D^2P - D^2T_2h\right) \overset{\text{fixed point}}{=} TrA\left(D^2P - D^2T_2h\right) = TrAD^2P - TrA\left(D^2T_2h\right) \\ &\geq M^{-}\left(D^2P\right) - M^{+}\left(D^2T_2h\right). \end{aligned}$$

It follows

$$M^{-}\left(D^2P\right) \leq M^{+}\left(D^2T_2h\right) \leq C\left(n,\mu\right)\chi_{Q_1}.$$

Thus

$$1 \leq C\left(n,\mu\right)\left\{\int_{w=\Gamma(w)}\left[C\left(n,\mu\right)\chi_{Q_1}\right]^{n}\right\}^{1/n} \leq C\left(n,\mu\right)\left|\left\{w=\Gamma\left(w\right)\right\}\cap Q_1\right|^{1/n}.$$

Also

$$\left\{w=\Gamma\left(w\right)\right\} \subset \left\{w<0\right\} \subset \left\{u < -h \leq \sup -h \overset{\text{def}}{=} M\left(n,\mu\right)\right\}.$$

Finally we have

$$\begin{aligned} & \frac{\left|\left\{u<M\right\}\cap Q_1\right|}{\left|Q_1\right|} = \left|\left\{u<M\right\}\cap Q_1\right| > \eta\left(n,\mu\right) = \left[\frac{1}{C\left(n,\mu\right)}\right]^{n} - 0.0000001 \text{ or} \\ & \frac{\left|\left\{u\geq M\right\}\cap Q_1\right|}{\left|Q_1\right|} \leq 1-\eta. \end{aligned}$$

Step 2. Claim: If

$$\begin{aligned} u &\in \bar{S}(\mu,0), \text{ i.e. } M^-\left(D^2u\right) \le 0 \\ u &\ge 0 \quad \text{in } Q_{4\sqrt{n}} \\ \inf_{Q_3} u &\le 1, \end{aligned}$$

then

$$\left|\left\{u \ge M^k\right\} \cap Q_1\right| \le (1-\eta)^k.$$

Step 1 shows $k=1$ is true. Suppose the decay estimate is true for $k-1$, we show "so is k''. Let

$$A = \left\{u \ge M^k\right\} \cap Q_1 \quad B = \left\{u \ge M^{k-1}\right\} \cap Q_1.$$

Already

$$|A| \le |\{u \ge M\} \cap Q_1| \le 1-\eta.$$

C-Z cube figure

We prove $|A| \le (1-\eta)|B|$ at every "effective" small scale via (Calderon-Zygmund) dyadic splitting Q_1 according to A.

Keeping case: $\frac{|Q\cap A|}{|Q|} > 1-\eta$, keep Q;

Splitting case: $\frac{|Q\cap A|}{|Q|} \le 1-\eta$, continue splitting Q.

Let $\{Q^j\}$ be the collection, for the predecessor Q^{j*} of each Q^j, we show that $Q^{j*} \subset B$, that is, $u \ge M^{k-1}$ in Q^{j*}. Suppose $Q^{j*} \not\subseteq B$ or $\inf_{Q^{j*}} u \le M^{k-1}$. We have

$$\begin{aligned} \frac{|Q^{j*} \cap A|}{|Q^{j*}|} &\le 1-\eta \\ \frac{|Q^{j} \cap A|}{|Q^{j}|} &> 1-\eta. \end{aligned} \tag{*}$$

Now

$$\begin{aligned} 0 \le \frac{u}{M^{k-1}} &\in \bar{S}(\mu,0) \\ \inf_{Q^{j*}} \frac{u}{M^{k-1}} &\le 1. \end{aligned}$$

Apply Step 1 to u/M^{k-1}, we get

$$\frac{\left|\left\{\frac{u}{M^{k-1}} \ge M\right\} \cap Q^j\right|}{|Q^j|} \le 1-\eta$$

which contradicts (*). Hence $Q^{j*} \subset B$. We can then finish the decay estimate

$$\begin{aligned} |A| &\overset{\text{Lebesgue}}{=} \sum_j \left|Q^j \cap A\right| \le \sum_{\substack{j' \\ \text{disjoint } Q^{j'*} \text{ cover all } Q^j}} \left|Q^{j'*} \cap A\right| \\ &\overset{\text{case splitting}}{\le} (1-\eta)\sum_{j'} \left|Q^{j'*}\right| \overset{Q^{j*}\subset B}{\le} (1-\eta)|B|. \end{aligned}$$

Therefore,

$$\left|\left\{u \geq M^k\right\} \cap Q_1\right| \leq (1-\eta) \mid \left\{u \geq M^{k-1}\right\} \cap Q_1\big| .$$

Corollary 2 (Krylov-Safonov's weak Harnack) *Let* $0 \leq u \in \bar{S}(\mu, 0)$. *Then*

$$\left(\int_{Q_1} u^{\varepsilon}\right)^{1/\varepsilon} \leq C(n,\mu) \min_{Q_3} u \leq C(n,\mu)\ u(0).$$

RMK. One immediate consequence of this corollary is the strong minimum principle for super solutions.

Step 3. Claim: For continuous $u \in S(\mu, 0) = \bar{S}(\mu, 0) \cap \underline{S}(\mu, 0)$, we have

$$\operatorname*{osc}_{Q_1} u \leq \theta \operatorname*{osc}_{Q_{4\sqrt{n}}} u \quad \text{with positive } \theta = \theta(n,\mu) < 1.$$

In fact let

$$w = \frac{u - \min_{Q_{4\sqrt{n}}} u}{\operatorname{osc}_{Q_{4\sqrt{n}}} u},$$

then $w \in S(\mu, 0)$ and $0 \leq w \leq 1$.

Case $|\{w \geq 1/2\} \cap Q_1| \geq 1/2$. By the corollary applied to $w \in \bar{S}$,

$$\frac{1}{2}\left(\frac{|Q_1|}{2}\right)^{1/\varepsilon} \leq \left(\int_{Q_1} w^{\varepsilon}\right)^{1/\varepsilon} \leq C(n,\mu) \min_{Q_3} w \leq C(n,\mu) \min_{Q_1} w.$$

Then

$$\min_{Q_1} w \geq \frac{\left(\frac{1}{2}\right)^{1+\frac{1}{\varepsilon}}}{C(n,\mu)} = \delta(n,\mu) \in (0,1).$$

Consequently

$$\operatorname*{osc}_{Q_1} w \leq 1 - \delta$$

or

$$\operatorname*{osc}_{Q_1} u \leq (1-\delta) \operatorname*{osc}_{Q_{4\sqrt{n}}} u.$$

Case $|\{w \geq 1/2\} \cap Q_1| < 1/2$. Apply the corollary to

$$1 - w \in \bar{S} \quad \text{with } |\{1 - w > 1/2\} \cap Q_1| > 1/2,$$

and repeat the argument in the first case, we get

$$\operatorname*{osc}_{Q_1} w = \operatorname*{osc}_{Q_1}(1-w) \leq 1 - \delta$$

or

$$\operatorname*{osc}_{Q_1} u \leq (1-\delta) \operatorname*{osc}_{Q_{4\sqrt{n}}} u.$$

The theorem is completely proved.

Corollary 3 (Krylov-Safonov's Liouville) *Let continuos u be a viscosity solution to*

$$\sum_{i,j} a_{ij}(x) D_{ij} u = 0 \quad \text{in } \mathbb{R}^n$$

with the continuos coefficients $a_{ij}(x)$ satisfying $\mu I \leq (a_{ij}) \leq \mu^{-1} I$ and

$$|u| \leq C.$$

Then u is constant.

The proof goes as follows.

$$\operatorname*{osc}_{Q_1} u \leq (1-\delta) \operatorname*{osc}_{Q_{4\sqrt{n}}} u \leq \cdots \leq (1-\delta)^k \operatorname*{osc}_{Q_{(4\sqrt{n})^k}} u \to 0, \quad \text{as } k \to \infty.$$

Theorem 4 (Krylov-Safonov) *Let continuos u be a solution in the viscosity sense to*

$$\sum_{i,j} a_{ij}(x) D_{ij} u = 0 \quad \text{in } B_{4\sqrt{n}}$$

with the continuos coefficients $a_{ij}(x)$ satisfying $\mu I \leq (a_{ij}) \leq \mu^{-1} I$. Suppose u satisfies

$$u \geq 0 \quad \text{in } B_{4\sqrt{n}}.$$

Then

$$\max_{Q_{1/2}} u \leq C(n,\mu)\, u(0), \quad \text{and } \max_{Q_{1/2}} u \leq C(n,\mu) \min_{Q_3} u$$

Proof. Step 1. As in the divergence case, we can also "flip" the large distribution decay estimate in Step 2 to obtain the Harnack inequality. Say $u(0) = 1$, if $\max_{Q_1} u > M$, then there exist $x_1, x_2, x_3, \cdots$ goes to $x_* \in Q_2$ such that $u(x_k) > l^{k-1} M$ goes to ∞. A contradiction.

Step 2. Local Maximum Principle. Let $u \in \underline{S}(\mu, 0)$ in $B_{4\sqrt{n}}$. Then for any $p > 0$, we have

$$\max_{Q_{1/2}} u \leq C(p,n,\mu) \left[\int_{Q_1} \left(u^+\right)^p \right]^{1/p}.$$

Exercise: Proof this LMP. Hints: Indeed, by scaling $u / \|u^+\|_{L^p(Q_1)}$, we assume $\|u^+\|_{L^p(Q_1)} = 1$. Then

$$|\{u > t\} \cap Q_1| \leq \int_{Q_1} \frac{(u^+)^p}{t^p} \leq \frac{1}{t^p}.$$

Note $u^+ = \max\{u, 0\}$ is a subsolution, we have the large distribution decay estimate by "look down" version of Step2. If $\max_{Q_{1/2}} u^+ > M$, then similar to the "blow-up" argument for the Harnack, there exist $x_1, x_2, x_3, \cdots$ goes to $x_* \in Q_1$ such that $u(x_k) > l^{k-1} M$ goes to ∞. A contradiction.

Weak Harnack then implies the full version.