Lecture 17 Dirichlet problem for special Lagrangian equations–a model case

∘ continuity method
∘ a priori estimate

We have answered Dirichlet problem for minimal surface equation with smooth boundary data. Now we solve Monge-Ampere equations and special Lagrangian equations. Let

$$f(\lambda)=\begin{cases}\ln\lambda_1+\cdots+\ln\lambda_n\\ \arctan\lambda_1+\cdots+\arctan\lambda_n-\Theta, \quad \text{for } \Theta\geq(\mathbf{n-2})\frac{\pi}{2}.\end{cases}$$

When $n=2$, $\ln\lambda_1+\ln\lambda_2=0\Leftrightarrow\arctan\lambda_1+\arctan\lambda_2=\frac{\pi}{2}$.

Theorem 1 *There exists a unique solution $u\in C^{2,\alpha}(\bar{B}_1)$ to*

$$\begin{cases} f(\lambda(D^2u))=0 \ in\ B_1\subset\mathbb{R}^n\\ u=\phi\in C^4(\partial B_1)\end{cases}\tag{E}$$

RMK. For subcritical special Lagrangian equations ($|\Theta|<(n-2)\frac{\pi}{2}$), even with analytic boundary data, the C^0 viscosity solution may be only $C^{1,\varepsilon}$, NO better; see the recent work [Wang-Yuan]. In the "ln" case, the solution is convex from the continuity process.

Proof.
The uniqueness is an easy exercise.
For existence, consider a family of equations

$$\begin{cases} f(\lambda)=0 \text{ in } B_1\subset\mathbb{R}^n\\ u=t\phi\in C^4(\partial B_1)\end{cases}\tag{E_t}$$

Let

$$I=\left\{t\in[0,1]\ \middle|\ E_t \text{ has a solution } u_t\in C^{2,\alpha}(\bar{B}_1),\ \alpha=\alpha(\phi,n)>0\right\}.$$

Step 0. $0\in I$.

$$u_0=\begin{cases}\frac{1}{2}(|x|^2-1)\exp\left(\frac{0}{n}\right) & f=\ln\lambda\\ \frac{1}{2}(|x|^2-1)\tan\left(\frac{\Theta}{n}\right) & f=\arctan\lambda\end{cases}$$

Step 1. I is open. Suppose $t_0\in I$, the linearized equation near u_{t_0} is

$$\begin{cases}F_{m_{ij}}(D^2u_{t_0})D_{ij}v=0 \text{ in } B_1\\ v=\varphi\in C^{2,\alpha} \text{ on } \partial B_1\end{cases}$$

and $\mu(||u_{t_0}||_{C^2})\leq(F_{m_{ij}})\leq\mu^{-1}(||u_{t_0}||_{C^2})$. It follows from Schauder theory that the equation is solvable for any $\varphi\in C^{2,\alpha}(\partial B_1)$ with solution $v\in C^{2,\alpha}(\bar{B}_1)$.

[0]February 11, 2020

By Implicit Function Theorem, there exists solution $u_t \in C^{2,\alpha}(\bar{B}_1)$ to the equation E_t for t close to t_0.

Step 2. I is closed. We show that

$$||u_t||_{C^{2,\alpha}(\bar{B}_1)} \leq C(\phi),$$

independent of t for all $C^{2,\alpha}(\bar{B}_1)$ solutions to E_t. Then Ascoli-Arzela theorem implies I is closed.

We will show

$$||u_t||_{C^{1,1}(\bar{B}_1)} \leq C(\|\phi\|_{C^4}, n, \Theta),$$

then the concave equation is $\mu(C(\phi, n, \Theta))$-elliptic.

By interior Evans-Krylov-(Safonov) and boundary Krylov (which we did not prove),

$$||D^2 u_t||_{C^\alpha(\bar{B}_1)} \leq C(\|\phi\|_{C^4}, n, \Theta).$$

For simplicity, we skip the index t in u_t and $t\phi$ in the following.

2.1 L^∞ bound.

We have

$$\underbrace{-\|\phi\|_{L^\infty} + \frac{\exp\left(\frac{0}{n}\right) \text{ or } \tan\left(\frac{\Theta}{n}\right)}{2}\left(|x|^2 - 1\right)}_{\underline{u}} \leq u \leq \underbrace{\|\phi\|_{L^\infty} + \frac{\exp\left(\frac{0}{n}\right) \text{ or } \tan\left(\frac{\Theta}{n}\right)}{2}\left(|x|^2 - 1\right)}_{\overline{u}} \quad \text{on} \quad \partial B_1$$

and

$$f\left(\lambda\left(D^2\underline{u}\right)\right) = f\left(\lambda\left(D^2 u\right)\right) = f\left(\lambda\left(D^2\overline{u}\right)\right) \quad \text{in } B_1.$$

By the comparison principle

$$||u||_{L^\infty(B_1)} \leq ||\phi||_{L^\infty(\partial B_1)} + \frac{\exp\left(\frac{0}{n}\right) \text{ or } \tan\left(\frac{\Theta}{n}\right)}{2}.$$

2.2 Lipschitz bound.

For any (unit) direction $e \in R^n$, we have

$$F_{m_{ij}} D_{ij} u_e = 0,$$

where $F\left(D^2 u\right) = f\left(\lambda\left(D^2 u\right)\right)$. The maximum principle leads

$$\sup_{B_1} |Du| = \sup_{\partial B_1} |Du| \leq \sup_{\partial B_1}(|u_r| + |\phi_\theta|).$$

Next we estimate the boundary normal derivative u_r. Fix $y \in \partial B_1$. Since ∂B_1 is strongly convex, $\phi \in C^2(\partial B_1)$, there exist two linear functions $L^\pm$ whose C^1 norms depend on $C^{1,1}$ norm of ϕ so that (see the Minimal surface equation lecture notes.)

$$L^- \leq \phi \leq L^+ \text{ on } \partial B_1 \text{ and " = " at } y.$$

Let

$$B^{\pm} = L^{\pm} + \frac{\exp\left(\frac{0}{n}\right) \text{ or } \tan\left(\frac{\Theta}{n}\right)}{2}\left(|x|^2 - 1\right).$$

Then

$$\begin{cases} F(D^2 B^{\pm}) = F(D^2 u) & \text{in } B_1 \\ B^- \leq u \leq B^+ & \text{on } \partial B_1 \text{ and `` = '' at } y \end{cases}.$$

It follows from the comparison principle, $B^- \leq u \leq B^+$ in $\bar{B}_1$. Hence

$$\frac{B^- - u(y)}{x_1 - y_1} \leq \frac{u - u(y)}{x_1 - y_1} \leq \frac{B^+ - u(y)}{x_1 - y_1}.$$

Let $x_1 \to y_1^+$, we get

$$\left|\frac{\partial}{\partial x_1} u(y)\right| \leq C(||\phi||_{C^2})$$

Thus

$$||Du||_{L^\infty(B_1)} \leq ||Du||_{L^\infty(\partial B_1)} \leq C(||\phi||_{C^2}).$$

2.3 $C^{1,1}$ bound.

First observe

convex level set over tangent plane figures

$$\triangle u \geq n \exp\left(\frac{0}{n}\right) \quad \text{or } n \tan\left(\frac{\Theta}{n}\right),$$

then an upper bound for $D^2 u$ would lead to a corresponding lower bound, which we estimate next.

Second, since $u \in C^{2,\alpha}$ (no bound yet), Schauder implies $u \in C^{3,\alpha}$, and then $C^{4,\alpha}$ (Hard Exercise). Thus we can differentiate the equation twice,

$$F_{m_{ij}} D_{ij} u_{ee} + \underbrace{F_{m_{ij}, m_{kl}} D_{ij} u_e D_{kl} u_e}_{\leq 0} = 0.$$

By the concavity of F, we have

$$F_{m_{ij}} D_{ij} u_{ee} \geq 0.$$

Maximal principle then implies

$$\sup_{B_1} u_{ee} \leq \sup_{\partial B_1} u_{ee}.$$

The only thing left is the boundary $C^{1,1}$ (upper) estimate for u in terms of the boundary data ϕ. There are tangential derivative and normal derivative on the boundary of the circle:

$$u_{TT},\ \text{say}\ \ u_{11} = \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r = \phi_{\theta\theta} + u_r \le C(||\phi||_{C^2});$$
$$u_{TN},\ \text{say}\ \ u_{n1} = \frac{1}{r}u_{r\theta} - \frac{1}{r^2}u_\theta = u_{r\theta} - \phi_\theta.$$

We show that $|u_{r\theta}(y)| \le C(||\phi||_{C^3})$. Apply

$$\partial_\theta = x_n\partial_{x_1} - x_1\partial_{x_n}.$$

to the equation $F(D^2u) = 0$ (exercise), we get

$$\begin{cases} F_{ij}D_{ij}u_\theta = 0 \\ u_\theta = \phi_\theta \text{ on } \partial B_1 \end{cases}.$$

Since $\phi_\theta \in C^2(\partial B_1)$ and ∂B_1 strongly convex, we have

$$L^- \le \phi_\theta \le L^+ \quad \text{(as in the Minimal surface equation lecture notes)}$$
$$\sum F_{ij}D_{ij}L^\pm = 0.$$

The comparison principle implies

$$L^- \le u_\theta \le L^+ \text{ in } B_1$$
$$\frac{L^- - u_\theta(y)}{x_n - (-1)} \le \frac{u_\theta - u_\theta(y)}{x_n - (-1)} \le \frac{L^+ - u_\theta(y)}{x_n - (-1)} \text{ in } B_1$$

Let $x_n \to -1^+$, we get $u_{r\theta} = \left|\frac{\partial}{\partial x_n}u_\theta(y)\right| \le C(||\phi||_{C^3})$.

Thus only the upper bound of double normal derivative is left to estimate.

Idea: we have, $F_{ij}D_{ij}(ru_r - 2u) = 0 = F_{ij}D_{ij}L^-$, exercise! Now if $ru_r - 2u \ge L^-(x', x_n)$ on ∂B_1. Then

$$ru_r - 2u \ge L^- \text{ in } B_1$$
$$\frac{ru_r - 2u - L^-(y)}{r-1} \le \frac{L^- - L^-(y)}{r-1} \Rightarrow u_{rr} \le L^-_r - u_r \ (*)$$

But L^- coefficients involve C^3 norms of u on ∂B_1, which is not available yet!

We get around in the following (Trudinger) way. We can have (*) at "minimal" u_{TT} (or rather $f(u_{TT})$). Then by the equation (still heuristic)

$$f(u_{TT}) + f(u_{rr}) = 0$$

we would get the upper bound

$$f\left(u_{rr}(y)\right) = -f(u_{TT}\left(y\right)) \le -f(u_{TT}\left(y_{min}\right)) = f(u_{rr}(y_{min})) \le C.$$

Realization:

$$D^2u = \left[\begin{array}{cc} u_{TT} & u_{Tr} \\ u_{rT} & u_{rr} \end{array}\right] \sim \left[\begin{array}{cc} \lambda' & u_{Tr} \\ u_{rT} & u_{rr} \end{array}\right],$$

where tangent vector T acts as

$$u_{TT} = \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = \phi_{\theta\theta} + u_r,$$

say $\phi(r,\theta) = \phi(\theta)$ is homogeneous order zero.

We estimate the lower bound of $trD^2u|_T = \lambda'_1 + \cdots + \lambda'_n$. Suppose y_{min} is one minimal point for $f'(\lambda')|_{\partial B_1}$, where

$$f'(\lambda') = \begin{cases} \ln \lambda'_1 + \cdots + \ln \lambda'_{n-1} \\ \arctan \lambda'_1 + \cdots + \arctan \lambda'_{n-1} - \Theta \end{cases}$$

Let $\lambda'_0 = \lambda'(x_{min})$. Then the $f'(\lambda'_0)$-level set of the function $f'(\lambda')$ is convex. Indeed $f'(\lambda') \geq f'(\lambda'_0) > \Theta - \pi/2$ or $0 - C$ by the following linear algebra lemma, thus λ' is in a convex set

convex set figure

$$\left\{ \lambda' : \ \arctan \lambda' \geq \arctan \lambda'_0 > \Theta - \frac{\pi}{2} \geq (n-1-2)\,\frac{\pi}{2} \right\} \quad \text{or}$$
$$\left\{ \lambda' : \ \ln \lambda' \geq \ln \lambda'_0 > 0 - C \right\}.$$

(Note the above inequality holds without the full concavity of function $f'(\lambda')$ in arctan case.) We conclude

$$\langle Df'(\lambda_0), \lambda' - \lambda'_0 \rangle \geq 0$$

or

$$\langle Df'(\lambda_0), \lambda' \rangle \geq \langle Df'(\lambda_0), \lambda'_0 \rangle = c_0 \text{ (not necessarily +)} \quad \text{and “=” at } y'_{\min};$$

in matrix form

$$tr \left(\frac{\partial f'\left(D^2 u\left(y_{\min}\right)|_T\right)}{\partial D^2 u|_T} \right) \left(D^2 u\left(y\right)|_T\right) \geq c_0.$$

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Recall $f'(\lambda')$ is a symmetric function of λ'. After symmetrizing λ', we get

$$\langle Df'(\lambda_0), \frac{tr\ \lambda'}{n-1}(1,\ldots,1)\rangle \geq c_0, \quad \text{FLAWED}$$

that is

$$\frac{1}{n-1}\left(f'_1(\lambda_0) + \cdots + f'_{n-1}(\lambda_0)\right) tr\ \lambda' \geq c_0.$$

It follows that

$$tr\ D^2u|_T = tr\ \lambda' \geq \frac{(n-1)\,c_0}{f'_1(\lambda_0) + \cdots + f'_{n-1}(\lambda_0)} = c_0(||\phi||_{C^2}).$$

Then

$$(n-1)u_r + tr\ D^2\phi|_T = tr\ D^2u|_T \geq c_0$$

or

$$ru_r \geq \frac{r}{n-1} c_0 - \frac{r}{n-1} tr\ D^2\phi|_T \quad \text{on } \partial B_1 \text{ and "=" at } y_{\min}.$$

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It turns out the above "cute" symmetrization is FLAWED. We go back to the original plain argument.

Note that $D^2u|_T = (u_r - \phi_r)\, II_{\partial B_1} + D^2\phi|_T = (u_r - \phi_r)\, I' + D^2\phi|_T$. Then

$$ru_r\,(y) \geq r\phi_r + \frac{r}{f_1'(\lambda_0) + \cdots + f_{n-1}'(\lambda_0)} \left[c_0 - tr \left(\frac{\partial f'\,(D^2u\,(y_{\min})\,|_T)}{\partial D^2u|_T} \right) \left(D^2\phi\,(y)\,|_T \right) \right]$$

on ∂B_1 and "=" at $y_{\min}$.

As in the Minimal surface equation lecture, for ru_r and also $ru_r - 2u$, we can find linear barrier L^-, whose C^1 norm now depends on $C^{3,1}$ norm of ϕ, so that

$$ru_r - 2u \geq L^-(x', x_n) \ \text{ on } \partial B_1 \text{ and still " = " at } y_{min}.$$

Recall

$$F_{ij}D_{ij}(ru_r - 2u) = 0 = F_{ij}D_{ij}L^- \ \text{ in } \ B_1.$$

The comparison principle implies

$$ru_r - 2u \geq L^- \text{ in } B_1,$$

then for $r < 1$

$$\frac{ru_r - 2u - L^-(y_{min})}{r-1} \leq \frac{L^- - L^-(y_{min})}{r-1} \ \text{ in } B_1.$$

Let $r \to 1^-$, we get

$$(u_{rr} - u_r)(y_{min}) \leq C(||\phi||_{C^4}).$$

Because we have already bounded Du in terms of the $C^{1,1}$ norm of ϕ, we thus obtain

$$u_{rr}(y_{min}) \leq \bar{C}(||\phi||_{C^4}).$$

Lemma 2 (Linear algebra lemma) *Let*

$$M = \begin{bmatrix} \lambda_1' & & & a_1 \\ & \ddots & & \vdots \\ & & \lambda_{n-1}' & a_{n-1} \\ a_1 & \cdots & a_{n-1} & a \end{bmatrix}$$

where $\lambda_1', \ldots, \lambda_{n-1}'$ *are fixed,* $|a_i| \leq C$ *and* $|a| \to +\infty$.

Then the eigenvalues of M behave like

$$\lambda_1' + o(1), \ldots, \lambda_{n-1}' + o(1), a + O(1),$$

where $o(1)$ and $O(1)$ are uniform as $a \to \infty$.

We proceed separately for Monge-Ampere equation and special Lagrangian equation.

M-A: In case

$$0 \le D^2 u(y_{min}) \le c(||\phi||_{C^4})$$

$$\ln \lambda_1 + \cdots \ln \lambda_n = 0$$

$$\Rightarrow \lambda_i(y_{min}) \ge c(||\phi||_{C^4}) > 0$$

$$\Rightarrow \begin{bmatrix} \lambda_1' & & \\ & \ddots & \\ & & \lambda_{n-1}' \end{bmatrix} \sim D^2 u|_T(y_{min}) \ge \min_i \lambda_i(y_{min}) \ge c(||\phi||_{C^4}) > 0.$$

Then from the definition of $y_{\min}$, we have

$$(\ln \lambda_1' + \cdots + \ln \lambda_{n-1}')(y) \ge (\ln \lambda_1' + \cdots + \ln \lambda_{n-1}')(y_{min}) \ge -C(||\phi||_{C^4}).$$

Recall we have estimated $\lambda'(y) \le C(||\phi||_{C^2})$, then we get

$$\lambda_i'(y) \ge c(||\phi||_{C^4}) > 0,\ \forall y \in \partial B_1.$$

Finally choose $K = K(\lambda_i'(y)) = K(c(||\phi||_{C^4}))$ large, to be determined. If

$$u_{rr}(y) \le K,$$

then OK. Otherwise by the linear algebra lemma

$$\begin{bmatrix} \lambda_1' & & & u_{1n} \\ & \ddots & & \vdots \\ & & \lambda_{n-1}' & u_{n-1,n} \\ u_{n1} & \cdots & a_{n,n-1} & u_{nn} \end{bmatrix}(y) \sim \begin{bmatrix} \lambda_1' + o(1) & & & \\ & \ddots & & \\ & & \lambda_{n-1}' + o(1) & \\ & & & u_{nn} + O(1) \end{bmatrix}(y)\,.$$

From equation (E_t), we have at y

$$\ln(\lambda_1' + o(1)) + \cdots + \ln(\lambda_{n-1}' + o(1)) + \ln(u_{nn} + O(1)) = 0$$

Now we choose K large enough so that at y

$$\ln(\lambda_1' + o(1)) + \cdots + \ln(\lambda_{n-1}' + o(1)) = \ln \lambda_1' + \cdots + \ln \lambda_{n-1}' + o(1) \ge -C(||\phi||_{C^4}).$$

Thus it follows that

$$u_{nn}(y) \le C(||\phi||_{C^4}).$$

Special Lagrangian case:

$$\left| D^2 u(y_{min}) \right| \le C(||\phi||_{C^4})$$

At y_{min},

$$f(D^2 u + 100 e_n \otimes e_n) - f(D^2 u) = \left\langle \text{“}\nabla^2 F\,(*)\text{”}, 100 e_n \otimes e_n \right\rangle = \delta_1(||\phi||_{C^4}) > 0.$$

Also we have

$$\lim_{a\to\infty} f(D^2u + a\cdot e_n\otimes e_n) \geq f(D^2u+100e_n\otimes e_n) \geq f(D^2u)+\delta_1 = \Theta+\delta_1.$$

It follows from the linear algebra lemma

$$\sum \arctan \lambda_i'(y_{min}) \geq \Theta + \delta_1 - \frac{\pi}{2}$$

As in the M-A case we choose $K = K(||\phi||_{C^4})$ large enough to be determined shortly. If

$$u_{NN}(y) \leq k.$$

then OK. Otherwise, we have at y

$$\begin{aligned}
\Theta = f(D^2u) &= f(\lambda' + o(1)) + f(u_{nn} + O(1)) \\
&= \sum_{i=1}^{n-1} \arctan \lambda_i'(y) + o(1) + \arctan(u_{nn}+O(1)) \\
&\geq \Theta + \delta_1 - \frac{\pi}{2} - \frac{\delta_1}{2} + \arctan(u_{nn}+O(1)).
\end{aligned}$$

We now take K large enough, then

$$u_{nn}(y) \leq \tan(\frac{\pi}{2} - \frac{\delta_1}{2}) - O(1) \leq C(||\phi||_{C^4}).$$

Therefore

$$||u||_{C^{1,1}(\bar{B}_1)} \leq C(||\phi||_{C^4}).$$

Our proof is complete.

RMK. Our adapted presentation from [T] is shorter and works simultaneously for both critical and supercrticial phases, whose corresponding equations are type I (the origin-level-set cone has λ_i-axis on its boundary) and type II (the origin-level-set cone is larger than the positive cone) respectively. Type I and II equations were handled separately in [CNS] and [T?]. Note that the pioneering paper [CNS] solves Slag equation, the convex branch of

$$\begin{aligned}
0 &= \operatorname{Im} \prod \left(1+\sqrt{-1}\lambda_i\right) = \operatorname{Im} \sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)} \exp\left(\sqrt{-1}\Theta\right) \\
&= \sqrt{(1+\lambda_1^2)\cdots(1+\lambda_n^2)} \sin\Theta,
\end{aligned}$$

which corresponds to

$$\begin{cases} \Theta = (n-1)\frac{\pi}{2} & n \text{ is odd} \\ \Theta = (n-2)\frac{\pi}{2} & n \text{ is even} \end{cases}.$$