Lecture 9 $W^{1,2}$ or H^1 space, trace

- ∘ $W^{1,2}$ space, different characterizations
- ∘ Trace
- ∘ Chain rule

Sobolev space $W^{1,2}$, different characterizations.

Approximation: $u \in W^{1,2}(\Omega)$ if there exist a sequence $u_k \in C^1(\Omega)$ s.t. $u_k \to u$ in $L^2(\Omega)$ and $\{Du_k\}$ is a Cauchy sequence in $L^2(\Omega)$.

IBP: $u \in W^{1,2}(\Omega)$ if $u \in L^2(\Omega)$ and $\int u D_e \varphi \leq C \|\varphi\|_{L^2(\Omega)}$ for all $\varphi \in C_0^1(\Omega)$ and $e \in S^{n-1}$.

Difference quotient: $u \in W^{1,2}(\Omega)$ if $u \in L^2(\Omega)$ and $\left\| \frac{u(x+he) - u(x)}{h} \right\|_{L^2(\Omega)} \leq C$ for h-free C and all $e \in S^{n-1}$.

Proposition. The above approximation, IBP, and difference quotient characterizations of Sobolev space $W^{1,2}(\Omega)$ are equivalent.

Proof. Approximation⇒IBP

We have

$$\int u_k D_e \varphi = -\int D_e u_k \ \varphi \leq \|D_e u_k\|_{L^2} \|\varphi\|_{L^2} \leq C \|\varphi\|_{L^2}.$$

Take limit of the left side, we have

$$\int u D_e \varphi \leq C \|\varphi\|_{L^2}.$$

IBP⇒Difference quotient

To produce differenec quotient from test functions, one integrates the difference of two delta functions along the given direction. We construct $\Phi \in W_0^{1,\infty}(R^n)$ so that $\Phi_e = \phi$ where

$$\phi = \frac{1}{|B_\varepsilon|} \left[\chi_{B_\varepsilon(he)} - \chi_{B_\varepsilon(0)} \right].$$

figureS: graphs of ϕ and Φ

graph of Φ cylinder like w/ height $-\frac{1}{\varepsilon^{n-1}}$ on $B_\varepsilon^{n-1} \times (0, h)$

Note that

$$\int |\Phi| \sim h.$$

To capture the integral norm of the difference quotient, we use duality. The duality test function $\varphi \in C_0^1(\Omega)$ is sneaked in as convolution with the primitive Φ

$$\int u(y) D_e (\varphi * \Phi) dy \leq C_u \|\varphi * \Phi\|_{L^2} \leq C_u \|\varphi\|_{L^2} \|\Phi\|_{L^1} \leq C_u \ h \ \|\varphi\|_{L^2},$$

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and the left hand side equals

$$\int u(y)\left(\int \varphi(x)\phi(y-x)\,dx\right)dy = \int \varphi(x)\left(\int u(y)\phi(y-x)\,dy\right)dx$$
$$= \int \varphi(x)\left(\int u(y+x)\phi(y)\,dy\right)dx = \int \varphi(x)\left(\int u(y+x)\frac{1}{|B_\varepsilon|}\left[\chi_{B_\varepsilon(he)} - \chi_{B_\varepsilon(0)}\right]dy\right)dx.$$

Let $\varepsilon \to 0$, we have

$$\int \varphi(x)\left[u(he+x) - u(x)\right] \leq C_u \; h \;\; \|\varphi\|_{L^2}$$

that is

$$\int \varphi(x)\left[\frac{u(he+x)-u(x)}{h}\right] \leq C_u \;\; \|\varphi\|_{L^2}.$$

So by duality

$$\left\|\frac{u(x+he)-u(x)}{h}\right\|_{L^2(\Omega)} \leq C_u.$$

Difference quotient $\Rightarrow$ Approximation

As $L^2(\Omega)$ space is reflexive and separable, the sequence $\frac{u(x+he)-u(x)}{h}$ has a weak limit v in $L^2(\Omega)$, that is for all $\varphi \in C_0^1(\Omega)$

$$\int \frac{u(x+he)-u(x)}{h}\varphi(x) \to \int v\varphi.$$

The left hand side equals

$$\int u(x)\frac{\varphi(x-he)-\varphi(x)}{h} \to -\int u(x)D_e\varphi(x) \quad \text{as } h \to 0.$$

So

$$-\int u(x)D_e\varphi(x) = \int v\varphi.$$

Next we construct a Cauchy sequence by taking φ to be a usual mollifier $\rho \in C_0^\infty(R^n)$ with $\int \rho = 1$.

Claim: $D_e(u*\rho) = v*\rho$.

Indeed

$$\begin{aligned} D_e(u*\rho) &= \int u(y)D_{e(x)}\rho(x-y)\,dy = \int u(y)\left[-D_{e(y)}\rho(x-y)\right]dy \\ &= \int v(y)\rho(x-y)\,dy \;\; \text{by the derived IBP identity above the claim.} \\ &= v*\rho. \end{aligned}$$

It is also true that $D_e(u*\rho_\varepsilon) = v*\rho_\varepsilon$ with $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^n$. Certainly $v*\rho_\varepsilon \to v$ in $L^2(\Omega)$. Therefore we have a Cauchy sequence with $u*\rho_\varepsilon \to u$ and $D_e(u*\rho_\varepsilon) \to v$ in $L^2(\Omega)$. Replace e with all axis directions we have the desired full gradient version.

Trace

figure: Q =flat piece Γ in x-R^{n-1} $\times$ δ height in y-R^1

We show that $u \in W^{1,2}$ has a well-defined restriction (trace) on the boundary. We first assume $u \in C^1$, then use approximation to reach the general conclusion. From

$$u(x,y) - u(x,0) = \int_0^y u_y(x,s)\,ds$$

it follows that

$$\begin{aligned}\int_\Gamma |u(x,y) - u(x,0)|^2\,dx &\le \int_\Gamma \left[\int_0^y u_y(x,s)\,ds\right]^2 dx \\ &\le \int_\Gamma y \int_0^y u_y^2(x,s)\,dsdx \\ &\le y \int_Q |Du|^2\end{aligned}$$

and

$$\begin{aligned}\int_0^\delta \int_\Gamma u^2(x,0)\,dxdy &= \int_0^\delta \int_\Gamma \left[-u(x,y) + \int_0^y u_y(x,s)\,ds\right]^2 dxdy \\ &\le 2\int_0^\delta \int_\Gamma u^2(x,y) + \left[\int_0^y u_y(x,s)\,ds\right]^2 dxdy \\ &\le 2\int_0^\delta \int_\Gamma \left[u^2(x,y) + \delta \int_0^\delta u_y^2(x,s)\,ds\right] dxdy \\ &\le 2\int_Q u^2 + 2\delta^2 \int_\Gamma \int_0^\delta u_y^2(x,s)\,dsdx \\ &\le 2\int_Q u^2 + 2\delta^2 \int_Q |Du|^2.\end{aligned}$$

The last inequality becomes

$$\int_\Gamma u^2(x,0)\,dx \le \frac{2}{\delta}\int_Q u^2 + 2\delta \int_Q |Du|^2.$$

Therefore $u(\cdot,y)$ is a Cauchy sequence in $L^2(\Gamma)$ as $y \to 0$. The limit of the Cauchy sequence is defined as the trace of u on Γ.

RMK. Inspecting the above argument, we can also go along $W^{1,1}$ function to leave a trace, as long as notice that the integral of L^1 function over a small measure set is small:

$$\int_\Gamma |u(x,y) - u(x,0)|\,dx \le \int_\Gamma \int_0^y |Du|\,dsdx \to 0 \quad \text{as } y \to 0.$$

Chain Rule

Given $f\in C^1(R^1)$ with $f'\in L^\infty(R^1)$ and $u\in W^{1,2}(\Omega)$, then $f(u)\in W^{1,2}(\Omega)$ and $Df(u)=f'(u)Du$.

Proof. We use the approximation version of $W^{1,2}$ space. Let u_k be an approximating sequence for u in $W^{1,2}$ norm, that is $u_k\to u$ in $L^2(\Omega)$ and $Du_k\to Du$ or Du_k is a Cauchy sequence in $L^2(\Omega)$. We have

$$\int_\Omega |f(u_k)-f(u)|^2\,dx\le\int_\Omega |Df(*)|^2|u_k-u|^2\,dx\le\sup_R|Df|^2\int_\Omega|u_k-u|^2\,dx\to 0$$

and

$$\begin{aligned}&\int_\Omega |f'(u_k)Du_k-f'(u)Du|^2\,dx\\&\le 2\int_\Omega [f'(u_k)]^2\ |Du_k-Du|^2\,dx+2\int_\Omega|f'(u_k)-f'(u)|^2|Du|^2\,dx\\&\to 0\quad\text{(1st limit by Cauchy } Du_k,\text{ 2nd limit by Lebesgue dom. con.).}\end{aligned}$$

Prop. For $u\in W^{1,2}$, we have $u\vee 0\in W^{1,2}$ and $D(u\vee 0)=\begin{cases}Du & u>0\\ 0 & u\le 0\end{cases}$ a.e. .

Proof. Smooth out the function $f(t)=\begin{cases}t & t>0\\ 0 & t\le 0\end{cases}$ by convolution with an even (radial) mollifier ρ_ε in R^1. Then the mollified function $\tilde f_\varepsilon(t)=\begin{cases}t & t>\varepsilon\\ \sim\varepsilon & |t|\le\varepsilon\\ 0 & t<-\varepsilon\end{cases}$. Let

$$f_\varepsilon(t)=\tilde f_\varepsilon(t+\varepsilon).$$

By the Chain Rule, $Df_\varepsilon(u)=f'_\varepsilon(u)Du$. Hence for any $\varphi\in C_0^1(\Omega)$ (by Lebesgue D.C.T twice)

$$-\int_\Omega f(u)D\varphi=\lim_{\varepsilon\to0}-\int_\Omega f_\varepsilon(u)D\varphi=\lim_{\varepsilon\to0}\int_\Omega Df_\varepsilon(u)\varphi=\int_\Omega Du\chi_{\{u>0\}}\varphi.$$

Therefore (by IBP characterization of $W^{1,2}$: $f(u)\in W^{1,2}$ then $Df(u)=f'(u)Du$)

$$u\vee 0=f(u)=\begin{cases}u & u>0\\ 0 & u\le 0\end{cases}\in W^{1,2}.$$

RMK. One can just take $f_\varepsilon(t)=\begin{cases}\sqrt{t^2+\varepsilon^2}-\varepsilon & t>0\\ 0 & t\le 0\end{cases}$ as in [GT].

Cor. If u and v are in $W^{1,2}$, then so are the following: $u\wedge 0=-(-u\vee 0)$, u^+ and u^- with $u=u^+-u^-$, $|u|=u^++u^-$, and also $u\vee v=(u-v)\vee 0+v=\begin{cases}u & u>v\\ v & v\ge u\end{cases}$, $u\wedge v=(u-v)\wedge 0+v=\begin{cases}u & u<v\\ v & v\le u\end{cases}$.

eg. $\int|Du|^2=\int|Du^+|^2+\int|Du^-|^2\ge\int|Du^+|^2$.