

Lecture 3 Minimal surface equations–Bernstein problem and monotonicity

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Minimal surface equation

Consider the variational problem for area functional $A[f] = \int \sqrt{1 + |Df|^2} dx$

$$\inf_{f=\varphi \text{ on } \partial U} A[f],$$

any critical f satisfies for all $\eta \in C_0^\infty(U)$

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon} A[f + \varepsilon\eta] \Big|_{\varepsilon=0} = \int_U \frac{Df \cdot D\eta}{\sqrt{1 + |Df|^2}} dx \\ &= \int_U -\operatorname{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) \eta dx. \end{aligned}$$

So

$$\text{mean curvature } H \triangleq \operatorname{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) = 0.$$

Note also

$$\begin{aligned} H &= \frac{\Delta f}{\sqrt{1 + |Df|^2}} - \frac{\langle Df, D^2 f \cdot Df \rangle}{\left(\sqrt{1 + |Df|^2}\right)^3} \\ &= \frac{1}{\left(\sqrt{1 + |Df|^2}\right)^3} [(1 + |Df|^2) \Delta f - \langle Df, D^2 f \cdot Df \rangle] \\ &\stackrel{2d}{=} \frac{1}{\left(\sqrt{1 + |Df|^2}\right)^3} [(1 + f_2^2) f_{11} - 2f_1 f_2 f_{12} + (1 + f_1^2) f_{22}] \\ &= \frac{1}{\left(\sqrt{1 + |Df|^2}\right)^3} Lf, \end{aligned}$$

where $L = (1 + f_2^2) \partial_{11} - 2f_1 f_2 \partial_{12} + (1 + f_1^2) \partial_{22}$.

Explicit examples of minimal surfaces

⁰November 12, 2014

RMK. Solutions, in particular explicit ones are hard to come by for nonlinear equations.)

catenoid: $|x, y| = \cosh z$

helicoid: $z = \arctan \frac{y}{x}$

Scherk's surface: $z = \ln \frac{\cos y}{\cos x}$

figure minimal surfaces

Exercise: invariance for minimal surface equation?

Bernstein. Let smooth f satisfies $\operatorname{div} \left(\frac{Df}{\sqrt{1+|Df|^2}} \right) = 0$ in R^2 . Then f is linear.

Bernstein's proof 1910s–40s

Strange obs. $L \arctan f_1 = 0!$ Only in 2d.

Stunning Theorem. Bounded global saddle surface is flat, really horizontal.

That is $\arctan f_1 = \text{const}$. Similarly $\arctan f_2 = \text{const}$. Thus f is linear.

Lewy 1930s

In studying the Monge-Ampere equations $\det D^2u = 1$, really Darboux equation

$$\det_g \nabla^2 u = K_g (1 - |\nabla_g u|^2)$$

for the isometric embedding problem, introduced the/his transformation $\eta(\xi) = \text{Lewy}[u(x)]$ with

$$\begin{cases} \xi_1 = 1 + u_1 \\ \xi_2 = 1 + u_2 \end{cases},$$

and $u - \eta$ satisfying the contact transformation

$$0 = u + \frac{1}{2} (x_1^2 + x_2^2) + \eta - x_1 \xi_1 - x_2 \xi_2.$$

In fact, Lewy rotation is just the usual Legendre transformation of function $u + \frac{1}{2} (x_1^2 + x_2^2)$. Lewy was trying to get a priori estimates (in order to solve the equations).

Jörgens 1954

$$\det D^2u = 1 \quad \text{in } R^2 \Rightarrow u \text{ is quadratic.}$$

Jörgens used Heinz's "hard" estimates on the 3rd order derivatives.

Exercise: Verify $v = x_1^3 + \frac{x_2^2}{12x_1}$ satisfies the 2d M-A equation.

Heinz 1952 observed, there exists a scalar function u such that

$$\frac{1}{\sqrt{1+|Df|^2}} \left[I + (Df)(Df)^T \right] = \frac{1}{\sqrt{1+|Df|^2}} \left[\begin{array}{cc} 1+f_1^2 & f_1 f_2 \\ f_1 f_2 & 1+f_2^2 \end{array} \right] = D^2u$$

and

$$\det D^2u = 1.$$

The second equation is easy, just note

$$\frac{1}{\sqrt{1+|Df|^2}}g \sim \frac{1}{\sqrt{1+|Df|^2}} \begin{bmatrix} 1+|Df|^2 & \\ & 1 \end{bmatrix}.$$

The first super potential part is a little hard.

Geometrically, we know $\Delta_g(x_1, x_2, f) = \vec{H} = 0$, in fact

$$\Delta_g f = 0 \Leftrightarrow \Delta_g x_1 = 0 \Leftrightarrow \Delta_g x_2 = 0,$$

where

$$\Delta_g = \frac{1}{\sqrt{g}} \operatorname{div} (\sqrt{g} g^{-1} D) = \frac{1}{\sqrt{1+|Df|^2}} \operatorname{div} \left(\frac{1}{\sqrt{1+|Df|^2}} \begin{bmatrix} 1+f_2^2 & -f_1 f_2 \\ -f_1 f_2 & 1+f_1^2 \end{bmatrix} D \right)$$

figure mean curvature

Then $\Delta_g x_1 = 0$ implies

$$\partial_1 \left(\frac{1+f_2^2}{\sqrt{1+|Df|^2}} \right) - \partial_2 \left(\frac{f_1 f_2}{\sqrt{1+|Df|^2}} \right) \stackrel{*}{=} 0$$

and $\Delta_g x_2 = 0$ implies

$$-\partial_1 \left(\frac{f_1 f_2}{\sqrt{1+|Df|^2}} \right) + \partial_2 \left(\frac{1+f_1^2}{\sqrt{1+|Df|^2}} \right) \stackrel{*}{=} 0.$$

We'll also verify these two identities directly. It follows that

$$\begin{bmatrix} \frac{1+f_2^2}{\sqrt{1+|Df|^2}} & \frac{-f_1 f_2}{\sqrt{1+|Df|^2}} \\ \frac{-f_1 f_2}{\sqrt{1+|Df|^2}} & \frac{1+f_1^2}{\sqrt{1+|Df|^2}} \end{bmatrix} = \begin{bmatrix} DF \\ DG \end{bmatrix}.$$

As $F_2 = G_1$, there exists u such that $(F, G) = Du$. Thus the existence of the double potential u .

Direct verification of $*$.

Let $V = \sqrt{1 + |Df|^2}$,

$$\begin{aligned}
LHS &= \partial_1 \left(\frac{V^2 - f_1^2}{V} \right) - \partial_2 \left(f_1 \frac{f_2}{V} \right) \\
&= \partial_1 \left[V - f_1 \left(\frac{f_1}{V} \right) \right] - \partial_2 \left[f_1 \left(\frac{f_2}{V} \right) \right] \\
&= \partial_1 V - f_{11} \left(\frac{f_1}{V} \right) - \underbrace{f_1 \partial_1 \left(\frac{f_1}{V} \right)}_{-f_{12} \left(\frac{f_2}{V} \right)} \\
&\quad - f_{12} \left(\frac{f_2}{V} \right) - f_1 \underbrace{\partial_2 \left(\frac{f_2}{V} \right)}_{= 0} \\
&= \frac{f_1 f_{11} + f_2 f_{21}}{V} - \frac{f_1 f_{11} + f_2 f_{21}}{V} - f_1 \operatorname{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) \\
&= 0.
\end{aligned}$$

In summary:

Heinz

$$\begin{aligned}
\operatorname{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) &= 0 \text{ in } R^2 \Rightarrow \text{there exists } u \text{ such that in } R^2 \\
\det D^2 u &= 1 \text{ and } \frac{1}{\sqrt{1 + |Df|^2}} \begin{bmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{bmatrix} = D^2 u.
\end{aligned}$$

Jörgens result implies Bernstein theorem in 2d.

RMK. A by product divergence $\Delta_g = \frac{1}{\sqrt{g}} \operatorname{div} (\sqrt{g} g^{-1} D) = \sum g_{ij} \partial_{ij}$ nondivergence on minimal graphs.

Nitsche's proof of Jorgens' Th'm via Lewy rotation (1956).

Now new interpretation of Nitsche's proof. (2001)

Geometric way.

Step 1. Set-up

$$\lambda_1 \lambda_2 = 1 \Leftrightarrow \arctan \lambda_1 + \arctan \lambda_2 = \frac{\pi}{2} \text{ or } \theta_1 + \theta_2 = \frac{\pi}{2}.$$

figure $\pi/4$ rotation

Step 2. $0 < \theta_i < \frac{\pi}{2}$ graph over $x-R^2$ plane

Make a $U(2)$ rotation

$$\begin{bmatrix} e^{-\sqrt{-2}\pi/4} & \\ & e^{-\sqrt{-2}\pi/4} \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \begin{cases} z_1 = x_1 + \sqrt{-1}y_1 \\ z_2 = x_2 + \sqrt{-1}y_2 \end{cases}$$

Obs. $U(2)$ rotation preserves the Lagrangian structure i.e. J Tangent space= Normal Space or $iT = N$. This is because $iUT =UiT = UN$. Locally Lagrangian means the graph has a “gradient” structure.

Obs. This $U(2)$ rotation decreases the angles

$$-\frac{\pi}{4} < \bar{\theta}_i = \theta_i - \frac{\pi}{4} < \frac{\pi}{4} \longleftrightarrow -1 < \tan \bar{\theta}_i = \bar{\lambda}_i < 1$$

Then $(x, Du(x))$ still a graph over \bar{x} -R² plane. In fact a Lagrangian graph $(\bar{x}, D\bar{u}(\bar{x}))$ with bounded Hessian $D^2\bar{u}$.

Step 3. $\bar{\theta}_1 + \bar{\theta}_2 = 0 \Leftrightarrow \Delta \bar{u} = 0$. Also $-I < D^2\bar{u} < I$. By Liouville, \bar{u} is quadratic. Then $(\bar{x}, D\bar{u})$ is a plane, finally u is quadratic in terms of x .

RMK. In justifying the rotation $e^{-\sqrt{-1}\pi/4}$, we assumed D^2u is diagonal, this can be achieved by another $U(2)$ rotation induced from the $O(2)$ rotation on x -R² plane

$$Rx + \sqrt{-1}Ry \quad \text{or} \quad [R]_{2 \times 2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Analytic way.

Step 1. Set up $\lambda_1\lambda_2 = 1$, say $\lambda_i > 0$.

figure $\pi/4$ rotation

Step 2. Change of variables

Now $(x, Du(x))$ represented by $(\Phi(x), \Psi(x))$ in $\bar{x} - \bar{y}$ coordinate system

$$\begin{aligned} \bar{x} &= \Phi(x) = \frac{1}{\sqrt{2}}(x + Du(x)) \\ \bar{y} &= \Psi(x) = \frac{1}{\sqrt{2}}(-x + Du(x)) \end{aligned}$$

Note $\frac{\partial \Phi}{\partial x} = \frac{1}{\sqrt{2}}(I + D^2u) \geq \frac{1}{\sqrt{2}}I$. Then Φ is $\frac{1}{\sqrt{2}}$ distance expanding and an open map. It follows that

* Φ is globally 1-1 and onto from x -R² to \bar{x} -R²

* $(x, Du(x))$ is still a graph over \bar{x} -plane.

Instead of this infinitesimal argument, we argue without derivative.

$$\begin{aligned} |\bar{x}^P - \bar{x}^Q|^2 &= \frac{1}{2} |x^P - x^Q + Du(x^P) - Du(x^Q)|^2 \\ &= \frac{1}{2} |x^P - x^Q|^2 + \frac{1}{2} |Du(x^P) - Du(x^Q)|^2 + \underbrace{\langle x^P - x^Q, Du(x^P) - Du(x^Q) \rangle}_{\geq 0, \text{ since } u \text{ is convex}} \\ &\geq \frac{1}{2} |x^P - x^Q|^2 + \frac{1}{2} |Du(x^P) - Du(x^Q)|^2 = \frac{1}{2} |P - Q|^2. \end{aligned}$$

So different points P and Q have different projections on \bar{x} -plane. So $(x, Du(x))$ is still a graph over \bar{x} -plane.

Checking the Lagrangian structure

$$\operatorname{curl}_{\bar{x}} \Psi = \frac{\partial \Psi^2}{\partial \bar{x}_1} - \frac{\partial \Psi^1}{\partial \bar{x}_2} = 0 \Leftrightarrow 0 = \underbrace{d\bar{x}_1 \wedge d\Psi^1 + d\bar{x}_2 \wedge d\Psi^2}_{\bar{x} \text{ parametrization}} = \underbrace{d\Phi^1 \wedge d\Psi^1 + d\Phi^2 \wedge d\Psi^2}_{x \text{ parametrization}}.$$

Now

$$\begin{aligned} & d(x_1 + u_1) \wedge d(-x_1 + u_1) + d(x_2 + u_2) \wedge d(-x_2 + u_2) \\ &= dx_1 \wedge du_1 - du_1 \wedge dx_1 + dx_2 \wedge du_2 - du_2 \wedge dx_2 \\ &= 2(dx_1 \wedge du_1 + dx_2 \wedge du_2) \\ &= 2(u_{12}dx_1 \wedge dx_2 + u_{21}dx_2 \wedge dx_1) = 0. \end{aligned}$$

Calculating the new Hessian $D^2\bar{u}$, and another way of checking “gradient” structure.

$$\begin{aligned} \begin{bmatrix} \frac{\partial \Psi^1}{\partial \bar{x}_1} & \frac{\partial \Psi^2}{\partial \bar{x}_1} \\ \frac{\partial \Psi^1}{\partial \bar{x}_2} & \frac{\partial \Psi^2}{\partial \bar{x}_2} \end{bmatrix} &= \frac{\partial \Psi}{\partial \bar{x}} = \frac{\partial \Psi}{\partial x} \frac{\partial x}{\partial \bar{x}} = \frac{\partial \Psi}{\partial x} \left(\frac{\partial \bar{x}}{\partial x} \right)^{-1} = \frac{\partial \Psi}{\partial x} \left(\frac{\partial \Phi}{\partial x} \right)^{-1} \\ &= \underbrace{(-I + D^2u)}_{\text{symmetric, thus } \frac{\partial \Psi^1}{\partial \bar{x}_2} = \frac{\partial \Psi^2}{\partial \bar{x}_1}} \left(I + D^2u \right)^{-1} \sim \begin{bmatrix} \frac{-1+\lambda_1}{1+\lambda_1} & \frac{-1+\lambda_2}{1+\lambda_2} \end{bmatrix}. \end{aligned}$$

figure graph $\frac{-1+\lambda}{1+\lambda}$

So

$$-I < (D^2\bar{u}) = \frac{\partial \Psi}{\partial \bar{x}} < I.$$

Step 3. Equation

$$\begin{aligned} \Delta \bar{u} &= \bar{\lambda}_1 + \bar{\lambda}_2 = \frac{-1+\lambda_1}{1+\lambda_1} + \frac{-1+\lambda_2}{1+\lambda_2} = \\ &= \frac{2\lambda_1\lambda_2 - 2}{(1+\lambda_1)(1+\lambda_2)} = 0! \end{aligned}$$

We have a harmonic function with bounded Hessian on R^2 . Liouville th'm implies that \bar{u} is quadratic, then so is u .

Complex way toward Bernstein w/o Jorgens, but via Heinz and Lewy.

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Conformal way toward Bernstein via Bernstein's strange observation.

Using $U(2)$ change of variables \bar{x} in step 2 of Analytic way, the induce metric on

$X = (x_1, x_2, f(x))$ in terms of parameter \bar{x} is

$$\begin{aligned}
g(\bar{x}) &= \frac{\partial X}{\partial \bar{x}} \left(\frac{\partial X}{\partial \bar{x}} \right)^T = \frac{\partial x}{\partial \bar{x}} \frac{\partial X}{\partial x} ()^T \\
&= \underbrace{\frac{\partial x}{\partial \bar{x}} \frac{\partial X}{\partial x}}_{g(x)} \left(\frac{\partial X}{\partial x} \right)^T \left(\frac{\partial x}{\partial \bar{x}} \right)^T \\
&\stackrel{\text{Heinz}}{=} \left(\frac{I + D^2 u}{\sqrt{2}} \right)^{-1} \sqrt{1 + |Df|^2} D^2 u \left(\frac{I + D^2 u}{\sqrt{2}} \right)^{-1} \\
&= 2\sqrt{1 + |Df|^2} (I + D^2 u)^{-1} D^2 u (I + D^2 u)^{-1} \\
&\sim 2\sqrt{1 + |Df|^2} \left[\frac{\frac{\lambda_1}{(1+\lambda_1)^2}}{\frac{\lambda_2}{(1+\lambda_2)^2}} = \frac{\lambda_1^2 \lambda_2}{\lambda_1^2 (1+\lambda_2)^2} = \frac{\lambda_1}{(1+\lambda_1)^2} \right] \\
&= 2\sqrt{1 + |Df|^2} \underbrace{\frac{\lambda_1}{(1+\lambda_1)^2}}_{=\lambda_1 \geq \lambda_2} I
\end{aligned}$$

Thus

$$g(\bar{x}) = 2\sqrt{1 + |Df|^2} \frac{\lambda_1}{(1+\lambda_1)^2} I \text{ is conformally flat.}$$

Bernstein

$$\begin{aligned}
0 &= L \arctan f_1 = \Delta_g \arctan f_1 = \Delta_{\bar{g}} \arctan f_1 \\
&= \frac{1}{\sqrt{\bar{g}}} \Delta_{\bar{x}} \arctan f_1.
\end{aligned}$$

The bounded \bar{x} -harmonic function $\arctan f_1$ is constant.

Verification of $L \arctan f_1 = 0$ from the minimal surface equation

$$Lf = (1 + f_2^2) f_{11} - 2f_1 f_2 f_{12} + (1 + f_1^2) f_{22} = 0.$$

Bernstein's way of computation is take derivative of the equation with respect to x_1 and eliminate the f_{22} term in the resulting equation by the equation:

$$(1 + f_2^2) f_{111} - 2f_1 f_2 f_{121} + (1 + f_1^2) f_{221} + 2\underbrace{f_2 f_{21} f_{11}}_{\rightarrow} - 2 \left(\underbrace{f_{11} f_2}_{\rightarrow} + \underbrace{f_1 f_{21}}_{\rightarrow} \right) f_{12} + 2f_1 f_{11} f_{22} = 0$$

and

$$(1 + f_2^2) \left(f_{111} - \frac{2f_1 f_{11} f_{11}}{1 + f_1^2} \right) - 2f_1 f_2 \left(f_{121} - \frac{2f_1 f_{11} f_{12}}{1 + f_1^2} \right) + (1 + f_1^2) \left(f_{221} - \frac{2f_1 f_{21} f_{12}}{1 + f_1^2} \right) = 0.$$

Divide by $(1 + f_1^2)$, we see $L \arctan f_1 = 0$.

Formulas related to submanifolds, in particular minimal surfaces

Laplace

$$\begin{aligned}\Delta_g u &= \operatorname{div}_g (\operatorname{grad}_g u) = \operatorname{div}_g (g^{1j} u_j, g^{2j} u_j, \dots, g^{nj} u_j) \\ &= \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} u_j) \\ &= g^{ij} u_{ij} + \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij}) u_j\end{aligned}$$

$$\begin{aligned}\Delta_g u &= \operatorname{tr}_g \nabla^2 u = \operatorname{tr}_g [(u_{ij} - \Gamma_{ij}^k u_k) dx^i \otimes dx_j] \\ &= g^{ij} u_{ij} - g^{ij} \Gamma_{ij}^k u_k\end{aligned}$$

Invariance of $|\nabla_g u|^2$ and $\sqrt{g} dp$, then Δ_g .

Mean curvature formula

$$\begin{aligned}\vec{H} &= \Delta_g X = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j X) \in NM \\ &= g^{ij} \partial_{ij} X - g^{ij} \Gamma_{ij}^k \partial_k X \\ &= (g^{ij} \partial_{ij} X)^N,\end{aligned}$$

where $g = \det(g_{ij})$ and the induced metric

$$g_{ij} = \langle \partial_i X, \partial_j X \rangle$$

for submanifold $M \subset R^{n+k}$ (locally) parametrized by

$$\begin{aligned}X : M &\rightarrow R^{n+k} \\ (p_1, \dots, p_n) &\mapsto X(p_1, \dots, p_n),\end{aligned}$$

for example $X = (p_1, \dots, p_n, F(p))$.

Invariance of $|\nabla_g u|^2$ and $\sqrt{g} dp$, then Δ_g .

For a difference parametrization or coordinates q

$$g(q) = \langle X_{q_i}, X_{q_j} \rangle = \left(\frac{\partial p}{\partial q} \right)' X_p (X_p)' \frac{\partial p}{\partial q} = \left(\frac{\partial p}{\partial q} \right)' g(p) \frac{\partial p}{\partial q}$$

(here position/vectors go horizontally, and derivatives go vertically) then

$$\sqrt{g(q)} dq = \sqrt{g(p)} \det \frac{\partial p}{\partial q} dq = \sqrt{g(p)} dp$$

and

$$\begin{aligned}|\nabla_g u|^2(q) &= u_{q_i} g^{ij}(q) u_{q_j} = (u_p)' \left(\frac{\partial p}{\partial q} \right) g^{-1}(q) \left(\frac{\partial p}{\partial q} \right)' u_p \\ &= (u_p)' g^{-1}(p) u_p = |\nabla_g u|^2(p).\end{aligned}$$

Consequently

$$\Delta_{g(q)} u(q) = \Delta_{g(p)} u(p)$$

as critical point of the invariant energy functional $\int |\nabla_g u|^2 dv_g = \int |\nabla_g u(p)|^2 \sqrt{g(p)} dp$. That the first variation $u + \varepsilon\varphi$ with compactly support φ leads

$$\begin{aligned} & \frac{d}{d\varepsilon} \int |\nabla_g(u + \varepsilon\varphi)|^2 dv_g \\ &= \frac{d}{d\varepsilon} \int g^{ij}(u_i + \varepsilon\varphi_i)(u_j + \varepsilon\varphi_j) \sqrt{g} dp \\ &\stackrel{\varepsilon=0}{=} 2 \int g^{ij}\varphi_i u_j \sqrt{g} dp \\ &= -2 \int \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} g^{ij} u_j) \varphi \sqrt{g} dp \\ &= -2 \int \Delta_g u \varphi dv_g. \end{aligned}$$

Variation way

$$vol(M) = \int \sqrt{g} dp.$$

First variation $X + \varepsilon\Phi(p)$ or $\frac{d}{d\varepsilon}X(p, \varepsilon) = \Phi(p) \in C_0^\infty(R^n, R^{n+k})$

$$\begin{aligned} \frac{d}{d\varepsilon} vol(M_\varepsilon) &= \int \frac{d}{d\varepsilon} \sqrt{g(\varepsilon)} dp \\ &= \int \frac{1}{2} \frac{1}{\sqrt{g}} gg^{ij} \frac{d}{d\varepsilon} g_{ij} dp \\ &= \int \frac{1}{2} \frac{1}{\sqrt{g}} gg^{ij} \frac{d}{d\varepsilon} \langle \partial_i X, \partial_j X \rangle dp = \int \sqrt{g} g^{ij} \left\langle \frac{d}{d\varepsilon} X_i, X_j \right\rangle dp \\ &= \int \langle \Phi_i, \sqrt{g} g^{ij} X_j \rangle dp \\ &= \int - \left\langle \Phi, \frac{1}{\sqrt{g}} \partial_i(\sqrt{g} g^{ij} X_j) \right\rangle \sqrt{g} dp. \end{aligned}$$

Thus (the defined) mean curvature formula.

Geometric way

$$0 = Hess X = D^2 X$$

This is true in all coordinates such as spherical one, or the curved one (p_1, \dots, p_n) .

$$\begin{aligned} 0 &= D_{\partial_{p_i} \partial_{p_j}}^2 X = (\partial_{p_i} \partial_{p_j} - D_{\partial_{p_i}} \partial_{p_j}) X = (\partial_{p_i} \partial_{p_j} - \nabla_{\partial_{p_i}} \partial_{p_j} - II(\partial_{p_i}, \partial_{p_j})) X \\ &= \nabla_{\partial_{p_i} \partial_{p_j}}^2 X - II(\partial_{p_i}, \partial_{p_j}) X = (\partial_{ij} - \Gamma_{ij}^k \partial_k) X - II(\partial_{p_i}, \partial_{p_j}) X. \end{aligned}$$

Thus

$$\nabla_{\partial_{p_i} \partial_{p_j}}^2 X = II(\partial_{p_i}, \partial_{p_j}) X = II(\partial_{p_i}, \partial_{p_j}) \in NM.$$

Note $\partial_e X = e$ for any vector in R^{n+k} . Take trace with respect to the metric g on M , one has

$$g^{ij} (\partial_{ij} X - \Gamma_{ij}^k \partial_k X) = \Delta_g X = \vec{H}.$$

Consistency of different formulations of Δ_g

$$\begin{aligned} \Delta_g X &= \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j X) = g^{ij} \partial_{ij} X + \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij}) \partial_j X \\ &= g^{ij} \partial_{ij} X - g^{ij} \Gamma_{ij}^k \partial_k X. \end{aligned}$$

This is because

$$\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij}) = -g^{kl} \Gamma_{kl}^j$$

which can be seen from the following intrinsic calculation.

$$\begin{aligned} \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij}) &= \frac{1}{\sqrt{g}} \partial_i (\sqrt{g}) g^{ij} + \partial_i (g^{ij}) = \frac{1}{2} (\partial_i \ln g) g^{ij} + \partial_k (g^{kj}) \\ &= \frac{1}{2} g^{kl} \partial_i g_{kl} g^{ij} - \underbrace{g^{kl} \partial_k g_{lb} g^{bj}}_{=0} = \frac{1}{2} g^{kl} \partial_b g_{kl} g^{bj} - \underbrace{\frac{1}{2} g^{kl} \partial_k g_{lb} g^{bj} - \frac{1}{2} g^{lk} \partial_l g_{kb} g^{bj}}_{=0} \\ &= g^{kl} \left[-\frac{1}{2} g^{jb} (\partial_k g_{lb} + \partial_l g_{kb} - \partial_b g_{kl}) \right] = -g^{kl} \Gamma_{kl}^j. \end{aligned}$$

Another intrinsic calculation

$$\begin{aligned} \underbrace{\frac{1}{\sqrt{g}} \partial_i (\sqrt{g}) g^{ij} + \partial_i (g^{ij})}_{=0} &= \frac{1}{2} g^{kl} \partial_i g_{kl} g^{ij} - g^{kl} \partial_k g_{lb} g^{bj} \\ &= \left[\frac{1}{2} g^{kl} (\Gamma_{ik}^a g_{al} + \Gamma_{il}^a g_{ka}) \right] g^{ij} - g^{kl} (\Gamma_{kl}^a g_{ab} + \Gamma_{kb}^a g_{al}) g^{bj} \\ &= \underbrace{[\Gamma_{ia}^a]}_{=0} g^{ij} - g^{kl} \Gamma_{kl}^j - \Gamma_{ab}^a g^{bj} = -g^{kl} \Gamma_{kl}^j. \end{aligned}$$

RMK. Derivation for Γ_{kl}^j :

$$\begin{aligned} \partial_i g_{kl} &= \partial_i \langle \partial_k, \partial_l \rangle = \langle \nabla_{\partial_i} \partial_k, \partial_l \rangle + \langle \partial_k, \nabla_{\partial_i} \partial_l \rangle \\ &= \langle \nabla_{\partial_k} \partial_i, \partial_l \rangle + \langle \partial_k, \nabla_{\partial_l} \partial_i \rangle \\ &= \partial_k \langle \partial_i, \partial_l \rangle - \langle \partial_i, \nabla_{\partial_k} \partial_l \rangle + \partial_l \langle \partial_k, \partial_i \rangle - \langle \nabla_{\partial_l} \partial_k, \partial_i \rangle \\ &= \partial_k g_{il} + \partial_l g_{ki} - 2\Gamma_{kl}^m g_{im}. \end{aligned}$$

multiplying both sides by g^{ji} yields

$$\Gamma_{kl}^j = \frac{1}{2} g^{ji} (\partial_k g_{il} + \partial_l g_{ki} - \partial_i g_{kl}).$$

Another geometric way to justify

$$\Delta_g X \in NM.$$

Note $\Delta_g X$ is invariant under Euclidean rotation and translation, we choose a tangent coordinate system near any point o

$$X = (p_1, \dots, p_n, F(p)) \in R^{n+k} \text{ with}$$

$$DF(o) = 0, \text{ and } g_{ij} = 1 + O(1)p^2$$

Then

$$\Delta_g X(o) = (0, \dots, 0, \Delta F) \in NM.$$

Divergence

$$\operatorname{div}_M Z = \operatorname{div}_g Z = \langle D_{e_i} Z, e_i \rangle = g^{ij} \langle D_{\partial_{p_i}} Z, \partial_{p_j} \rangle = \frac{1}{\sqrt{g}} \partial_{p_i} (\sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle) - \langle Z, \vec{H} \rangle,$$

where Z is any vector field on the ambient space (not necessarily R^{n+k}), $\{e_1, \dots, e_n\}$ orthonormal basis for TM .

Recall

$$e_i = (g^{-1/2})^{ai} \partial_{p_a}$$

then

$$\langle D_{e_i} Z, e_i \rangle = (g^{-1/2})^{ai} \langle D_{\partial_{p_a}} Z, \partial_{p_b} \rangle (g^{-1/2})^{bi} = (g^{-1})^{ab} \langle D_{\partial_{p_a}} Z, \partial_{p_b} \rangle = g^{ab} \langle D_{\partial_{p_a}} Z, \partial_{p_b} \rangle.$$

Thus we already see the invariance of $\operatorname{div}_g Z$ under coordinate change, as in the invariance of $|\nabla_g u| = g^{ij} u_i u_j$.

Further,

$$\begin{aligned} g^{ij} \langle D_{\partial_{p_i}} Z, \partial_{p_j} \rangle &= g^{ij} (\partial_{p_i} \langle Z, \partial_{p_j} \rangle - \langle Z, D_{\partial_{p_i}} \partial_{p_j} \rangle) \\ &= g^{ij} [\partial_{p_i} \langle Z, \partial_{p_j} \rangle - \langle Z, \partial_{p_k} \rangle \Gamma_{ij}^k - \langle Z, II_{ij} \rangle] \\ &= g^{ij} \partial_{p_i} \langle Z, \partial_{p_j} \rangle - g^{ik} \langle Z, \partial_{p_j} \rangle \Gamma_{ik}^j - \langle Z, \vec{H} \rangle \\ &= g^{ij} \partial_{p_i} \langle Z, \partial_{p_j} \rangle + \frac{1}{\sqrt{g}} \partial_{p_i} (\sqrt{g} g^{ij}) \langle Z, \partial_{p_j} \rangle - \langle Z, \vec{H} \rangle \text{ by "consistency"} \\ &= \frac{1}{\sqrt{g}} \partial_{p_i} (\sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle) - \langle Z, \vec{H} \rangle. \end{aligned}$$

Eg1. If M is the whole ambient space, then there is no II or \vec{H} term.

Eg2. If M^n is the level set of a scalar function $u(x_1, \dots, x_n)$ in R^{n+1} , then with $N = Du / |Du|$ and e_1, \dots, e_n being orthonormal tangent vectors to M^n , we have

$$\begin{aligned} \operatorname{div}_{R^{n+1}} \frac{Du}{|Du|} &= \sum_{i=1}^n \langle D_{e_i} N, e_i \rangle + \overrightarrow{\langle D_N N, N \rangle}^0 = \operatorname{div}_M N \\ &= \frac{1}{\sqrt{g}} \partial_{p_i} (\sqrt{g} g^{ij} \langle N, \partial_{p_j} \rangle) - \langle N, \vec{H} \rangle \\ &= -H. \end{aligned}$$

Divergence theorem

$$\int_M \operatorname{div}_M Z \ dv_g = \int_{\partial M} \langle Z, N_g \rangle \ dA_g$$

Indeed

$$\begin{aligned} \int_M \operatorname{div}_M Z \ dv_g &= \int_{\Omega} \frac{1}{\sqrt{g}} \partial_{p_i} (\sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle) \sqrt{g} dp = \int_{\Omega} \partial_{p_i} (\sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle) dp \\ &= \int_{\partial\Omega} \sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle (-1)^{i-1} dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n = \int_{\partial\Omega} \sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle N_i dA, \end{aligned}$$

where $N = (N_1, \dots, N_n)$ is unit outward normal of $\partial\Omega$ w.r.t. the “Euclidean” metric; recall

$$dA = i_N dp = N_i (-1)^{i-1} dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n;$$

also

$$N_i dA = (-1)^{i-1} dp_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n.$$

Now let

$$N_g = \frac{g^{ij} N_i \partial_{p_j}}{\sqrt{g^{ab} N_a N_b}},$$

then

$$\begin{aligned} dA_g &= i_{N_g} dv_g = \sqrt{g} i_{N_g} dp = \frac{\sqrt{g}}{\sqrt{g^{ab} N_a N_b}} g^{ij} N_i \underbrace{(-1)^{j-1} dp_1 \wedge \cdots \wedge \widehat{dp_j} \wedge \cdots \wedge dp_n}_{dA} \\ &= \frac{\sqrt{g}}{\sqrt{g^{ab} N_a N_b}} g^{ij} N_i \underbrace{N_j dA}_{dA} = \sqrt{g^{ab} N_a N_b} \sqrt{g} dA \end{aligned}$$

thus

$$\int_{\partial\Omega} \sqrt{g} g^{ij} \langle Z, \partial_{p_j} \rangle N_i dA = \int_{\partial\Omega} \langle Z, N_g \rangle \sqrt{g^{ab} N_a N_b} \sqrt{g} dA = \int_{\partial M} \langle Z, N_g \rangle dA_g.$$

Gauss and Codazzi equation via decomposition along tangent and normal direction.

Given a hypersurface $X(p)$ in R^{n+1} , then

$$X_{ij} = D_{X_{p_i}} X_{p_j} = \langle X_{ij}, X_a \rangle g^{ab} X_b + \langle X_{ij}, N \rangle N = \Gamma_{ij}^b X_b + II_{ij} N.$$

Already

$$\begin{aligned} II_{ij} N &= X_{ij} - \Gamma_{ij}^b X_b \text{ and} \\ \vec{H} &= g^{ij} II_{ij} N = g^{ij} (X_{ij} - \Gamma_{ij}^k X_k) = \Delta_g X. \end{aligned}$$

Also

$$\Gamma_{ij}^b = \langle X_{ij}, X_a \rangle g^{ab} = \frac{1}{2} [\partial_i g_{ja} + \partial_j g_{ia} - \partial_a g_{ij}] g^{ab}$$

as

$$\begin{aligned}\langle X_{ij}, X_a \rangle &= \partial_j \langle X_i, X_a \rangle - \langle X_i, X_{aj} \rangle = \partial_j \langle X_i, X_a \rangle - \partial_a \langle X_i, X_j \rangle + \langle X_{ia}, X_j \rangle \\ &= \partial_j \langle X_i, X_a \rangle - \partial_a \langle X_i, X_j \rangle + \partial_i \langle X_a, X_j \rangle - \langle X_a, X_{ij} \rangle.\end{aligned}$$

Take one more derivative with respect to p_k , we have

$$\begin{aligned}X_{ijk} &= \partial_k \Gamma_{ij}^a X_a + \Gamma_{ij}^b X_{bk} + \partial_k II_{ij} N + II_{ij} \partial_k N \\ &= (\Gamma_{ij}^b II_{bk} + \partial_k II_{ij}) N \\ &\quad + (\partial_k \Gamma_{ij}^a + \Gamma_{ij}^b \Gamma_{bk}^a) X_a + II_{ij} \langle \partial_k N, X_b \rangle g^{ba} X_a \\ &= (\Gamma_{ij}^b II_{bk} + \partial_k II_{ij}) N \\ &\quad + (\partial_k \Gamma_{ij}^a + \Gamma_{ij}^b \Gamma_{bk}^a) X_a - II_{ij} II_{bk} g^{ba} X_a.\end{aligned}$$

Switch j and k ,

$$\begin{aligned}X_{ikj} &= (\Gamma_{ik}^b II_{bj} + \partial_j II_{ik}) N \\ &\quad + (\partial_j \Gamma_{ik}^a + \Gamma_{ik}^b \Gamma_{bj}^a) X_a - II_{ik} II_{bj} g^{ba} X_a.\end{aligned}$$

Equate the normal and tangential part, we have Codazzi and Gauss equations:

$$\begin{aligned}\partial_k II_{ij} + \Gamma_{ij}^b II_{bk} &= \partial_j II_{ik} + \Gamma_{ik}^b II_{bj} \quad \text{or} \\ \partial_k II_{ij} - \Gamma_{ik}^b II_{bj} - \underline{\Gamma_{kj}^b II_{ib}} &= \partial_j II_{ik} - \Gamma_{ij}^b II_{bk} - \underline{\Gamma_{jk}^b II_{ib}}\end{aligned}$$

and

$$\partial_k \Gamma_{ij}^a + \Gamma_{ij}^b \Gamma_{bk}^a - \partial_j \Gamma_{ik}^a - \Gamma_{ik}^b \Gamma_{bj}^a = II_{ij} II_{bk} g^{ba} - II_{ik} II_{bj} g^{ba}$$

Riemann Curvature

$$\begin{aligned}R_{1234} &= g_{4l} (\Gamma_{23,1}^l - \Gamma_{13,2}^l + \Gamma_{23}^m \Gamma_{1m}^l - \Gamma_{13}^m \Gamma_{2m}^l) \\ &\stackrel{o}{=} \Gamma_{23,1}^4 - \Gamma_{13,2}^4 \quad \text{tangent/normal coordinates} \\ &\stackrel{o}{=} \frac{1}{2} (g_{13,24} + g_{24,13} - g_{23,14} - g_{14,23}) \quad \text{tangent coordinates} \\ &\stackrel{o}{=} g_{13,24} - g_{23,14} \quad \text{normal coordinates}\end{aligned}$$

Metric

$$\begin{aligned}g_{ij}(x) &= \delta_{ij} + \frac{1}{2} g_{ij,kl} x^k x^l + O(|x|^3) \quad \text{tangent coordinates} \\ &= \delta_{ij} + \frac{1}{3} R_{ikjl} x^k x^l + O(|x|^3) \quad \text{normal coordinates}\end{aligned}$$

Monotonicity or mean value equality

on minimal surfaces
Suppose $M^n \subset R^{n+k}$ is a minimal surface and u is a (smooth) function on M^n , then we have

$$\frac{1}{s^n} \int_{B_s \cap M} u dv_g \Big|_{s_1}^{s_2} = \int_{(B_{s_2} \setminus B_{s_1}) \cap M} \frac{|X^N|^2}{|X|^{n+2}} u dv_g + \int_{s_1}^{s_2} \frac{1}{2s^{n+1}} \int_{B_s \cap M} (s^2 - |X|^2) \Delta_g u dv_g,$$

where $B_s = B_s^{n+k}$ is a ball in R^{n+k} with radius s and center 0 and X^N is the normal projection of the position vector X . Note the center could be anywhere X_0 , and in this case, X is replaced by $X - X_0$.

RMK. When M^n is R^n , certainly this is the mean value equality for Laplace equation.

We follow the argument for Laplace equation starting from $\Delta_g |X|^2 = 2 \langle \Delta_g X, X \rangle + 2 \langle \nabla_g X, \nabla_g X \rangle = 2n$.

$$\begin{aligned} 2n \int_{B_s \cap M} u dv_g &= \int_{B_s \cap M} u \Delta_g |X|^2 dv_g \\ &= \int_{B_s \cap M} |X|^2 \Delta_g u dv_g + \int_{\partial(B_s \cap M)} u \partial_{\gamma_g} |X|^2 - |X|^2 \partial_{\gamma_g} u dA_g \\ &= \int_{B_s \cap M} (|X|^2 - s^2) \Delta_g u dv_g + \int_{\partial(B_s \cap M)} u \left\langle \nabla_g |X|^2, \frac{\nabla_g |X|}{|\nabla_g |X||} \right\rangle dA_g \\ &= \dots + \int_{\partial(B_s \cap M)} u \left\langle 2X^T, \frac{X^T}{|X^T|} \right\rangle dA_g \\ &= \int_{B_s \cap M} (|X|^2 - s^2) \Delta_g u dv_g + \int_{\partial(B_s \cap M)} 2u |X^T| dA_g. \end{aligned}$$

We now calculate the changing rate of the "mean" value with respect to s . To take the derivative of the integral, we use co-area formula with level function $s = |X|$

$$\begin{aligned} \frac{d}{ds} \left[\frac{1}{s^n} \int_{B_s \cap M} u dv_g \right] &= -ns^{-n-1} \int_{B_s \cap M} u dv_g + s^{-n} \frac{d}{ds} \int^s \int_{\{|X|=s\} \cap M} u \frac{1}{|\nabla_g |X||} dA_g \\ &= -ns^{-n-1} \int_{B_s \cap M} u dv_g + s^{-n} \int_{\partial(B_s \cap M)} u \frac{1}{|\nabla_g |X||} dA_g. \end{aligned}$$

We proceed by replacing solid integral with the above resulting identity after Green's formula,

$$\begin{aligned} \frac{d}{ds} \left[\frac{1}{s^n} \int_{B_s \cap M} u dv_g \right] &= \frac{1}{2s^{n+1}} \int_{B_s \cap M} (s^2 - |X|^2) \Delta_g u dv_g + \int_{\partial(B_s \cap M)} \frac{-u |X^T|}{s^{n+1}} + u \frac{1}{s^n} \frac{1}{|\nabla_g |X||} dA_g \\ &= \dots + \int_{\partial(B_s \cap M)} \frac{-|X^T| |\nabla_g |X|| + s}{s^{n+1}} \frac{u}{|\nabla_g |X||} dA_g \\ &= \dots + \int_{\partial(B_s \cap M)} \frac{-|X^T|^2 + s^2}{s^{n+2}} \frac{u}{|\nabla_g |X||} dA_g \\ &= \frac{1}{2s^{n+1}} \int_{B_s \cap M} (s^2 - |X|^2) \Delta_g u dv_g + \int_{\partial(B_s \cap M)} \frac{|X^N|^2}{|X|^{n+2}} u \frac{1}{|\nabla_g |X||} dA_g. \end{aligned}$$

Integrate with respect to s , we obtain the mean value identity.

Eg1. When $u = 1$, we have the usual increasing monotonicity of the density of the minimal surface.

Eg2. When the density $\frac{1}{s^n} \int_{B_s \cap M} u dv_g$ is constant, then the minimal surface is a cone, with vertex at X_0 .