Lecture 4 Mean curvature flow, monotonicity, and self shrinking solutions

○ mean curvature flow
○ monotonicity formula
○ self shrinking solutions

Mean curvature flow

Recall the area/volume variation E for hypersurface/surface in Euclidean space

$$
\begin{aligned}
\frac{d}{d\varepsilon} vol\left(M_\varepsilon\right)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_M dv_{g(\varepsilon)} = \int_M \left\langle E_i, g^{ij} X_j \right\rangle dv_g \\
&= \int_M \operatorname{div}_M E dv_g = \int_M \operatorname{div}_M E^T + \operatorname{div}_M E^N dv_g \\
&= \int_M \operatorname{div}_M E^T - \left\langle E^N, \frac{1}{\sqrt{g}}\partial_i \left(\sqrt{g} g^{ij} X_j\right) \right\rangle dv_g.
\end{aligned}
$$

We deform the surface along the negative gradient of the volume

$$
X_t = \triangle_g X = \frac{1}{\sqrt{g}}\partial_i \left(\sqrt{g} g^{ij} X_j\right) = g^{ij} X_{ij} - g^{ij}\Gamma_{ij}^k X_k.
$$

The volume element changes as

$$
\frac{d}{dt}\sqrt{g} = \operatorname{div}_{M_t} X_t \cdot \sqrt{g} = -\left\langle H, \triangle_g X \right\rangle \sqrt{g} = -\left|\triangle_g X\right|^2 \sqrt{g}.
$$

We call this the mean curvature flow.

For example, in the graph setting, if we insist the equation for the height $(x, f(x,t)) \subset R^{n+1}$, the effect deformation, namely the normal projection has to be exactly the mean curvature

$$
\left(\triangle_g X, \frac{(-Df, 1)}{\sqrt{1+|Df|^2}} \right) = \operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}} \right).
$$

Tracing back, the height changing rate has to be "larger"

$$
\begin{aligned}
f_t &= \sqrt{1+|Df|^2}\operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}} \right) \\
&= \sum \left(\delta_{ij} - \frac{f_i f_j}{1+|Df|^2} \right) D_{ij} f = \sum g^{ij} D_{ij} f.
\end{aligned}
$$

This parabolic equation is in fact strictly parabolic. Recall the grim reaper solution to this equation in R^2 : $f(x,t) = t - \ln\cos x$.

Degenerate parabolic system

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The mean curvature flow system has the following form under parametrization $X(p, t)$

$$X_t = g^{ij} \left[X_{ij} - \Gamma^b_{ij} X_b \right] = g^{ij} \left[X_{ij} - \langle X_{ij}, X_a \rangle g^{ab} X_b \right]$$

from the decomposition

$$X_{ij} = D_{X_{p_i}} X_{p_j} = \langle X_{ij}, X_a \rangle g^{ab} X_b + \langle X_{ij}, N \rangle N = \Gamma^b_{ij} X_b + II_{ij} N.$$

From this we see the equation is degenerate along tangential directions, or the principle symbol could vanish. For example, manipulate the tangentials X_a or reparametrize, we make the coefficient of X_{11} zero at a point.

In order to get short time existence for the degenerate parabolic mean curvature flow system, one trick is to add the tangential part $g^{ij} \Gamma^b_{ij} X_b$ to the "effective" mean curvature flow, then we work with the nondegenerate parabolic system

$$X_t = g^{ij} X_{ij}.$$

Apparently, this trick is already hidden in the evolution equation of the height of the above insisted non-parametric or graph representation $(x, f(x, t))$.

We first solve the strictly parabolic system

$$\begin{cases} Y_t(x, t) = g^{ij}(x, t) Y_{x_i x_j}(x, t) \\ \quad Y(x, 0) = X(x) = X(p) \end{cases}$$

to get a short time solution $Y(x, t)$. Then we solve the ODE system

$$\begin{cases} \frac{dx_k}{dt} = -g^{ij}(x) \Gamma^k_{ij}(x) = -g^{ij}(x) \langle Y_{x_i x_j}, Y_{x_a} \rangle g^{ak}(x) \\ x_l(0) = p \end{cases}.$$

Now the re-parametrized surface $Y(x(p, t), t) = X(p, t)$ satisfies

$$\begin{aligned} X_t(p, t) &= \frac{d}{dt} Y(x(p, t), t) = Y_t(x(p, t), t) + D_{x_k} Y \frac{dx_k}{dt} \\ &= g^{ij}(x, t) Y_{x_i x_j}(x, t) - D_{x_k} Y \, g^{ij}(x) \Gamma^k_{ij}(x) \\ &= \triangle_{g(x, t)} Y(x(p, t), t) \\ &= \triangle_{g(p, t)} X(p, t) \quad \text{(invariance of } \triangle_g \text{)} \end{aligned}$$

as desired.

It is important to recognize that the first non-divergent parabolic system and the ODE system are both invariant under the orthogonal change of variable in R^{n+k}, which we take as the tangent coordinates of the hypersurface (surface). Accordingly, we assume the initial hypersurface/surface $X(p, t)$ is parametrized globally by tangent coordinates. At least in the compact and co-dimension one graph cases, everything fits together. In general noncompact complete case, certain assumptions should be thrown in.

Lastly let us make co-dimension one graph case $X = (x, f(x, t))$ explicit. As before, the strictly parabolic system becomes a single equation

$$f_t(x, t) = \sum \left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2} \right) D_{ij} f.$$

From the decomposition

$$\Gamma_{ij}^k X_k = X_{x_i x_j} - II_{ij} N = (0, f_{ij}) - \frac{f_{ij}}{\sqrt{1 + |Df|^2}} \frac{(-Df, 1)}{\sqrt{1 + |Df|^2}}$$

$$= \frac{f_{ij}}{1 + |Df|^2} \left(Df, |Df|^2\right) = \frac{f_{ij}}{1 + |Df|^2} f_k X_{x_k}$$

we have

$$\Gamma_{ij}^k (x, t) = \frac{f_{ij}}{1 + |Df|^2} f_k.$$

Then the ODE system becomes

$$\frac{dx_k}{dt} = -\left(\delta_{ij} - \frac{f_i f_j}{1 + |Df|^2}\right) \frac{f_{ij}}{1 + |Df|^2} f_k = -\frac{f_t}{1 + |Df|^2} f_k.$$

Monotonicity for mean curvature flow.

Recall the familiar heat equation $\triangle u - u_t = 0$ in $R^n \times (0, \infty)$, the solution (with quadratic exponential growth) can be determined from all previous time $t \in [0, t_0)$ via convolution with heat kernel

$$u(x_0, t_0) = \int_{R^n} u(x, t) \frac{1}{[4\pi (t_0 - t)]^{n/2}} \exp\left(\frac{|x_0 - x|^2}{-4(t_0 - t)}\right) dx$$

$$= \int_{R^n} u(x, t) \Phi(x - x_0, t - t_0) dx,$$

where we denote the heat kernel term in terms of the backward heat kernel $\triangle_{R^n} \Phi + \Phi_t = 0$. This representation is independent of the time t, that is $\frac{d}{dt} u(x_0, t_0) = 0$. Let us also take the time derivative of the right hand side

$$\frac{d}{dt} \int_{R^n} u(x, t) \Phi(x - x_0, t - t_0) dx = \int_{R^n} u_t \Phi + u \Phi_t dx = \int_{R^n} u_t \Phi - u \triangle \Phi dx$$

$$= \int_{R^n} (u_t - \triangle u) \Phi dx = 0.$$

We follow this route to derive Huisken's monotonicity formula for mean curvature flow of n-dim hypersurface (same for any codimensions) in $R^{n+1} \times (0, T)$.

Huisken. Let $M_t = X(p, t) \subset R^{n+1}$ be a smooth solution (compact or noncompact complete) to the mean curvature flow $X_t = \triangle_g X$ for $t \in (0, T)$. Let $u(p, t)$ be a smooth function on M_t for $t \in (0, T)$. Then we have for $t < t_0$

$$\frac{d}{dt} \int_{M_t} u(p, t) \Phi(X - X_0, t - t_0) dv_g = \int_{M_t} \left\{ (u_t - \triangle_g u) - \left| \triangle_g X + \frac{(X - X_0)^N}{2(t_0 - t)} \right|^2 \right\} \Phi dv_g,$$

where we assume all integrals are finite, in particular they are so when M_t is compact.

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For simplicity of notation, let us assume $X_0 = 0$ and $t_0 = 0$. Thus $t < 0$ from now on. we calculate

$$\frac{d}{dt} \int_{M_t} u(p,t) \Phi(X,t) \, dv_g = \int_{M_t} \left(u_t \Phi + u\frac{d}{dt}\Phi - u\Phi |H|^2 \right) dv_g.$$

We convert time derivative of $\Phi(X,t) = (-4\pi t)^{-n/2} \exp\left(|X|^2/4t\right)$ to space derivatives. Note now

$$(\triangle_{R^{n+1}} + \partial_t) \frac{\Phi}{\sqrt{-t}} = 0.$$

We continue

$$\frac{d}{dt}\Phi(X,t) = \partial_t \left[\frac{1}{(-t)^{1/2}} \Phi \cdot (-t)^{1/2} \right] + \langle D\Phi, X_t \rangle_{R^{n+1}}$$

$$= -\triangle_{R^{n+1}} \left[\frac{1}{(-t)^{1/2}} \Phi \right] \cdot (-t)^{1/2} + \frac{1}{(-t)^{1/2}} \Phi \cdot \frac{1}{2}(-t)^{\frac{1}{2}-1}(-1) + \langle D\Phi, X_t \rangle_{R^{n+1}}$$

$$= -\triangle_{R^{n+1}} \Phi + \frac{\Phi}{2t} + \langle D\Phi, X_t \rangle$$

$$= -\left[\triangle_g \Phi - \left\langle D\Phi, \vec{H} \right\rangle + \left\langle D^2\Phi N, N \right\rangle \right] + \frac{\Phi}{2t} + \left\langle D\Phi, \vec{H} \right\rangle$$

$$= -\triangle_g \Phi + 2\left\langle D\Phi, \vec{H} \right\rangle - \left\langle D^2\Phi N, N \right\rangle + \frac{\Phi}{2t}.$$

Then

$$u_t\Phi + u\frac{d}{dt}\Phi - u\Phi \left|\vec{H}\right|^2 = u_t\Phi - u\triangle_g\Phi + u\left[2\left\langle D\Phi, \vec{H} \right\rangle - \Phi\left|\vec{H}\right|^2 - \left\langle D^2\Phi N, N \right\rangle + \frac{\Phi}{2t} \right]$$

$$= u_t\Phi - u\triangle_g\Phi + u\left[-\left|\vec{H} - \frac{D^N\Phi}{\Phi}\right|^2 \Phi + \frac{\left|D^N\Phi\right|^2}{\Phi} - \left\langle D^2\Phi N, N \right\rangle + \frac{\Phi}{2t} \right]$$

$$= u_t\Phi - u\triangle_g\Phi - u\left|\vec{H} - \frac{D^N\Phi}{\Phi}\right|^2 \Phi,$$

where the last three terms in the second line sum up to 0. At the current point X, we rotated coordinates so that $N = \partial_{x_{n+1}}$ and $\partial_{x_1}, \cdots \partial_{x_n}$ are along the tangent space. Then

$$\frac{\left|D^N\Phi\right|^2}{\Phi} - \left\langle D^2\Phi N, N \right\rangle + \frac{\Phi}{2t} = \frac{\left(\Phi_{x_{n+1}}\right)^2}{\Phi} - \Phi_{x_{n+1},x_{n+1}} + \frac{\Phi}{2t}$$

$$= \frac{\left(\Phi\frac{x_{n+1}}{2t}\right)^2}{\Phi} - \left[\frac{\Phi}{2t} + \Phi\left(\frac{x_{n+1}}{2t}\right)^2 \right] + \frac{\Phi}{2t}$$

$$= 0.$$

RMK. For k co-dimension submanifolds in R^{n+k}, the normal N means $N^1, \cdots, N^k$ or $\partial_{x_{n+1}}, \cdots, \partial_{x+k}$ at this particular point under the tangent coordinates.

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We collect all terms together

$$\frac{d}{dt}\int_{M_t} u\left(p,t\right)\Phi\left(X,t\right)dv_g = \int_{M_t}\left[u_t\Phi - u\triangle_g\Phi - u\left|\vec{H} - \frac{D^N\Phi}{\Phi}\right|^2\Phi\right]dv_g$$

$$= \int_{M_t}\left[u_t\Phi - \Phi\triangle_g u - u\left|\vec{H} - \frac{D^N\Phi}{\Phi}\right|^2\Phi\right]dv_g$$

$$\overset{IBP}{=} \int_{M_t}\left[\left(u_t - \triangle_g u\right) - u\left|\vec{H} - \frac{D^N\Phi}{\Phi}\right|^2\right]\Phi\,dv_g.$$

Lastly note

$$D^N\Phi = \langle D\Phi, N\rangle N = \Phi\left\langle\frac{X}{2t}, N\right\rangle N = \Phi\frac{X^N}{2t}$$

we arrive the desired formula.

Self shrinking solutions out of "minimal" blow up rate.

First we list a few evolution formulas under mean curvature flow of hypersurfaces $X_t\left(p,t\right) = \triangle_g X\left(p,t\right)$ in R^{n+1} :

$$\partial_t g_{ij} = -2hA_{ij}$$
$$\partial_t N = -h_i g^{ij}X_j = -\bigtriangledown h$$
$$\partial_t A_{ij} = \triangle_g A_{ij} + 2hA_{ia}g^{ab}A_{bj} - |A|^2 A_{ij}$$
$$\partial_t h = \triangle_g h - h|A|^2$$
$$\partial_t |A|^2 = \triangle_g |A|^2 - 2|\bigtriangledown_g A|^2 + 2|A|^4.$$

Actually the first two are straight forward. The maneuver is to switch higher order derivatives to lower ones by the perpendicular relation. We compute

$$\partial_t g_{ij} = \partial_t\langle X_i, X_j\rangle = \langle(X_t)_i, X_j\rangle + \langle X_i, (X_t)_j\rangle = -\langle\triangle_g X, X_{ji}\rangle - \langle X_{ij}, \triangle_g X\rangle = -2hA_{ij},$$

$$\partial_t N = \langle\partial_t N, X_a\rangle g^{ab}X_b = -\langle N, X_{at}\rangle g^{ab}X_b = -\langle N, (hN)_a\rangle g^{ab}X_b$$
$$= -h_a g^{ab}X_b = \nabla_g h.$$

What prevents the mean curvature flow continues is that the second fundamental form become infinity. Thus the maximal existence time T is either (ii) finite and $\max_{X(\cdot,t)}|A|$ blows up as t goes to T or (ii) $T = \infty$. In the finite time blow up case, let us see the asymptotic behavior of $|A|$.

Proof. The blow up rate for $\max_{X(\cdot,t)}|A|^2$ satisfies

$$\max_{X(\cdot,t)}|A|^2 \geq \frac{1}{2\left(T-t\right)}.$$

Proof. Denote $u\left(t\right) = \max_{X(\cdot,t)}|A|^2$ (as max of Lipschitz function, still Lip). From the evolution equation for $|A|^2$

$$\frac{du}{dt} \leq 2u^2 \quad\text{or}\quad \frac{d\frac{1}{u}}{dt} \geq -2.$$

Integrate, we have

$$0 - \frac{1}{u(t)} \geq -2(T - t)$$

that is

$$u(t) \geq \frac{1}{2(T - t)}.$$

So we distinguish two types of blow up rate or types of singularity, minimal $\frac{1}{2(T-t)}$ rate or large ones.

Type I. $\frac{1}{2(T-t)} \leq \max_{X(\cdot,t)} |A|^2 \leq \frac{C}{2(T-t)}$;

Type II. $\max_{X(\cdot,t)} |A|^2 (T - t) \to \infty$, as $t \to T$.

In the following, we extract a subconvergence sequence of the evolving hypersurface to self shrinking surface. We rescale the evolving surface or blow it up near the type I singular point P, where the whole surface collapses, $X(p, t) \to P$ as $t \to T$. Set

$$Y(p, t) = \frac{1}{\sqrt{T - t}} (X(p, t) - P).$$

Observe $Y(p, t)$ is bounded, as

$$|X(p, t) - P| = \left| \int_t^T X_t dt \right| \leq \int_t^T |h| \, dt \leq \int_t^T \frac{C}{\sqrt{T - t}} dt = 2C\sqrt{T - t}.$$

Further note the whole second fundamental of the rescaled surface now becomes bounded, as $|A(Y)| = \sqrt{T - t} |A(X)| \leq C$.

Once we manage to find a subconvergence sequence $Y(\cdot, t_k)$ (doable because of the above two observation and higher order derivative estimates Schauder), then any limit $Z(p)$ satisfies our self shrinking equation

$$\vec{H} + \frac{1}{2} Z^N = 0 \quad \text{or} \quad \triangle_g Z = -\frac{1}{2} Z^N.$$

Indeed, now $P = (0, T)$, here we just assume the space component of the singular point P is 0, by the above Huisken's monotonicity formula with heat kernel weight

$$\frac{d}{dt} \int_{X(t)} \frac{1}{(T - t)^{n/2}} e^{-\frac{|X|^2}{4(T-t)}} dv_{g(X)} = \int_{X(t)} - \left| \triangle_{g(X)} X + \frac{X^N}{2(T - t)} \right|^2 \frac{1}{(T - t)^{n/2}} e^{-\frac{|X|^2}{4(T-t)}} dv_{g(X)}$$

$$= \frac{-1}{T - t} \int_{Y(t)} \left| \triangle_{g(Y)} Y + \frac{1}{2} Y^N \right|^2 e^{-\frac{|Y|^2}{4}} dv_{g(Y)}.$$

It follows that

$$-\int_{X(0)} \frac{1}{T^{n/2}} e^{-\frac{|X|^2}{4T}} dv_{g(X)} \leq \int_{X(t)} \frac{1}{(T - t)^{n/2}} e^{-\frac{|X|^2}{4(T-t)}} dv_{g(X)} \Bigg|_0^{t_k}$$

$$= \int_0^{t_k} \frac{-1}{T - t} \int_{Y(t)} \left| \triangle_{g(Y)} Y + \frac{1}{2} Y^N \right|^2 e^{-\frac{|Y|^2}{4}} dv_{g(Y)} dt$$

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or

$$\int_0^{t_k} \frac{1}{T-t} \int_{Y(t)} \left| \triangle_{g(Y)} Y + \frac{1}{2} Y^N \right|^2 e^{-\frac{|Y|^2}{4}} dv_{g(Y)} dt \leq \int_{X(0)} \frac{1}{T^{n/2}} e^{-\frac{|X|^2}{4T}} dv_{g(X)} < \infty.$$

As $\int_0^T \frac{1}{T-t} dt = \infty$, we conclude that any limit $Z(p)$ of $Y(p, t_k)$ as t_k goes to T should satisfy our self shrinking equation.

One can also present this self similar equation derivation via a change of time. The rescaled surface satisfies

$$Y_t = \frac{1}{\sqrt{T-t}} X_t + \frac{1}{2} (T-t)^{-3/2} (X(p,t) - P)$$

$$= \frac{1}{\sqrt{T-t}} \triangle_{g(X)} X + \frac{1}{2} (T-t)^{-1} Y$$

$$= \triangle_{g(\underline{X})} Y + \frac{1}{2} (T-t)^{-1} Y$$

$$= \frac{1}{\left(\sqrt{T-t}\right)^2} \triangle_{g(\underline{Y})} Y + \frac{1}{2} (T-t)^{-1} Y,$$

that is

$$(T-t) Y_t = \triangle_{g(Y)} Y + \frac{1}{2} Y.$$

We change time scale

$$\tau = -\log \frac{T-t}{T} \in [0, \infty).$$

Then

$$Y_\tau = \frac{1}{\frac{d\tau}{dt}} Y_t = (T-t) Y_t = \triangle_{g(Y)} Y + \frac{1}{2} Y.$$

Huisken. (Monotonicity with weight $\exp\left(-|Y|^2/4\right)$). The rescaled flow $Y_\tau = \triangle_{g(Y)} Y + \frac{1}{2} Y$ satisfies

$$\frac{d}{d\tau} \int_Y e^{-\frac{|Y|^2}{4}} dv_g = -\int_Y \left| \triangle_g Y + \frac{1}{2} Y^N \right|^2 e^{-\frac{|Y|^2}{4}} dv_g.$$

Proof. Let $\phi(Y) = \exp\left(-|Y|^2/4\right)$. We compute

$$\frac{d}{d\tau} \int_Y \phi dv_g = \int_Y \langle D\phi, Y_\tau \rangle + \phi \operatorname{div}_Y Y_\tau dv_g = \int_Y \underline{\langle D\phi, Y_\tau \rangle} + \operatorname{div}_Y \phi Y_\tau - \underline{\langle D^T \phi, Y_\tau \rangle} dv_g$$

$$= \int_Y \underline{\langle D^N \phi, Y_\tau \rangle} + \operatorname{div}_Y \phi Y_\tau^T + \operatorname{div}_Y \phi Y_\tau^N dv_g$$

$$= \int_Y \phi \left\langle -\frac{1}{2} Y^N, Y_\tau \right\rangle - \phi \left\langle Y_\tau^N, \vec{H} \right\rangle dv_g + \underline{\int_Y \operatorname{div}_Y \phi Y_\tau^T dv_{g0}}$$

$$= -\int_Y \phi \left\langle \vec{H} + \frac{1}{2} Y^N, Y_\tau \right\rangle dv_g = -\int_Y \phi \left| \vec{H} + \frac{1}{2} Y^N \right|^2 dv_g.$$

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One consequence is

$$\int_{Y(\infty)} e^{-\frac{|Y|^2}{4}} dv_g - \int_{Y(0)} e^{-\frac{|Y|^2}{4}} dv_g = -\int_0^\infty \int_Y \phi \left| \vec{H} + \frac{1}{2} Y^N \right|^2 dv_g d\tau$$

and then

$$\int_0^\infty \int_Y \phi \left| \vec{H} + \frac{1}{2} Y^N \right|^2 dv_g d\tau < \int_{Y(0)} e^{-\frac{|Y|^2}{4}} dv_g < \infty.$$

Once we manage to find a subconvergence sequence $Y(\cdot, \tau_k)$ with $\int_{Y(\tau_k)} \phi \left| \vec{H} + \frac{1}{2} Y^N \right|^2 dv_g \to$ 0 (doable), then limit $Z(p)$ satisfies our self shrinking equation

$$\vec{H} + \frac{1}{2} Z^N = 0 \quad \text{or} \quad \triangle_g Z = -\frac{1}{2} Z^N.$$

Rigidity of entire self shrinking graph in R^{n+1}.

We need another evolution formula in terms the normal N with the second fundamental showing up. Under the non-divergence flow of hyper surface $X_t(x,t) = g^{ij} \partial_{ij} X(x,t)$, we have

$$\left(\partial_t - g^{ij} \partial_{ij} \right) N = |A|^2 N.$$

Indeed, we have obtained the N_t part for the divergence flow, now we go with the non-divergence one

$$N_t = \langle N_t, X_a \rangle g^{ab} X_b = - \langle N, X_{ta} \rangle g^{ab} X_b = \underbrace{- \langle N, X_t \rangle_a} g^{ab} X_b + \underline{\langle N_a, X_t \rangle} g^{ab} X_b$$

$$\left(= -\nabla_g h + \langle N_a, X_t \rangle g^{ab} X_b \right).$$

Next for the $\partial_{ij} N$ part,

$$N_i = \langle N_i, X_a \rangle g^{ab} X_b = - \langle N, X_{ai} \rangle g^{ab} X_b$$

$$N_{ij} = \left[- \langle N, X_{aij} \rangle - \langle N_j, X_{ai} \rangle \right] g^{ab} X_b - \langle N, X_{ai} \rangle \partial_j g^{ab} X_b - \langle N, X_{ai} \rangle g^{ab} X_{bj}$$

and

$$g^{ij} N_{ij} = \left[- \langle N, g^{ij} X_{aij} \rangle - g^{ij} \langle N_j, X_{ai} \rangle \right] g^{ab} X_b - g^{ij} \langle N, X_{ai} \rangle \partial_j g^{ab} X_b - g^{ij} \langle N, X_{ai} \rangle g^{ab} X_{bj}$$

$$= \left[\underbrace{- \langle N, g^{ij} X_{ij} \rangle_a} + \underline{\langle N_a, g^{ij} X_{ij} \rangle} + \partial_a g^{ij} \langle N, X_{ij} \rangle - g^{ij} \langle N_j, X_{ai} \rangle \right] g^{ab} X_b$$

$$- g^{ij} \langle N, X_{ai} \rangle \partial_j g^{ab} X_b - g^{ij} \langle N, X_{ai} \rangle g^{ab} X_{bj}.$$

Then

$$\left(\partial_t - g^{ij} \partial_{ij} \right) N = \left[-\partial_a g^{ij} A_{ij} + g^{ij} \langle N_j, X_{ai} \rangle \right] g^{ab} X_b + g^{ij} A_{ai} \partial_j g^{ab} X_b + g^{ij} A_{ai} g^{ab} X_{bj}$$

Note at the tangent point under the tangent coordinates, $\partial_a g = 0, \partial_a g^{-1} = 0$, and $X_{ab} \parallel N$, in turn, the evolution for N is already valid. Recall our non-divergence system is invariant under orthogonal rotation in R^{n+1}, consequently we have proved the desired equation for the normal.

Here we also include a direct proof:

$$
\begin{aligned}
\left(\partial_t - g^{ij}\partial_{ij}\right) N &= \left[-\partial_a g^{ij} A_{ij} + g^{ij}\langle N_j, X_{ai}\rangle\right] g^{ab} X_b + g^{ij} A_{ai}\partial_j g^{ab} X_b + g^{ij} A_{ai} g^{ab} X_{bj} \\
&= \left[g^{i\alpha}\left(\langle X_{\alpha a}, X_\beta\rangle + \langle X_\alpha, X_{\beta a}\rangle\right) g^{\beta j} A_{ij} - g^{ij} A_{j\alpha} g^{\alpha\beta}\langle X_\beta, X_{ai}\rangle\right] g^{ab} X_b \\
&\quad - g^{ij} A_{ai} g^{a\alpha}\left(\langle X_{\alpha j}, X_\beta\rangle + \langle X_\alpha, X_{\beta j}\rangle\right) g^{\beta b} X_b + g^{ij} A_{ai} g^{ab} X_{bj} \\
&= \left[g^{i\alpha} A_{ij} g^{\beta j}\left(\underbrace{\langle X_\beta, X_{\alpha a}\rangle} + \underbrace{\langle X_\alpha, X_{\beta a}\rangle}\right) - \underline{g^{ij} A_{j\alpha} g^{\alpha\beta}\langle X_\beta, X_{ai}\rangle}\right] g^{ab} X_b \\
&\quad - g^{ij} A_{ai} g^{a\alpha} X_{\alpha j}^T - \underbrace{g^{ij} A_{ai} g^{a\alpha}\langle X_\alpha, X_{\beta j}\rangle g^{\beta b} X_b} + g^{ij} A_{ai} g^{ab} X_{bj} \\
&= -g^{ij} A_{ai} g^{ab} X_{bj}^T + g^{ij} A_{ai} g^{ab} X_{bj} = g^{ij} A_{ai} g^{ab} X_{bj}^N \\
&= g^{ij} A_{ai} g^{ab} A_{bj} N = |A|^2 N.
\end{aligned}
$$

RMK. Under the effective mean curvature flow $Y_t(p,t) = \triangle_g Y(p,t)$ for $Y(p,t) = X(x(p,t),t)$, the equation for $N(p,t)$ takes a similar form

$$
\left(\partial_t - \triangle_g X\right) N = |A|^2 N.
$$

This is just because of the ODE system for the reparametrization $x(p,t)$

$$
\frac{dx_k}{dt} = -g^{ij}(x)\Gamma_{ij}^k(x).
$$

Theorem(Lu Wang): Any ancient self-similar solution $(x, f(x,t)) = \left(x, \sqrt{-t}\,u\left(\frac{x}{\sqrt{-t}}\right)\right)$ to

$$
f_t = g^{ij}(Df)\partial_{ij}f \ \text{ in } R^n \times (-\infty, 0) \ \Leftrightarrow\ g^{ij}(Du)\partial_{ij}u = \frac{1}{2}x\cdot Du - \frac{1}{2}u \ \text{ in } R^n
$$

is linear, $f(x,t) = Du(0)\cdot x = u(x)$.

Lu Wang's original proof is in an integral way. We have a shorter pointwise subharmonic approach. Here we present yet an even shorter superharmonic argument.

Step1. Superharmonic inner product $W(x,t) = \langle N, e_{n+1}\rangle = 1/\sqrt{1 + |Df|^2} > 0$ satisfies

$$
\left(g^{ij}\partial_{ij} - \partial_t\right) W = -|A|^2 W \leq 0,
$$

from the above evolution equation for N. By self-similarity

$$
W(x,t) = w\left(x/\sqrt{-t}\right), \quad g_f(x,t) = g_u\left(x/\sqrt{-t}\right), \ \text{ and } \ |A_f(x,t)|^2 = \left|a_u\left(x/\sqrt{-t}\right)\right|^2 / (-t).
$$

The equation for W becomes

$$
g^{ij}\partial_{ij}w - \frac{1}{2}x\cdot Dw = -|a|^2 w \leq 0.
$$

Heuristic: As $g^{ij}\partial_{ij}w \leq \frac{1}{2}x\cdot Dw$, the amplifying force in the right forces w up near ∞. Otherwise, bounded w becomes unboundedly negative near ∞. Hence super solution w attains its min at a finite point, then constant.

Step2. The self-similar term rw_r with barrier like $v(x) = -\varepsilon\left(|x|^2 - 4n^2\right) + \min_{B_{2n}} w$ forces superharmonic w attains its global minimum at a finite point. This is because

$$g^{ij}\partial_{ij}v - \frac{1}{2}x \cdot Dv = -\varepsilon\left(2g^{ij}\delta_{ij} - |x|^2\right) \geq -\varepsilon\left(2n - |x|^2\right) \geq 0 \quad \text{for } |x| \geq 2n,$$

and the subharmonic function v is less than or equal to w at the boundary B_{2n} and near ∞. Then the maximum principle implies $w \geq v$. Let ε go to 0, we conclude

$$w(x) \geq \min_{B_{2n}} w \quad \text{for } |x| \geq 2n.$$

Hence we see w attains its global minimal at some finite point. Now the strong maximum principle then implies that $w \equiv const. > 0$.

Step3. By the equation for w, one concludes the second fundamental form $|a| = 0$.