○ equations
○ source for equations
○ examples, stationary & dynamical (symmetries)

<u>The equation</u>
Deform surface canonically (independent of coordinates ...) or geometrically, such as along mean curvature flow, Gauss curvature flow (needs convex surface, otherwise equation hyperbolic, even initial condition is obviouly posed), along the normal direction (tangential deformation has no effect on the shape change).

Surprisingly, in reality, mean curnvature flow resembles nature such as in metal melting...

In application, mean curvaure (and other curvature) flow provides very efficient way of image processing and restoring ....

$$X_t = \vec{H} = \triangle_g X$$

Mean curvature vector
$\lambda \dashrightarrow \kappa = (\kappa_1, \cdots, \kappa_n)$, principle curvatures of hypersurface $(x, u(x))$.
$\sigma_k(\kappa) = c,$

in particular, mean curvature $H = \kappa_1 + \cdots + \kappa_n = div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right),$

and Gauss curvature $K = \kappa_1 \cdots \cdots \kappa_n = \frac{\det D^2 u}{\left(1+|Du|^2\right)^{\frac{n}{2}+1}}$

Tangent way: Re-represent graph $(x, u(x))$ at $(x_0, u(x_0)) = (0,0)$ over its tangent plane, $\bar{u}(\bar{x}) = 0 + 0 \cdot \bar{x} + \frac{1}{2}\left(\kappa_1 \bar{x}_1^2 + \cdots + \kappa_n \bar{x}_n^2\right).$

$$figure$$

Assume $Du \overset{0}{=} (0, \cdots, 0, u_n)$ and $u_n = \tan\theta$, then

$$\bar{u}(\bar{x}) = [u(x) - \tan\theta \ x_n]\cos\theta$$

and

$$(x_1, \cdots, x_{n-1}, x_n) = (\bar{x}_1, \cdots, \bar{x}_{n-1}, \bar{x}_n \cos\theta).$$

Now at 0

$$D_{\bar{x}}^2 \bar{u}(\bar{x}) = \cos\theta \begin{bmatrix} u_{11} & & u_{1,n-1} & u_{1n}\cos\theta \\ & & & \\ u_{n-1,1} & & & \\ u_{n1}\cos\theta & & u_{n-1,1}\cos\theta & u_{nn}\cos^2 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & & \\ & 1 & \\ & & \cos\theta \end{bmatrix} D^2 u \begin{bmatrix} 1 & & \\ & 1 & \\ & & \cos\theta \end{bmatrix}.$$

Hence $H = \cos\theta \left[u_{11} + \cdots + u_{n-1,n-1} + u_{nn}\cos^2\theta \right] \overset{Du=(0,\cdots,0,u_n)}{=} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}} \right)$ and

$K = \cos^n\theta \cos^2\theta \det D^2 u = \dfrac{\det D^2 u}{\left(1+|Du|^2\right)^{\frac{n}{2}+1}}.$

Second fundamental way: $II = \dfrac{D^2 u}{\sqrt{1+|Du|^2}}$

$g = I + (Du)^T (Du) \quad g^{-1} = I - \dfrac{1}{1+|Du|^2}(Du)^T(Du)$

$g^{-1}II = B \ \text{ or } \ g^{-1/2}IIg^{-1/2} \sim \begin{bmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_n \end{bmatrix}$

$H = \kappa_1 + \cdots + \kappa_n = \dfrac{1}{\sqrt{1+|Du|^2}}\left(\triangle u - \dfrac{u_i u_j}{1+|Du|^2}u_{ij} \right) = \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}} \right)$

$K = \kappa_1 \cdots \kappa_n = \dfrac{1}{\left(1+|Du|^2\right)\left(1+|Du|^2\right)^{n/2}} \det D^2 u = \dfrac{1}{n}\partial_i \left[\dfrac{\partial\sigma_n(B)}{\partial B_{ij}}? \dfrac{\partial_j u}{\sqrt{1+|Du|^2}} \right]$

$\sigma_k(\kappa) = \dfrac{1}{k}\partial_i \left[\dfrac{\partial\sigma_k(B)}{\partial B_{ij}}? \dfrac{\partial_j u}{\sqrt{1+|Du|^2}} \right]$

Examples.

Stationary, $X_t = 0$ leads to $H = 0$ or minimal surface surface equation, its linear version is Laplace equation.

Dynamical, linear version is heat equation, special solutions could be generated via symmetry such as homothetic shrinking/expanding, translating, rotating.

eg1. Sphere
eg2. cylinder
eg3. torus, mean convex torus and Angent's torus

○ shrinker.
$X(x,t) = \sqrt{-t}F(x)$

$$-\dfrac{1}{2\sqrt{-t}}F^N = \dfrac{1}{\sqrt{-t}}\triangle_g F$$

or

$$\triangle_g F = -\dfrac{1}{2}F^N$$

In particular $F = (x, f(x)) \in R^n \times R^1$

$$\operatorname{div}\left(\dfrac{Df}{\sqrt{1+|Df|^2}} \right) = -\dfrac{1}{2}\dfrac{-x \cdot Df + f}{\sqrt{1+|Df|^2}}$$

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or

$$\triangle f - \frac{f_i f_j}{1 + |Df|^2} f_{ij} = \frac{1}{2} (x \cdot Df - f)$$

when f radial

$$\frac{f_{rr}}{1 + f_r^2} + \frac{n-1}{r} f_r = \frac{1}{2} (x \cdot Df - f)$$

examples: sphere, cylinder, shamrock ...

    ○ Translator

$X(x,t) = F(x) + te$

$$e^N = \triangle_g F$$

In particular $F = (x, f(x))$, $e = (0, \cdots, 0, 1)$

$$\text{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) = \frac{1}{\sqrt{1 + |Df|^2}}$$

eg. grim reaper $\frac{f_{xx}}{1 + f_x^2} = 1$, $f = -\ln \cos x$

    ○ Rotator

$X(x,t) = F(x) + t\partial_\theta$

1d $X(x,t) = F(x) + tJF(x)$

$$(JF)^N = \triangle_g F$$

In particular $F = (x, f(x))$

$$\text{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) = \left\langle (-f, x), \frac{(-f_x, 1)}{\sqrt{1 + f_x^2}} \right\rangle = \frac{f f_x + x}{\sqrt{1 + f_x^2}}$$

In polar coordinates $(r(\theta, t), \theta)$

1st mean curvature flow

$(r_t \partial_r)^N = H$

$N = JT / |T| = J[r_\theta (\cos\theta, \sin\theta) + r(-\sin\theta, \cos\theta)] / |T| = [r_\theta (-\sin\theta, \cos\theta) + r(-\cos\theta, -\sin\theta)] /$

$\beta = \measuredangle(T, \partial_x)$

$$-\frac{r_t r}{|T|} = \frac{d\beta}{ds} = \frac{d\beta}{d\theta} \frac{d\theta}{ds}$$

or

$$-r_t r = \frac{d\beta}{d\theta}$$

2nd rotator

    $r(\theta, t) = r(\theta + t)$

$$-r_\theta r = \frac{d\beta}{d\theta}$$

then

$$\frac{d\beta}{dr} = -r$$

and $\beta = -\frac{1}{2}r^2$ or

$$\arctan f_x = -\frac{1}{2}\left(x^2 + f^2\right).$$

By symmetry, $r = \sqrt{-2\beta}$ & $r = \sqrt{2(\beta - \pi)}$ form the Yin-Yang rotator.

RMK. One obvious thing

1d principle curvature of curve $(x, f(x))$

$$\kappa = \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} = \left(\frac{f_x}{\sqrt{1+f_x^2}}\right)_x$$

also $\int \kappa ds = \int \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3}\sqrt{1+f_x^2}dx = \int (\arctan f_x)_x\, dx = \arctan f_x|_\partial$

○ $H = 0$

catenoid: $|(x,y)| = \cosh z$

helicoid: $z = \arctan \frac{y}{x}$

Sherk's surface: $z = \ln \frac{\cos y}{\cos x}$

○ $H_k = const.$

unit sphere

○ $\triangle u = 0$

complex analysis in even d: $u = \operatorname{Re} z^k$, z^{-k}, e^z, $z_1^3 e^{z_2}$, $\cdots$

algebraic n-d $u = \sigma_k(x_1, \cdots, x_2)$

radial

$$\triangle u = \partial_r^2 u + \frac{n-1}{r}\partial_r u + \frac{1}{r^2}\triangle_{S^{n-1}} u$$

$u_{rr} + \frac{n-1}{r}u_r = 0$

$r^{n-1}u_{rr} + (n-1)r^{n-2}u_r = 0$ or $(r^{n-1}u_r)_r = 0$

$u_r = \frac{c}{r^{n-1}}$

$$u = \frac{c}{r^{n-2}} \ , \ \ln|(x_1, x_2)| \ , \ \text{or} \ |x_1|$$

Comparison principle

two solutions cannot touch each other

<div align="center"><i>figure</i></div>

$$\triangle u = 0$$
$$D^2 u_1 > D^2 u_2 \Rightarrow 0 = \triangle u_1 - \triangle u_2 > 0 \rightarrow\leftarrow$$

two solutions can cross each other

<div align="center"><i>figure</i></div>

In contrast to $w_{tt} - w_{xx} = 0$, $w_1 = x^2 + t^2$, $w_2 = 0$.