Lecture 2 Laplace and heat equations

∘ invariance
∘ mean value equality
maximum principle,
(higher order) derivative estimates and smoothing effect
Harnack inequality
Liouville
∘ strong maximum principle for general elliptic and parabolic equations

Laplace equation $\triangle u = 0$

complex analysis in even d: $u = \operatorname{Re} z^k$, z^{-k}, e^z, $z_1^3 e^{z_2}, \cdots$
algebraic n-d $u = \sigma_k(x_1, \cdots, x_2)$
radial

$$\triangle u = \partial_r^2 u + \frac{n-1}{r}\partial_r u + \frac{1}{r^2}\triangle_{S^{n-1}} u$$

$u_{rr} + \frac{n-1}{r} u_r = 0$
$r^{n-1} u_{rr} + (n-1) r^{n-2} u_r = 0$ or $(r^{n-1} u_r)_r = 0$
$u_r = \frac{c}{r^{n-1}}$

$$u = \frac{c}{r^{n-2}}\ ,\ \ln|(x_1, x_2)|\ ,\ \text{or}\ \ |x_1|$$

Comparison principle
two solutions cannot touch each other

$$figure$$

$$\triangle u = 0$$
$$D^2 u_1 > D^2 u_2 \Rightarrow 0 = \triangle u_1 - \triangle u_2 > 0 \rightarrow\leftarrow$$

two solutions can cross each other

$$figure$$

In contrast to $w_{tt} - w_{xx} = 0$, $w_1 = x^2 + t^2$, $w_2 = 0$.
Invariance for Harmonic functions, solutions to $\triangle u = 0$
· $u(x + x_0)$
· $u(Rx)$
· $u(tx)$
RMK. Equations don't know/care which coordinates they are in.
· $u + v$, au, where $\triangle v = 0$
· $\int u(x-y)\varphi(y)\,dy$

[0]November 2, 2014

$\cdot \frac{u(x+\varepsilon e)-u(x)}{\varepsilon} \to D_e u$, so is $D^k u$
$\cdot \frac{u(R\varepsilon x)-u(x)}{\varepsilon} \to D_\theta u = x_i u_j - x_j u_i$
$\cdot \frac{u((1+\varepsilon)x)-u(x)}{\varepsilon} \to Du(x) \cdot x = r u_r$, so are $r\partial_r (r u_r) = r u_r + \underline{r^2 u_{rr}}$, $r^3 u_{rrr}, \cdots$
$\cdot |x|^{2-n} u\left(\frac{x}{|x|^2}\right)$ Kelvin transformation

RNK. "Kelvin" transformation for the heat equation $u_t - \triangle u = 0$, $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} u\left(\frac{x}{t}, \frac{-1}{t}\right)$.

<u>Mean value equality</u>

Recall the divergence formula (the fundamental theorem of calculus)

$$\int_\Omega div\left(\vec{V}\right) dx = \int_{\partial\Omega} \left\langle \vec{V}, \gamma \right\rangle dA.$$

$\vec{V} = Du$, then $0 = \int_{\partial\Omega} u_\gamma dA$.
$\vec{V} = vDu$, then $\int_\Omega \langle Dv, Du\rangle + v \triangle u = \int_{\partial\Omega} v u_\gamma dA$.
$\vec{V} = uDv$, then $\int_\Omega \langle Du, Dv\rangle + u \triangle v = \int_{\partial\Omega} u v_\gamma dA$.

$$\int_\Omega v \triangle u - u \triangle v = \int_{\partial\Omega} v u_\gamma - u v_\gamma dA.$$

Mean value case. Now $\triangle u = 0$ in B_1, take $v = |x|^{2-n}$, $\Omega = B_1 \backslash B_\varepsilon$,

$$B_1 \backslash B_\varepsilon \ figure$$

we then have $0 = \int_{\partial\Omega} v u_\gamma - u v_\gamma dA$, or

$$\overbrace{\int_{\partial(B_1\backslash B_\varepsilon)} v u_\gamma dA}^{0} = \int_{\partial(B_1\backslash B_\varepsilon)} u v_\gamma dA = \int_{\partial B_1} u \frac{(2-n)}{r^{n-1}} dA - \int_{\partial B_\varepsilon} u \frac{(2-n)}{r^{n-1}} dA. \qquad (*)$$

We get $\int_{\partial B_1} u dA = \int_{\partial B_\varepsilon} u \frac{1}{\varepsilon^{n-1}} dA \overset{\varepsilon\to 0}{\to} |\partial B_1| u(0)$. So $u(0) = \frac{1}{|\partial B_1|} \int_{\partial B_1} u dA$.

RMK. In hindsight one just takes $v = \dfrac{-1}{(n-2)|\partial B_1|} \dfrac{1}{|x|^{n-2}} \overset{\text{def}}{=} \Gamma$.

Also

$$u(0) = \frac{1}{|\partial B_r|} \int_{\partial B_r} u dA.$$

Take a weight function $|\partial B_r|$, $u(0)|B_1| = \int_0^1 u(0)|\partial B_r| dr = \int_0^1 \int_{\partial B_r} u dA dr = \int_{B_1} u dx$. So $u(0) = \frac{1}{|B_1|} \int_{B_1(0)} u dx$.

Also

$$u(0) = \frac{1}{|B_r|} \int_{B_r(0)} u dx.$$

RMK1. Tracing the sign of $\triangle u$, one gets mean value inequalities for superharmonic functions $\triangle u \leq 0 : u(0) \leq \fint u$ and subharmonic functions $\triangle u \geq 0 : u(0) \leq \fint u$.

RMK2. "$\cdots$ all the women are strong, all the men are good-looking, and all the children are above average." –A Prairie Home Companion with Garrison Keillor.

Application 1. **Strong maximum principle** (No toughing).

$$\begin{aligned} \triangle u_1 &= \triangle u_2 = 0 \\ u_1 &\geq u_2, \ \text{“}=\text{''} \text{ at } 0 \end{aligned}$$

then

$$0 = u_1(0) - u_2(0) = \frac{1}{|B_r|} \int_{B_r} (u_1 - u_2)\, dx \geq 0.$$

It follows that $u_1 \equiv u_2$.

Application 2. **Smooth effect and derivative estimates.**

Take radial weight $\varphi(y) = \varphi(|y|) \in C_0^\infty(R^n)$ such that $1 = \int \varphi(y)\, dy = \int_0^\infty \varphi(r) |\partial B_r|\, dr$. Then

$$\begin{aligned} \int_{R^n} u(y)\, \varphi(x-y)\, dy &= \int_0^\infty \int_{\partial B_r(x)} u(y)\, \varphi(x-y)\, dA dr \\ &= \int_0^\infty u(x)\, \varphi(r) |\partial B_r|\, dr = u(x) \int \varphi(y)\, dy \\ &= u(x). \end{aligned}$$

Consequence $u(x) = \int_{R^n} u(y)\, \varphi(x-y)\, dy$ is smooth for continuous initial $u(y)$, and

$$D^k u(0) = \int u(y)\, D_x^k \varphi(x-y)\, dy = (-1)^k \int u(y)\, D_y^k \varphi(x-y)\, dy.$$

Thus, (say support of φ is in B_1)

$$\left|D^k u(0)\right| \leq C(k, n, \varphi) \|u\|_{L^1(B_1)}$$

and also the point-to-point version for $u \geq 0$

$$\left|D^k u(0)\right| \leq C(k, n, \varphi) \int_{B_1} u dx = C(k, n, \varphi)\, u(0)$$

Scaled version

$$\left|D^k u(0)\right| \leq \begin{cases} \frac{C(k,n,\varphi)\|u\|_{L^1(B_R)}}{R^{n+k}} \\ \frac{C(k,n,\varphi)\|u\|_{L^\infty(B_R)}}{R^k} \end{cases}$$

and also

$$\left|D^k u(0)\right| \leq \frac{C(k, n, \varphi)}{R^k} u(0) \quad \text{provided } u \geq 0.$$

That is the larger the domain, the flatter the harmonic graph.

Liouville. Every bounded or even one side bounded entire harmonic function in R^n is constant. Similarly every entire polynomial growth harmonic function is a polynomial.

Application 3. **Harnack inequality**–a quantitative version of the strong maximum principle.

eg. Consider positive harmonic functions r^{2-n}, $x_1 r^{-n}$ on $\{x_1 > 0\}$.

$$r^{2-n}, x_1 r^{-n} \text{ graph } figure$$

eg. In general for $\triangle \mathrm{u} = 0,\ u > 0$ in $B_1(0)$, we have

$$u(x) = \frac{1}{\left|B_{1-|x|}\right|} \int_{B_{1-|x|}(x)} u dx \leq \frac{1}{\left|B_{1-|x|}\right|} \int_{B_1(0)} u dx = \frac{|B_1|}{\left|B_{1-|x|}\right|} u(0) = \frac{1}{(1-|x|)^n} u(0).$$

RNK. As those two examples suggest, from estimating the kernel of Poisson representation, we have a sharper comparison

$$\frac{(1-|x|)}{2^{n-1}} u(0) \leq u(x) \leq \frac{2}{(1-|x|)^{n-1}} u(0).$$

Harnack. Suppose $\triangle \mathrm{u} = 0,\ u > 0$ in $B_r(x_0)$. Then we have

$$\sup_{B_{r/4}(x_0)} u \leq 3^n \inf_{B_{r/4}(x_0)} u.$$

Discrete way. In fact

$$4 \text{ circle figure } \mathrm{B}_1, B_{1/4}, B_{1/4}(x_{\max}), B_{3/4}(x_{\min})$$

$$\begin{aligned}
\max_{B_{1/4}} u = u(x_{\max}) &= \frac{1}{\left|B_{1/4}\right|} \int_{B_{1/4}(x_{\max})} u dx \\
&\leq \frac{1}{\left|B_{1/4}\right|} \int_{B_{3/4}(x_{\min})} u dx \\
&= 3^n u(x_{\min}) = 3^n \min_{B_{1/4}} u.
\end{aligned}$$

Continuous way. Suppose $\triangle \mathrm{u} = 0,\ u > 0$ in $B_2(0)$. Then we have

$$\max_{B_1(0)} u \leq C(n) \min_{B_1(0)} u.$$

Indeed by the point-to-point version of gradient estimate, $|Du(x)|/u(x) \leq C(n)$ in $B_1(0)$. We measure the ratio between $u_{\max}$ and $u_{\min}$ in $B_1(0)$ by integration

$$\begin{aligned}\log \frac{u(x_{\max})}{u(x_{\min})} &= \int_{x_{\min}}^{x_{\max}} d\log u(x) = \int_{x_{\min}}^{x_{\max}} \frac{u_e}{u}(x_{\min} + t(x_{\max} - x_{\min}))\, dt \\ &\leq \int_{x_{\min}}^{x_{\max}} \frac{|Du|}{u}(x_{\min} + t(x_{\max} - x_{\min}))\, dt \leq |x_{\max} - x_{\min}| \cdot C(n).\end{aligned}$$

Then

$$\max_{B_1(0)} u \leq e^{2\cdot C(n)} \min_{B_1(0)} u.$$

Consequences $\cdots$, for example one sided Liouville for entire harmonic functions.

RMK. Harnack inequality is in fact a quantitative version of the strong maximum principle. It measures how much the minimum leaps when moving inside, or after flipping around, how much the maximum drops when moving inside. For example, to move inside $B_{1/4}$ from B_1,

$$\min_{B_{1/4}}(u - m_1) \geq 3^{-n} \max_{B_{1/4}}(u - m_1)$$

or

$$m_{1/4} \geq m_1 + 3^{-n}\left(M_{1/4} - m_1\right).$$

The flip version is

$$\min_{B_{1/4}}(M_1 - u) \geq 3^{-n} \max_{B_{1/4}}(M_1 - u)$$

or

$$M_1 \geq M_{1/4} + 3^{-n}\left(M_1 - m_{1/4}\right).$$

(This should be Moser's observation: subtracting the leap from the drop, one has oscillation decay of the "harmonic" function.)

Co-area way for solid mean value equality.

Recall the derivation of the "solid" mean value formula for harmonic functions

$$\int_{B_R} u \triangle v - v \triangle u = \int_{\partial B_R} u v_\gamma - v u_\gamma$$

Set $v = \underbrace{\frac{-1}{(n-2)|\partial B_1|}}_{c_n} \left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}}\right)$

$$u(0) = \int_{\partial B_R} u v_\gamma dA = \int_{\Gamma = \frac{-1}{c_n R^{n-2}} = l} u\,|D\Gamma|\, dA$$

where $\Gamma = \frac{-1}{c_n}\frac{1}{|x|^{n-2}}$.

As R goes from 1 to 0,the level of Γ runs from $-1/c_n$ to $-\infty$, we seek a weight satisfying $1=\int_{-\infty}^{-1/c_n} w(l)\,dl$, where $w(l)=c(-l)^{\alpha}$ to be determined.

$$\begin{aligned} u(0) &= \int_{-\infty}^{-1/c_n} w(l) \int_{\Gamma=l} u\,|D\Gamma|\,dAdl \\ &= \int_{B_1} u\; w(l)\,|D\Gamma|^2\,dx, \end{aligned}$$

where we used change of variable or co-area formula: $dx = dvol = dA\,\frac{dl}{|D\Gamma|}$

figure gradient length=dl/ $|D\Gamma|$

Now let $w(l)\,|D\Gamma|^2 = 1/|B_1|$, then

$$w(l) = \frac{1}{|B_1|}\left[\frac{c_n}{(n-2)}R^{n-1}\right]^2.$$

At $R=1$ and $l=-1/c_n$, we have

$$c\,(1/c_n)^{\alpha} = \frac{1}{|B_1|}\left[\frac{c_n}{(n-2)}\right]^2.$$

The integral for the weight implies

$$-c\frac{1}{1+\alpha}\left(\frac{1}{c_n}\right)^{1+\alpha} = 1.$$

Thus $-(1+\alpha) = \frac{c_n}{|B_1|(n-2)^2} = n/(n-2)$, then

$$w(l) = \frac{n}{n-2}(c_n)^{-n/(n-2)}(-l)^{-\frac{2(n-1)}{n-2}}.$$

RMK. Old weight way, $1=\int_0^1 nr^{n-1}dr \overset{r=(-c_n l)^{-1/(n-2)}}{=} \frac{n}{n-2}c_n^{\frac{-n}{n-2}}\int_{-\infty}^{-1/c_n}(-l)^{-\frac{2(n-1)}{n-2}}\,dl$. And pleasantly $w(l)\,|D\Gamma|^2 = 1/|B_1|$!

As the curved Laplace operator is variable coefficients

$$\triangle_g = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right) = g^{ij}\partial_{ij} + \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\right)\partial_j,$$

we need to deal with general elliptic operator

$$L = \sum a_{ij}(x)\,\partial_{ij} + \sum b_i(x)\,\partial_i$$

where elliptic means

$$(a_{ij}(x)) \le \lambda I > 0.$$

Weak Maximum Principle. Assume $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$,

- Max. If $Lu \geq 0$ in Ω, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.
- Min. If $Lu \leq 0$ in Ω, then $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.
- Comparison. If $Lu \geq Lv$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. It suffices to handle the max case, the other cases follow by applying the first case to $-u$ and $u - v$ respectively.

Strict case. $Lu > 0$.

Suppose u attains its maximum at an interior point x_*, then $Du(x_*) = 0$ and $D^2u(x_*) \leq 0$. But

$$\begin{aligned} Lu(x_*) &= \sum a_{ij}(x_*) D_{ij}u(x_*) + \sum b_i(x_*) D_i u(x_*) \\ &= Tr\left[a_{ij}(x_*) D^2u(x_8)\right] \leq 0. \end{aligned}$$

This contradicts $Lu(x_*) > 0$. Thus $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

General case. $Lu \geq 0$.

We do analysis by approximating it with strict inequalities. Let

$$u_\varepsilon = u + \varepsilon e^{Kx_1}.$$

Then

$$Lu_\varepsilon = Lu + \varepsilon e^{Kx_1}\left(a_{11}K^2 + b_1 K\right) \geq \varepsilon e^{Kx_1}\left(\lambda K^2 - \|b\|_{L^\infty} K\right) > 0$$

provided constant K is large enough. By the strict case

$$\max_{\bar{\Omega}}\left(u + \varepsilon e^{Kx_1}\right) = \max_{\partial\Omega}\left(u + \varepsilon e^{Kx_1}\right).$$

Take $\varepsilon \to 0$, we find $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

Weak Maximum Principle with $c \leq 0$. Assume $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c(x) \leq 0$ in Ω

- Max. If $Lu + c(x)u \geq 0$ in Ω, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.
- Min. If $Lu + c(x)u \leq 0$ in Ω, then $\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u^-$.
- Comparison. If $Lu + c(x)u \geq Lv + c(x)v$ in Ω and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Again it suffices to handle the first case. We just ignore the negative piece of u, and reduces it to the weak maximum principle without zeroth order term. Let

$$\Omega^+ = \{x \in \Omega : \ u(x) > 0\}.$$

We have

$$Lu \geq -c(x)u \geq 0 \text{ in } \Omega^+.$$

By the maximum principle without zeroth order term

$$\max_{\bar{\Omega}^+} u = \max_{\partial\bar{\Omega}^+} u = \max_{\partial\Omega} u^+.$$

It follows that

$$\max_{\bar{\Omega}} u \leq \max_{\Omega^+} u = \max_{\partial\Omega} u^+.$$

C-eg with $c > 0$.

$$\begin{aligned} u &= \cos x \\ u'' + u &= 0 \text{ in } \Omega = (-\pi/2, \pi/2) \\ \max_{\Omega} u &= 1 \not< 0 = \max_{\partial\Omega} u. \end{aligned}$$

C-eg for necessity of the positive part in $\max_{\partial\Omega} u^+$.

$$\begin{aligned} u &= -chx \\ u'' - u &= 0 \ \text{ in } \Omega = (-1, 1) \\ \max_{\Omega} u &= -1 \not\leq -ch1 = \max_{\partial\Omega} u. \end{aligned}$$

C-eg for necessity of the negative part in $\min_{\partial\Omega} u^-$.

$$\begin{aligned} u &= chx \\ u'' - u &= 0 \ \text{ in } \Omega = (-1, 1) \\ \min_{\Omega} u &= 1 \not\geq ch1 = \min_{\partial\Omega} u. \end{aligned}$$

Strong Maximum Principle (E. Hopf). Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and Ω is open and connected.

• Max. If $Lu \geq 0$ in Ω and u attains its global maximum at an interior point, then u is constant.

• Min. If $Lu \leq 0$ in Ω and u attains its global minimum at an interior point, then u is constant.

It suffices to prove only one case, say minimum one; the maximum case follows by considering $-u$. The inconsistency: if "strict" minimum occurs, then gradient at the point is not zero. Hopf achieved it by constructing a barrier.

eg. Consider two positive harmonic functions

$$u = xy \ \text{ in } \{x > 0, y > 0\},$$

$$u = \operatorname{Im} z^3 = r^3 \sin(3\theta) = 3yx^2 - y^3 \ \text{ in } \left\{y > 0, y \leq \sqrt{3}x\right\}.$$

Observe the inner normal derivatives of the two along the boundary except the corner points are all strictly positive. This is not an accident, indeed it is the content of the Hopf boundary lemma: For every positive "harmonic" function vanishing at a boundary point, on which an interior ball touches, the inner normal derivative is the strictly positive at this boundary point satisfying the interior ball condition.

Proof. Step1. Set-up.

Suppose u attains its global minimum $m = \min_{\bar{\Omega}} u$ at an interior point. Set

$$\Omega_m = \{x \in \Omega : u(x) = m\}.$$

If $u(x)$ is not constant m, then $\Omega_m^c = \{x \in \Omega : u(x) > m\}$ is non-empty. Next we choose a ball touching the boundary of Ω_m and inside Ω. There exists an interior

point $y \in \Omega_m^c$ such that $\rho = dist\,(y, \partial\Omega_m) < dist\,(y, \partial\Omega)$. Enlarge the ball centered at y until it hits the boundary $\partial\Omega_m$, say at x_0, before hitting $\partial\Omega$. The largest radius is ρ.

figure

For simplicity of notation, we assume $y = 0$. What we have now is

$$B_\rho^0(0) \subset \Omega_m^c \quad \text{and} \quad x_0 \in \partial B_\rho(0) \cap \Omega_m.$$

Step2. **Hopf boundary lemma.**

We construct a subsolution v such that $u \underset{=\text{at } x_0}{\geq} v$ near x_0 and $v_r(x_0) < 0$. This forces $u_r(x_0) < 0$ or $Du(x_0) \neq 0$, a contradiction to interior minimum $u(x_0)$ at x_0.

First take

$$w(x) = e^{-K|x|^2} - e^{-K\rho^2}$$

with large constant K to be determined later. We compute

$$\begin{aligned} Lw &= e^{-K|x|^2}\left[\sum a_{ij}\left(4K^2 x_i x_j - 2K\delta_{ij}\right) + \sum b_i\left(-2Kx_i\right)\right] \\ &> 0 \quad \text{in } B_\rho(0)\backslash B_{\rho/2}(0) \end{aligned}$$

provided K is taken large enough.

RMK. Another barrier is $|x|^{-K}$.

Notice $u > m$ in $B_\rho^0(0)$. Then we can take small enough ε so that $v(x) = \varepsilon w(x) + u(x_0)$ stays below $u(x)$ on the boundary of the annulus

$$u(x) \geq v(x) \quad \text{on } \partial B_{\rho/2}(0) \cup \partial B_\rho(0).$$

By the construction

$$Lu \leq 0 \leq Lv \quad \text{in } B_\rho(0)\backslash B_{\rho/2}(0).$$

From the comparison principle (weak maximum principle)

$$u(x) \geq v(x) \quad \text{in } B_\rho(0)\backslash B_{\rho/2}(0).$$

Remember $u(x_0) = v(x_0)$, taking radial derivatives, we get

$$-u_r(x_0) \geq -v_r(x_0) = \varepsilon K\rho e^{-K\rho^2} > 0.$$

This contradicts the fact that $Du(x_0) = 0$ at minimum interior point x_0.

Therefore, $u(x) \equiv m$.

Strong maximum principle with $c \leq 0$. Assume $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

• Max. If $Lu + cu \geq 0$ in Ω and u attains its nonnegative global maximum at an interior point, then u is constant.

• Min. If $Lu + cu \leq 0$ in Ω and u attains its nonpositive global minimum at an interior point, then u is constant.

Proof. Again it suffices to justify only one case, say the minimum one.

Now

$$m = \min_{\bar{\Omega}} u \leq 0.$$

We go over the same two steps as in proof of the strong maximum principle without c term. Then for $v = \varepsilon\left(e^{-K|x|^2} - e^{-K\rho^2}\right) + m$

$$\begin{aligned}
&Lv + cv \\
&= e^{-K|x|^2}\left[\sum a_{ij}\left(4K^2 x_i x_j - 2K\delta_{ij}\right) + \sum b_i\left(-2Kx_i\right) + c\right] \\
&+ cm \\
&\geq e^{-K|x|^2}\left[\sum a_{ij}\left(4K^2 x_i x_j - 2K\delta_{ij}\right) + \sum b_i\left(-2Kx_i\right) + c\right] \\
&> 0 \quad \text{in } B_\rho(0)\setminus B_{\rho/2}(0)
\end{aligned}$$

provided K is taken large enough. Then

$$\begin{gathered}
Lu + cu \leq 0 \leq Lv + cv \quad \text{in } B_\rho(0)\setminus B_{\rho/2}(0), \\
u(x) \geq v(x) \quad \text{on } \partial B_{\rho/2}(0) \cup \partial B_\rho(0).
\end{gathered}$$

The remaining argument goes through.

Strong maximum principle with no sign restriction on c but with one sign restriction on solution. Assume $u \in C^2(\Omega) \cap C\left(\bar{\Omega}\right)$ and the arbitrary sign of $c(x)$ in Ω.

• Max. If $Lu + cu \geq 0$ in Ω and $\max\limits_{\bar{\Omega}} u = 0$ attains at an interior point, then $u \equiv 0$.

• Min. If $Lu + cu \leq 0$ in Ω and $\min\limits_{\bar{\Omega}} u = 0$ attains at an interior point, then $u \equiv 0$.

Proof. Again, we only need to handle one case, say the minimum case. Note that $u \geq 0$, then splitting $c = c^+ - c^-$, we have

$$Lu - c^- u \leq -c^+ u \leq 0.$$

We are back to the above strong maximum principle with $c \leq 0$.

C-eg to $c \leq 0$.

$$\begin{aligned}
u &= \cos x \\
u'' + u &= 0 \ \text{ in } \ (-\pi/2, \pi/2) \\
u(0) &\ \text{ is an interior global maximum.}
\end{aligned}$$

C-eg to nonpositive minimum.

$$\begin{aligned}
u &= chx \\
u'' - u &= 0 \\
\text{positive } u(0) &= 1 \text{ is an interior global minimum.}
\end{aligned}$$

Heat equation $u_t - \triangle u = 0$.

Invariance for caloric functions, solutions to $u_t - \triangle u = 0$

$\cdot\ u(x+x_0, t+t_0)$
$\cdot\ u(Rx, t)$
$\cdot\ u(sx, s^2 t)$
RMK. Equations don't know/care which coordinates they are in.
$\cdot\ u+v$, au, where $v_t - \triangle v = 0$
$\cdot\ \int u(x-y, t-s)\,\varphi(y,s)\,dyds$
$\cdot \frac{u(x+\varepsilon e,t)-u(x,t)}{\varepsilon} \to D_e u$, $\frac{u(x,t+\varepsilon)-u(x,t)}{\varepsilon} \to D_t u$, so is $D_x^k D_t^m u$
$\cdot \frac{u(R\varepsilon x,t)-u(x,t)}{\varepsilon} \to D_\theta u = x_i u_j - x_j u_i$
$\cdot\ \frac{u\left((1+\varepsilon)x,(1+\varepsilon)^2 t\right)-u(x,t)}{\varepsilon} \to Du(x)\cdot x + 2u_t = ru_r + 2u_t$, so are $(r\partial_r + 2\partial_t)(ru_r + 2u_t), \cdots$
$\cdot\ \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} u\left(\frac{x}{t}, \frac{-1}{t}\right)$ Kelvin transformation
Examples.
$\cdot$ all harmonic functions are caloric, $x_1x_2x_3$, $1/|x|^{n-2}$, $\mathrm{Re}/\mathrm{Im}\, e^{3x_1+4x_2+i5x_3}$.
$\cdot\ t + |x|^2/2n, t^2 + t|x|^2/2n + (x_1^4 + \cdots + x_n^2)/24n$.
$\cdot\ \mathrm{Re}/\mathrm{Im}\, e^{i\xi\cdot x - |\xi|^2 t}, e^{\xi\cdot x + |\xi|^2 t}$
RMK. $e^{\xi\cdot x}, \mathrm{Re}/\mathrm{Im}\, e^{i(\xi\cdot x+|\xi|t)}, u(\xi\cdot x + |\xi|\, t)$ are wave functions. But e^{x_1+t} are both caloric and wave functions.
$\cdot$ caloric polynomials
$\Phi(x,t) = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}}$, then

$$\Phi(x,t)\, D_x^\alpha D_t^l \Phi\left(\frac{x}{t}, \frac{-1}{t}\right) \quad \text{is a caloric polynomial of degree } |\alpha| + l.$$

eg.

$$D_{123}\Phi(x,t) = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \frac{x_1x_2x_3}{(-2t)^3}\Bigg|_{\left(\frac{x}{t},\frac{-1}{t}\right)} \to (-t)^{n/2}\, e^{\frac{|x|^2}{4t}} \frac{x_1x_2x_3}{8}$$
$$\overset{\Phi(x,t)\ \cdot}{\to} (-1)^{n/2} \frac{x_1x_2x_3}{8}$$

$$D_{111}\Phi(x,t) = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \frac{-x_1^3 + 6tx_1}{8t^3}\Bigg|_{\left(\frac{x}{t},\frac{-1}{t}\right)} \to (-t)^{n/2}\, e^{\frac{|x|^2}{4t}} \frac{x_1^3 + 6tx_1}{8}$$
$$\overset{\Phi(x,t)\ \cdot}{\to} (-1)^{n/2} \frac{x_1^3 + 6tx_1}{8}.$$

$$D_t\Phi(x,t) = \frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} \left(\frac{-n/2}{t} + \frac{|x|^2}{4t^2}\right)\Bigg|_{\left(\frac{x}{t},\frac{-1}{t}\right)} \to (-t)^{n/2}\, e^{\frac{|x|^2}{4t}} \left(\frac{n}{2}t + \frac{1}{4}|x|^2\right)$$
$$\overset{\Phi(x,t)\ \cdot}{\to} (-1)^{n/2} \left(\frac{n}{2}t + \frac{1}{4}|x|^2\right)$$

<u>Self-similar way</u>
$*$ self similar solution $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{r}{\sqrt{t}}\right)$ to $u_t - \triangle u =$ "$\delta(x,t)$" or 0

$u_t = -\alpha \frac{1}{t^{\alpha+1}} v + \frac{1}{t^\alpha} \frac{-r}{2t^{3/2}} v' = \frac{1}{t^{\alpha+1}} \left(-\alpha v - \frac{r}{2\sqrt{t}} v' \right)$

$\triangle u = \frac{1}{t^\alpha} \left(\partial_{rr} + \frac{n-1}{r} \partial_r \right) v \left(r/\sqrt{t} \right)$

$\quad = \frac{1}{t^\alpha} \left(\frac{1}{t} v'' + \frac{n-1}{r} \frac{1}{\sqrt{t}} v' \right) = \frac{1}{t^{\alpha+1}} \left(v'' + \frac{n-1}{r/\sqrt{t}} v' \right)$

$v'' \left(\frac{r}{\sqrt{t}} \right) + \frac{n-1}{\frac{r}{\sqrt{t}}} v' \left(\frac{r}{\sqrt{t}} \right) + \frac{1}{2} \frac{r}{\sqrt{t}} v' \left(\frac{r}{\sqrt{t}} \right) + \alpha v \left(\frac{r}{\sqrt{t}} \right) = 0$

Set $\rho = \frac{r}{\sqrt{t}}$, then

$v''(\rho) + \frac{n-1}{\rho} v'(\rho) + \frac{1}{2} \rho v'(\rho) + \alpha v(\rho) = 0$

$$(\rho^{n-1} v')' + \underbrace{\frac{1}{2} \rho^n v' + \alpha \rho^{n-1} v}_{=\left(\frac{1}{2}\rho^n v\right)' \text{ when } \alpha = n/2} = 0$$

* Let $\alpha = n/2$

$\rho^{n-1} v' + \frac{1}{2} \rho^n v = c$ or $v' + \frac{\rho}{2} v = c/\rho^{n-1}$ or $\left(v e^{\frac{\rho^2}{4}} \right)' = c e^{\frac{\rho^2}{4}} / \rho^{n-1}$

Then $v(\rho) = c' e^{-\frac{\rho^2}{4}} + c\, e^{-\frac{\rho^2}{4}} \int e^{\frac{\rho^2}{4}} / \rho^{n-1} d\rho$ and

$$u(x,t) = \frac{1}{t^{n/2}} \left[c' e^{-\frac{|x|^2}{4t}} + \underbrace{c e^{-\frac{|x|^2}{4t}} \left(\int e^{\frac{\rho^2}{4}} / \rho^{n-1} d\rho \right) \Big|_{\rho = r/\sqrt{t}}} \right]$$

After some testing for the fundamental solution, we find $c = 0$.

Mean value equality

Recall the derivation of the "solid" mean value formula for harmonic functions

$$\int_{B_R} u \triangle v - v \triangle u = \int_{\partial B_R} u v_\gamma - v u_\gamma$$

Set $v = \underbrace{\frac{-1}{(n-2)\,|\partial B_1|}}_{c_n} \left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}} \right)$

$$u(0) = \int_{\partial B_R} u v_\gamma dA = \int_{\Gamma = \frac{-1}{c_n R^{n-2}} = l} u\, |D\Gamma|\, dA$$

where $\Gamma = \frac{-1}{c_n} \frac{1}{|x|^{n-2}}$.

As R goes from 0 to 1,the level of Γ runs from $-\infty$ to $-1/c_n$, we seek a weight satisfying $1 = \int_{-\infty}^{-1/c_n} w(l)\, dl$, where $w(l) = c(-l)^\alpha$ to be determined.

$$\begin{aligned} u(0) &= \int_{-\infty}^{-1/c_n} w(l) \int_{\Gamma = l} u\, |D\Gamma|\, dA dl \\ &= \int_{B_1} u\; w(l)\, |D\Gamma|^2\, dx, \end{aligned}$$

where we used change of variable or co-area formula: $dx = dvol = dA\, \frac{dl}{|D\Gamma|}$

figure gradient "height"=dl/ $|D\Gamma|$

Now let $w(l)\left|D\Gamma\right|^2 = 1/\left|B_1\right|$, then

$$\begin{aligned} w(l) &= \frac{1}{|B_1|}\left[\frac{c_n}{(n-2)}r^{n-1}\right]^2 \overset{\frac{-1}{c_n r^{n-2}}=l}{=} \frac{1}{|B_1|}\left[\frac{c_n}{(n-2)}\left(-c_n l\right)^{-\frac{n-1}{n-2}}\right]^2 \\ &= n(n-2)^{\frac{2}{n-2}}\left|\partial B_1\right|^{\frac{2}{n-2}}(-l)^{-\frac{2(n-1)}{n-2}}. \end{aligned}$$

RMK. Old weight way, $1 = \int_0^1 nr^{n-1}dr \overset{r=(-c_n l)^{-1/(n-2)}}{=} \frac{n}{n-2}c_n^{\frac{-n}{n-2}}\int_{-\infty}^{-1/c_n}(-l)^{-\frac{2(n-1)}{n-2}}\,dl$. And pleasantly $w(l)\left|D\Gamma\right|^2 = 1/\left|B_1\right|$!

Mean value equality for caloric functions

$$u(0,0) = \frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi=(4\pi R^2)^{-n/2}} u\frac{|x|^2}{\sqrt{4t^2|x|^2 + \left(2nt+|x|^2\right)^2}}dA.$$

figure: heat sphere

One "solid" version

$$u(0,0) = \frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi\geq(4\pi R^2)^{-n/2}} u\frac{|x|^2}{4t^2}dA,$$

where

$$\Phi(x_0,t_0;x,t) = \frac{1}{\left[4\pi(t_0-t)\right]^{n/2}}e^{-\frac{|x_0-x|^2}{4(t_0-t)}}$$

figure: heat kernel graphs

Derivation of the hollow version.

$$\begin{aligned} \int_{U_T} div(uDv) &= \int_{U_T}(div, D_t)(uDv,0) = \int_{\partial U_T}(uDv,0)\cdot(\gamma_x,\gamma_t)\,dA \\ -)\int_{U_T} div(vDu) &= \int_{U_T}(div, D_t)(uvDu,0) = \int_{\partial U_T}(vDu,0)\cdot(\gamma_x,\gamma_t)\,dA \\ +)\int_{U_T} D_t(uv) &= \int_{U_T}(div, D_t)(0,uv) = \int_{\partial U_T}(0,uv)\cdot(\gamma_x,\gamma_t)\,dA \end{aligned}$$

$\Rightarrow$

$$\int_{U_T} u\Big(\underbrace{D_t v + \triangle v}_{\text{“}\delta(0,0)''}\Big) + v\Big(\underbrace{D_t u - \triangle u}_{0}\Big) = \int_{\partial U_T} uv_{\gamma_x} - vu_{\gamma_x} + uv\gamma_t\ dA.$$

Take $v = \frac{1}{[4\pi(-t)]^{n/2}}e^{\frac{|x|^2}{4t}} - \frac{1}{[4\pi]^{n/2}}$, then

$$\begin{aligned} E &= \left\{\frac{1}{\left[4\pi(-t)\right]^{n/2}}e^{\frac{|x|^2}{4t}} \geq \frac{1}{[4\pi]^{n/2}}\right\} \overset{t\leq 0}{\Leftrightarrow} \left\{|x|^2 \leq 2nt\ln(-t)\right\} \\ U_T &= E\cap\{t\leq s\}. \end{aligned}$$

figure U_T

We have from the Green's identity

$$0 = \int_{\partial E \cap \{t \leq s\}} u v_{\gamma_x} + \underbrace{\int_{I_s} u v dA}_{\substack{\to u(0,0) \\ \text{as } s \to 0^-, \text{ say for } C^0 \ u}}$$

$$v_{\gamma_x} = D\Phi \cdot \gamma_x = D\Phi \cdot \frac{-D\Phi}{|(D\Phi, D_t\Phi)|} = \frac{-|D\Phi|^2}{|(D\Phi, D_t\Phi)|}$$
$$= \frac{-\Phi^2 \left|\frac{x}{2t}\right|^2}{\sqrt{\Phi^2 \left|\frac{x}{2t}\right|^2 + \Phi^2 \left[\frac{n}{2}\frac{1}{-t} - \frac{|x|^2}{4t^2}\right]^2}} = -\Phi \frac{|x|^2}{\sqrt{4t^2 |x|^2 + \left(2nt + |x|^2\right)^2}}.$$

Therefore

$$u(0,0) = \frac{1}{(4\pi)^{n/2}} \int_{\Phi = (4\pi)^{-n/2}} u \frac{|x|^2}{\sqrt{4t^2 |x|^2 + \left(2nt + |x|^2\right)^2}} dA$$
$$u(0,0) = \frac{1}{(4\pi R^2)^{n/2}} \int_{\Phi = (4\pi R^2)^{-n/2}} u \frac{|x|^2}{\sqrt{4t^2 |x|^2 + \left(2nt + |x|^2\right)^2}} dA$$
$$= \int_{\Phi = (4\pi R^2)^{-n/2} = l} u \frac{|D\Phi|^2}{|(D\Phi, D_t\Phi)|} dA$$

Having this sphere version, let's have a "solid" mean value formula. Set $w(l)$ s.t. $1 = \int_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty} w(l)\, dl$,

$$u(0,0) = \int_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty} w(l) \int_{\Phi = l} u \frac{|D\Phi|^2}{|(D\Phi, D_t\Phi)|} dA\ dl$$
$$= \int_{\Phi \geq \left(\frac{1}{4\pi}\right)^{n/2}} u w(l) |D\Phi|^2 dx dt$$
$$= \int_{\Phi \geq \left(\frac{1}{4\pi}\right)^{n/2}} u \underbrace{w(l) \Phi^2}_{\text{constant}} \left|\frac{x}{2t}\right|^2 dx dt$$

Let $w(l) = \left(\frac{1}{4\pi}\right)^{n/2} \frac{1}{l^2}$, then $\int_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty} w(l)\, dl = \left(\frac{1}{4\pi}\right)^{n/2} \frac{-1}{l}\Big|_{\left(\frac{1}{4\pi}\right)^{n/2}}^{\infty} = 1$. Thus

$$u(0,0) = \frac{1}{(4\pi)^{n/2}} \int_{\Phi \geq \left(\frac{1}{4\pi}\right)^{n/2}} u \ \frac{|x|^2}{4t^2} dx dt$$

or

$$u(x_0,t_0)=\frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi(x_0,t_0,x,t)\geq\left(\frac{1}{4\pi R^2}\right)^{n/2}} u\ \ \frac{|x_0-x|^2}{4(t_0-t)^2}dxdt$$

figure heat ball

In particular, for $u\equiv 1$

$$1=\frac{1}{(4\pi)^{n/2}}\int_{\frac{e^{\frac{|x|^2}{4t}}}{[4\pi(-t)]^{n/2}}\geq\frac{1}{(4\pi)^{n/2}}}\frac{|x|^2}{4t^2}dxdt=\int_{\underbrace{\Phi\geq 1}_{\text{say } 4\pi R^2=1}}\frac{|x|^2}{4t^2}dxdt.$$

RMK. Other choices of weight
eg. $w(l)=c/l^{2-\alpha}$, $\alpha<1$

$$u(0,0)=c\int_{\Phi\geq 1}u\ \ \Phi^{\alpha}\frac{|x|^2}{4t^2}dxdt,$$

the kernel is still singular.

Now that we have discovered mean value formulas for harmonic and caloric functions, we could provide a derivative way to verify those formulas (to impress others).

Verification of solutions to $\triangle u=0$ satisfying

$$\frac{1}{s^n}\int_{B_s(0)}udx=\text{constant}=u(0)\,|B_1|\,.$$

Indeed, we just prove

$$\begin{aligned}\frac{d}{ds}\left[\frac{1}{s^n}\int_{B_s(0)}udx\right]&=\frac{d}{ds}\left[\frac{1}{s^n}\int_0^s\int_{\partial B_s(0)}udAds\right]\\&=-ns^{-n-1}\int_{B_s(0)}udx+s^{-n}\int_{\partial B_s(0)}udA\\&=0.\end{aligned}$$

Note

$$\begin{aligned}\int_{B_s}2n\ udx&=\int_{B_s}u\triangle|x|^2dx=\int_{B_s}|x|^2\triangle udx+\int_{\partial B_s}u\partial_\gamma|x|^2-|x|^2\partial_\gamma u\ dA\\&=2s\int_{\partial B_s}udA+\int_{B_s}\left(|x|^2-s^2\right)\triangle u\ dx\ \ \left(\int_{\partial B_s}\partial_\gamma u=\int_{B_s}\triangle u\right)\\&=2s\int_{\partial B_s}udA.\end{aligned}$$

Dividing both sides by s^{n+1}, we see changing rate of the average of harmonic function is zero.

This mean value verification for harmonic functions leads to verification proofs of mean value formula for caloric functions, minimal surfaces (monotonicity formula), should also for mean curvature flow.

Verification of solutions to $\triangle u - u_t = 0$ satisfying

$$\frac{1}{R^n}\int_{\frac{1}{(-t)^{n/2}}e^{\frac{|x|^2}{4t}}\geq R^{-n}} u\frac{|x|^2}{4t^2}dxdt = \text{constant} = (4\pi)^{n/2}\, u\,(0,0)\,.$$

Indeed we just prove the derivative of the left hand side with respect to R is zero. In order to take the derivative of the integral, we re-formulate it by the co-area formula

$$\int_{\phi\leq R} u\frac{|x|^2}{4t^2}dxdt = \int_0^R\int_{\phi=l} u\frac{|x|^2}{4t^2}dA\frac{dl}{|\nabla\phi|},$$

where

$$\phi = (-t)^{-1/2}\, e^{\frac{-|x|^2}{4nt}}.$$

Then we need to prove

$$\begin{aligned}
&\frac{d}{dR}\left[\frac{1}{R^n}\int_{\phi\leq R} u\frac{|x|^2}{4t^2}dxdt\right]\\
&= -nR^{-n-1}\int_{\phi\leq R} u\frac{|x|^2}{4t^2}dxdt + R^{-n}\int_{\phi=R} u\frac{|x|^2}{4t^2}\frac{1}{|\nabla\phi|}dA\\
&= 0.
\end{aligned}$$

We look for a function φ such that $\triangle\varphi + \varphi_t = -\,|x|^2/4t^2$ (in order to apply Green's identity). Surprisingly, the log function of the backward heat kernel does it

$$\varphi = -\frac{n}{2}\ln(-t) + \frac{|x|^2}{4t} + n\ln R.$$

We proceed with Green's identity

$$\begin{aligned}
\int_{\phi<R} u\frac{-\,|x|^2}{4t^2}dxdt &= \int_{\phi<R} u\,(\triangle\varphi + \varphi_t)\,dxdt\\
&= \int_{\phi<R}\varphi\,(\triangle u - u_t)\,dxdt + \int_{\phi=R} u\varphi_{\gamma_x} - \varphi u_{\gamma_x} + u\varphi\gamma_t dA\\
&= \int_{\phi=R} u\varphi_{\gamma_x}dA \qquad (\text{Recall } \varphi = 0 \text{ on } \phi = R)\\
&= -\frac{R}{n}\int_{\phi=R} u\frac{|x|^2}{4t^2}\frac{1}{|\nabla\phi|}dA,
\end{aligned}$$

where we used

$$\varphi_{\gamma_x} = D\varphi \cdot \frac{D\phi}{|\nabla\phi|} = \frac{x}{2t} \cdot \frac{\frac{-x}{2nt}\phi}{|\nabla\phi|}$$
$$= -\frac{R}{n}\frac{|x|^2}{4t^2}\frac{1}{|\nabla\phi|} \quad \text{on the boundary } \phi = R.$$

This is exactly what we need in showing $\frac{d}{dR}\left[\frac{1}{R^n}\int_{\phi\leq R} u\frac{|x|^2}{4t^2}dxdt\right] = 0.$

Applications of mean value formulas

App1. Strong max principle:

Let $u \in C_1^2$ solution to $u_t - \triangle u = 0$ in U_T.

$\cdot$ $\max u$ only attains at the parabolic boundary of U_T;

$\cdot$ otherwise, if $\max u = u(x_0, t_0)$, where (x_0, t_0) is an interior or non-parabolic boundary of U_T, then we have $u(x,t) \equiv u(x_0,t_0)$ for all (x,t) in the closure of the connected set of $U_T \cap \{t \leq t_0\}$ by chain of downward heat balls.

Def: Parabolic boundary: the closure of those points that cannot center any backward heat ball inside the domain U_T.

Examples of U_T

figure parabolic bdry

Proof of the strong max principle.

Suppose $u(x_0,t_0) = \max_{U_T} u$ and (x_0,t_0) is not a parabolic boundary point, that is, (x_0,t_0) centers a heat ball in U_T. By the mean value formula in this ball

$$u(x_0,t_0) = \frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi(x_0,t_0,x,t)\geq(4\pi R^2)^{-n/2}} u(x,t) \quad \frac{|x_0-x|^2}{4(t_0-t)^2}dxdt$$
$$\overset{\text{kernel}\geq 0}{\leq} \frac{1}{(4\pi R^2)^{n/2}}\int_{\Phi(x_0,t_0,x,t)\geq(4\pi R^2)^{-n/2}} u(x_0,t_0) \quad \frac{|x_0-x|^2}{4(t_0-t)^2} \quad dxdt$$
$$\overset{\int \text{kernel}=1}{=} u(x_0,t_0).$$

Thus $u(x,t) \equiv u(x_0,t_0)$ in

$$\left\{(x,t)\middle| \ \frac{e^{-\frac{|x_0-x|^2}{4(t_0-t)}}}{(t_0-t)^{n/2}} \geq \frac{1}{R^n}\right\}.$$

RMK. The closure includes the points at horizontal level $\{t = t_0\}$, this is because such a point (y,t_0) is the limit of (y,s) as $s \uparrow t_0$, and the segment (x_0,t_0) -(y,s) can be covered a chain of heat balls.

figure downward segment

Then $u(y,t) = \lim u(y,s) = \lim u(x_0,t_0)$.

Uniqueness of caloric function on bounded domains

Let u, v be two $C_1^2(U_T) \cap C_0^0(\bar{U}_T)$ solutions to $w_t - \triangle w = 0$ in U_T, and $u = v$ on the parabolic boundary of U_T. THEN $u \equiv v$.

RMK. U_T including U_∞ domains like $\{t > \text{convex}(x)\}$, say

$$\text{figure } t \geq |x|^4.$$

Question. What happens to $R^n \times [0,T]$ or $R_+^1 \times [0,T]$?

App2. Regularity

Faking space dimension $R^n \to R^{n+m}$ will lead us to a C_1^2 (even better ones for larger m) kernel in the mean value formula:

$$u_t(x,t) - (\triangle_x + \triangle_y) u(x,t) = 0$$

$$\begin{aligned} u(x_0,t_0) &= \int_{\Phi(x_0,y_0,t_0,x,y,t) \geq (4\pi R^2)^{-n/2}} u(x,t) \ \frac{|x_0 - x|^2 + |y_0 - y|^2}{4(t_0 - t)^2} \, dydxdt \\ &= \int_{\Phi(x_0,t_0,x,t) \geq (4\pi R^2)^{-n/2}} u(x,t) \ K(x_0 - x, t_0 - t) \, dxdt \end{aligned}$$

where Kuptsov kernel

$$K(x_0 - x, t_0 - t) = \int_{|x_0-x|^2 + |y_0-y|^2 \leq 4(t-t_0)\left[\frac{(n+m)}{2}\ln(t_0-t) - (n+m)\ln R\right]} \frac{|x_0 - x|^2 + |y_0 - y|^2}{4(t_0 - t)^2} \, dy$$

is C_x^2 and C_t^1 for $m \geq 5$. Then start from L^1 function, satisfying the parabolic mean value formula, we immediately have C_1^2 solution to the heat equation (no need existence).

We can also get interior estimates via multiple-integrals.

First, by using a different argument via Green's identity, we show the solutions are C^∞ in x, t and C^ω in x. Recall the fundamental solution is not C^ω in t.

Green's identity

$$\int_{U_T} u\Big(\underbrace{D_t v + \triangle v}_{\text{“}\delta(0,0)''}\Big) + v\Big(\underbrace{D_t u - \triangle u}_{0}\Big) = \int_{\partial U_T} uv_{\gamma_x} - vu_{\gamma_x} + uv\gamma_t \, dA.$$

$v = \Phi(x,t;y,s) = e^{\frac{|x-y|^2}{4(t-s)}} / [4\pi(t-s)]^{n/2}$

$$
\begin{aligned}
u(x,t) = &-\int_{\partial U\times[0,t]} u(y,s) \underbrace{\Phi_{\gamma_y}(x,t;y,s)}_{C^\omega \text{ in } x \text{ not in } t,\ C^\infty \text{ in } t} dA \\
&+\int_{\partial U\times[0,t]} u_{\gamma_y}(y,s) \underbrace{\Phi(x,t;y,s)}_{C^\omega \text{ in } x \text{ not in } t,\ C^\infty \text{ in } t} dA \\
&+\int_U u(y,0) \underbrace{\Phi(x,t;y,0)}_{C^\omega \text{ in } x,t \text{ for } t\geq\delta_0} dy.
\end{aligned}
$$

So we conclude u is $C^\omega(x)$ and $C^\infty(t)$ in $U_T' \subset U_T$.

Interior estimates

$$
\max_{C_1} \left|D_x^k D_t^l u\right| \leq C(k,l,K) \max_{C_2} \left|D_x^2 u\right| + |D_t u| \leq C(k,l,K) \max_{C_3} |u|
$$

or

$$
\max_{C_R} \left|D_x^k D_t^l u\right| \leq \frac{C(k,l,K)}{R^{k+2l}} \max_{C_{3R}} |u|
$$

via scaling $v(x,t) = u(Rx, R^2 t)$, $D_x^k D_t^l v(x,t) = R^{k+2l}\ D_x^k D_t^l u|_{(Rx,R^2t)}$.

Liouville Th'm

Global (eternal) solution, say C_1^2 to $u_t - \triangle u = 0$ in $R^n \times (-\infty, +\infty)$ satisfying

$$
|u(x,t)| \leq A\left(|x|^k + |t|^l\right) \quad \text{for large } |x| + |t|
$$

must be a caloric polynomial of degree less than $k + 2l$.

Proof.

$$
\left|D_x^k D_t^l u(0,0)\right| \leq \frac{C(k',l',K)}{R^{k'+2l\prime}} A\ \left(R^k + R^{2l}\right) \overset{R\to\infty}{\to} 0
$$

for $k' + 2l\prime > k + 2l$. Note $(0,0)$ could be anywhere, so $u(x,t)$ is a caloric polynomial of degree $\leq k + 2l$.

App3. Harnack inequality

$u \geq 0$ solution to $u_t - \triangle u = 0$, then

$$
\max_{C_r^-} u \leq C(n) \min_{C_r^+} u.
$$

Here $C_r^- = B_r \times (-3r^2, -2r^2)$ and $C_r^- = B_r \times (-r^2, 0)$.

figure Harnack.

One proof is via "fake" dimension mean value formula, it is little "involved" in calculating the *positive weight*. However, everything is at calculus level.

eg. The following example shows the delay of time is necessary in the Harnack inequality. Take the heat kernel Φ in $Q = B_1(2) \times (-1,1)$, we have

$$0 < \max_Q \Phi \not< 0 = \min_Q \Phi.$$

Uniqueness for Cauchy problem with constraints in $R^n \times R^+$.

Max Principle: Let u be $C_1^2(R^n \times (0,T)) \cap C(R^n \times [0,T])$ solution to

$$\begin{cases} u_t - \triangle u = 0 \\ u(x,0) = g(x) \end{cases}.$$

Suppose $|u(x,t)| \le Ae^{a|x|^2}$ in $R^n \times [0,T]$. Then

$$|u(x,t)| \le \sup_{R^n} g(x) \quad \text{in } R^n \times [0,T].$$

Proof. We only need to prove $u(x,t) \le \sup_{R^n} g(x) \triangleq M$ for subcaloric solution $u_t - \triangle u \le 0$ with sub quadratic-exponential growth $u(x,t) \le Ae^{a|x|^2}$.

Step1.

figure t-direction thin domain.

For any $\mu > 0$, set

$$\begin{aligned} v &= M + \mu \frac{1}{\left(\frac{1}{8a} - t\right)^{n/2}} e^{\frac{|x|^2}{4\left(\frac{1}{8a} - t\right)}} \\ v_t - \triangle v &= 0 \ge u_t - \triangle u \\ v &\ge u \ \text{ on } \partial B_{R_\mu} \times \left[0, \frac{1}{16a}\right] \text{ for } R_\mu \text{ large} \\ v &\ge u \ \text{ on } R^n \times \{0\}. \end{aligned}$$

RMK. Invariance way to construct this barrier:

* $\frac{1}{\sqrt{t}} e^{\frac{|x|^2}{-4t}} \overset{\sqrt{-1}}{\dashrightarrow} \frac{1}{\sqrt{-t}} e^{\frac{|x|^2}{-4t}}$ still sol.

* the time shift sol v is quadratic-exponential growth at $t = 0$, it goes to ∞ as $t \nearrow \frac{1}{8a}$ for every x.

It follows from the maximum principle on the bounded domain,

$$u(x,t) \le M + \mu \frac{1}{\left(\frac{1}{8a} - t\right)^{n/2}} e^{\frac{|x|^2}{4\left(\frac{1}{8a} - t\right)}} \quad \text{in } B_{R_\mu} \times \left[0, \frac{1}{16a}\right].$$

So for any fixed (x_0, t_0) with $t_0 \leq \frac{1}{16a}$, we have

$$u(x_0, t_0) \leq M + \mu \frac{1}{\left(\frac{1}{8a} - t_0\right)^{n/2}} e^{\frac{|x_0|^2}{4\left(\frac{1}{8a} - t_0\right)}} \overset{\mu \to 0}{\to} M.$$

Step2. The above argument works equally well on $\left[\frac{1}{16a}, \frac{2}{16a}\right]$, $\left[\frac{2}{16a}, \frac{3}{16a}\right]$, $\cdots$, still $[0, T]$.

Corollary. The Cauchy problem with growth constraint

$$\begin{cases} u_t - \triangle u = f \\ u(x, 0) = g(x) \end{cases} \quad \text{in} \quad R^n \times (0, T)$$

with $|u(x, t)| \leq A e^{a|x|^2}$ in $R^n \times [0, T]$, has at most one solution.

Proof. The difference of any two solutions satisfies the condition in the max principle with $g = 0$ and difference is less $2Ae^{a|x|^2}$, so the difference is 0.

eg. The caloric function, or a solution to $\begin{cases} u_t - \triangle u = 0 \\ u(x, 0) = e^{a|x|^2} \end{cases}$

Integral way to construct the barrier:

$$\begin{aligned} u(x, t) &= \frac{1}{(4\pi t)^{n/2}} \int_{R^n} e^{\frac{|x-y|^2}{-4t}} e^{a|y|^2} dy \\ &= \frac{1}{\pi^{n/2}} \int_{R^n} e^{-|y|^2 + a\left|2\sqrt{t}y - x\right|^2} dy \\ &= \frac{1}{(1 - 4at)^{n/2}} e^{\frac{\alpha}{1-4at}|x|^2} \end{aligned}$$

is

* $a > 0$ unique in $R^n \times [0, \frac{1}{4a})$ with constraint $|u(x, t)| \leq \frac{1}{(1-4at)^{n/2}} e^{\frac{\alpha}{1-4at}|x|^2}$ and grows faster than $e^{a|x|^2}$ for $t > 0$;

* $a < 0$ uniqueness in $R^n \times [0, \infty)$ with constraint $|u(x, t)| \leq e^{100|x|^2}$ and grows faster than $e^{a|x|^2}$ for $t > 0$.

The message: the growth/decay rate is not preserved precisely.

Nonuniqueness of Cauchy problem $\begin{cases} u_t - \triangle u = 0 \text{ in } R^n \times [0, \infty) \\ u(x, 0) = 0 \end{cases}$

Tikhonov's counterexample.

Idea of construction:

* along $t = 0$, position $u(x, 0)$ alone determines all the derivatives (if analytic).

* along $x = 0$, position $u(0, t)$ and velocity $u_x(0, t)$ determines all the derivatives in x, (if analytic in x).

Now we solve a "real" Cauchy problem along the t-axis

$$\begin{cases} u(0, t) = g(t) \\ u_x(0, t) = 0 \end{cases}$$

$u_x(0,t)=0,\ u_{xxx}(0,t)=u_{xt}(0,t)=0,\ \cdots,D_x^{2k+1}u(0,t)=D_t^k u_x(0,t)=0$

$u_{xx}(0,t)=u_t(0,t)=g'(t),u_{xxxx}(0,t)=u_{tt}(0,t)=g''(t),\cdots,D_x^{2k}u(0,t)=D_t^k u(0,t)=D_t^k g(t)$.

Assuming u is C^ω in terms of x, then

$$u(x,t)=g(t)+\sum_{k=1}^{\infty}\frac{g^{(k)}(t)}{(2k)!}x^{2k}.$$

Technical realization:

$$\text{graph for } g(t)=\begin{cases} e^{-\frac{1}{t^\alpha}} & t>0 \\ 0 & t\le 0 \end{cases} \quad \text{need } \alpha>1.$$

How to control the derivatives

1st Direct try.

$g(t)=e^{-t^{-\alpha}}$

$g'=e^{-t^{-\alpha}}\alpha t^{-\alpha-1}$

$g''=e^{-t^{-\alpha}}\left[\left(\alpha t^{-\alpha-1}\right)^2-\alpha(\alpha+1)t^{-\alpha-2}\right]$

$g'''=e^{-t^{-\alpha}}\left[\left(\alpha t^{-\alpha-1}\right)^3+\cdots+\alpha(\alpha+1)(\alpha+2)t^{-\alpha-3}\right]$

$\cdots$

$g^{(k)}=e^{-t^{-\alpha}}\left[\left(\alpha t^{-\alpha-1}\right)^k+\cdots\pm\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)t^{-\alpha-k}\right]$

$\approx e^{-t^{-\alpha}}k!\left(\alpha t^{-\alpha-1}\right)^k$

$\frac{g^{(k)}}{(2k)!}\approx e^{-t^{-\alpha}}\frac{k!}{(2k)!}\left(\alpha t^{-\alpha-1}\right)^k$, and $\frac{k!}{(2k)!}=\frac{1}{2^k\ 1\cdot3\cdot5\cdots(2k-1)}\approx\frac{1}{2^{2k}\ k!}$

$|u(x,t)|\le g(t)+\sum_{k=1}^{\infty}\frac{e^{-t^{-\alpha}}}{k!\ 2^{2k}}\left(\alpha t^{-\alpha-1}\right)^k\ x^{2k}=e^{-t^{-\alpha}}e^{\frac{\alpha t^{-\alpha-1}}{4}x^2}=e^{-\frac{1}{t^\alpha}+\frac{\alpha x^2}{4}\frac{1}{t^{\alpha+1}}}\overset{t\to0+}{\longrightarrow}\infty$.

2nd complex try

Observe $e^{-t^{-\alpha}}$ is analytic when $t>0$

figure complex plan $z=t+is$

$g(t)=\frac{1}{2\pi i}\int_\gamma\frac{g(z)}{z-t}dz$

$g^{(k)}(t)=\frac{k!}{2\pi i}\int_\gamma\frac{g(z)}{(z-t)^{k+1}}dz$

Now by continuity of $z^{-\alpha}$ at 1, $\operatorname{Re}z^{-\alpha}\ge\frac{1}{2}$ for $|z-1|\le\mu$, where $\mu=\mu(1/2)<1$, then $\operatorname{Re}(tz)^{-\alpha}\ge\frac{1}{2}t^{-\alpha}$ for $|tz-t|\le\mu t$

or $\operatorname{Re}z^{-\alpha}\ge\frac{1}{2}t^{-\alpha}$ for $|z-t|\le\mu t$

and

$$\left|e^{-z^{-\alpha}}\right| = e^{\mathrm{Re}\,-z^{-\alpha}} \leq e^{-\frac{1}{2}t^{-\alpha}} \quad \text{for } |z-t| \leq \mu t.$$

So

$$\left|g^{(k)}(t)\right| \leq \frac{k!}{2\pi}\left|\int_{|z-t|=\mu t} \frac{g(z)}{(z-t)^{k+1}}dz\right| \leq \frac{k!}{2\pi}\frac{e^{-\frac{1}{2}t^{-\alpha}}}{(\mu t)^{k+1}}2\pi\mu t = \frac{k!\ e^{-\frac{1}{2}t^{-\alpha}}}{(\mu t)^{k}}$$

and

$$\begin{aligned}
|u(x,t)| &\leq g(t) + \sum_{k=1}^{\infty}\frac{1}{(2k)!}\frac{k!\ e^{-\frac{1}{2}t^{-\alpha}}}{(\mu t)^{k}}\ x^{2k} \\
&\leq e^{-t^{-\alpha}} + e^{-\frac{1}{2}t^{-\alpha}}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{x^2}{\mu t}\right)^{k} \\
&\leq e^{-\frac{1}{2}t^{-\alpha}+\frac{x^2}{\mu t}}\begin{cases} <\infty \ \text{ for all } (x,t) \text{ with } t>0, \\ \to 0 \ \text{ as } t\to 0+ \text{ for each fixed } x, \end{cases} \quad \text{provided } \alpha>1.
\end{aligned}$$

Then the Tikhonov's series converges, we've constructed a "super" quadratic-exponential caloric function such that $u_t - u_{xx} = 0$ and $u(x,0) = 0$, $u(x,t)$ is not identically 0 for $t > 0$.

RMK. Choosing

$$g(t) = \begin{cases} e^{-\frac{1}{t^{\alpha}}-\frac{1}{(1-t)^{\alpha}}} & t>0 \\ 0 & t\leq 0 \text{ or } t\geq 1 \end{cases},$$

figure complex plan $z = t + is$ for this g

we have another Tikhonov solution/caloric function vanishes before 0 or after 1.

figure Tikhonov double sided vanishing

RMK. Let $v = u^2$, where u is the above Tikhonov caloric function, then $v_t - \triangle v = 2uu_t - 2u \triangle u - 2|Du|^2 = -2|Du|^2 \leq 0$. This v is non-negative sub-caloric function vanishing at $t = 0$ and $t = 1$, yet doesn't vanish identically between $(0,1)$.

Nonanalytic, yet smooth solution in $R^n \times [0,\infty)$

eg.

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = e^{-x^4} \end{cases}$$

has the bounded solution/quadratic-exponential growth solution

$$\begin{aligned} u(x,t) &= \frac{1}{(4\pi t)^{1/2}} \int_{R^1} e^{-\frac{|x-y|^2}{4t}} e^{-|y|^4} dy \\ &= \frac{1}{\pi^{1/2}} \int_{R^1} e^{-y^2} e^{-\left|x-2\sqrt{t}y\right|^4} dy \end{aligned}$$

which is $C^\infty(R^1 \times [0,\infty))$, but NOT C^ω at $t=0$.

In fact, if $u(0,t)$ is analytic in terms of t near $t=0$, then

$$u(0,t) = \sum_{k=0}^{\infty} a_k t^k,$$

where

$$\begin{aligned} a_k = \frac{1}{k!} D_t^k u(0,0) = \frac{1}{k!} D_x^{2k} u(0,0) &= \frac{1}{k!} D_x^{2k} \sum_{m=0}^{\infty} \frac{(-x^4)^m}{m!} \Bigg|_{x=0} \\ &\overset{2k=4m}{=} \frac{1}{(2m)!} \frac{(4m)!}{m!} > m!. \end{aligned}$$

So

$$\left|a_{2m} t^{2m}\right| > m! t^{2m} \overset{m\to\infty}{\to} \infty \text{ for any fixed } t > 0.$$

Then the series diverges, $u(x,t)$ cannot be analytic in t at $(0,0)$.

Nonexistence of nonnegative solution to Cauchy problem

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = e^{x^4} \end{cases}$$

First note the representation

$$\frac{1}{(4\pi t)^{1/2}} \int_{R^1} e^{-|x-y|^2} e^{y^4} dy = \infty.$$

Overheated, the nonnegative solution blows up once time starts.

Now the proof.

figure for e^{x^2} and g_k

$g_k(x) \in C_0^\infty(B_{k+1})$ and $g_k(x) = e^{x^4}$ on B_k. The bounded C_1^2 solution to

$$\begin{cases} u_t - u_{xx} = 0 \\ u(x,0) = g_k \end{cases}$$

is

$$u_k(x,t) = \frac{1}{\pi^{1/2}} \int_{R^1} e^{-y^2} g_k\left(x - 2\sqrt{t}y\right) dy.$$

figure for e^{-y^2} and $g_k\left(R - 2\sqrt{t}y\right)$

For each fixed k and say, 0.9, there exists $R_k = R(k, 0.9)$ large so that

$$0 \le u_k(\pm R, t) \le 0.9 \quad \text{for } 0 \le t \le 1.$$

This is because

$$u_k(\pm R_k, t) \le \frac{1}{\pi^{1/2}} e^{-\left(\frac{R-k-1}{2\sqrt{t}}\right)^2} \; e^{k^4} \; \frac{k+1}{\sqrt{t}}.$$

Then as u is nonnegative, $u_k \le 0.9 + u$ on the parabolic boundary of the cylinder $B_{R_k} \times [0,1]$. The maximum principle implies $u_k \le 0.9 + u$ in $B_{R_k} \times [0,1]$. In particular

$$u_k(0,1) \le 0.9 + u(0,1) \quad \text{for all } k.$$

But $u_k(0,1)$ goes to $+\infty$, as k goes to $+\infty$. A contradiction!

In fact, $u(0,l)$ is forced to be ∞ for all small $l > 0$.

Question: Existence of sign-changing solutions? Answer: YES.

As the mean curvature flow operator is variable coefficients

$$\triangle_g - \partial_t = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j\right) = g^{ij} \partial_{ij} + \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij}\right) \partial_j,$$

we need to deal with general parabolic operator

$$L = \sum a_{ij}(x,t) \partial_{ij} + \sum b_i(x,t) \partial_i - \partial_t$$

where λ-parabolic means

$$(a_{ij}(x,t)) \le \lambda I > 0.$$

Notation $\Omega_T = \Omega \times (0,T] \subset R^n \times R^1$, the parabolic boundary of Ω_T consists of two parts, the bottom and lateral side

$$\partial_p \Omega_T = \Omega \times \{0\} \cup \partial\Omega \times [0,T].$$

Note that the top side $\Omega \times \{T\}$ is considered as interior of the "parabolic" domain Ω_T.

Weak Maximum Principle. Assume $u, v \in C^{2,1}(\Omega_T) \cap C\left(\bar{\Omega}_T\right)$,

- Max. If $Lu \ge 0$ in Ω, then $\max_{\bar{\Omega}_T} u = \max_{\partial_p \Omega} u$.
- Min. If $Lu \le 0$ in Ω, then $\min_{\bar{\Omega}_T} u = \min_{\partial_p \Omega} u$.
- Comparison. If $Lu \ge Lv$ in Ω_T and $u \le v$ on $\partial_p \Omega$, then $u \le v$ in Ω_T.

Proof. It suffices to handle the max case, the other cases follow by applying the first case to $-u$ and $u-v$ respectively.

Strict case. $Lu>0$.

Suppose u attains its maximum at an interior point (x_*,t_*), then $Du(x_*,t_*)=0$, $D^2u(x_*)\leq 0$, and $\partial_t u(x_*,t_*)\geq 0$ (not necessarily $\partial_t u(x_*,t_*)=0$). But

$$\begin{aligned} Lu(x_*,t_*) &= \sum a_{ij}(x_*,t_*)D_{ij}u(x_*,t_*)+\sum b_i(x_*,t_*)D_iu(x_*,t_*)-\partial_t u(x_*,t_*) \\ &= Tr\left[a_{ij}(x_*,t_*)D^2u(x_*,t_*)\right]\leq 0. \end{aligned}$$

This contradicts $Lu(x_*,t_*)>0$. Thus $\max_{\bar{\Omega}_T}u=\max_{\partial_p\Omega_T}u$.

General case. $Lu\geq 0$.

We do analysis by approximating it with strict inequalities. Let

$$u_\varepsilon = u-\varepsilon t$$

(or as in the elliptic case $u_\varepsilon=u+\varepsilon e^{Kx_1}$). Then

$$Lu_\varepsilon = Lu+\varepsilon>0.$$

By the strict case

$$\max_{\bar{\Omega}_T}(u-\varepsilon t)=\max_{\partial_p\Omega_T}(u-\varepsilon t).$$

Take $\varepsilon\to 0$, we find $\max_{\bar{\Omega}}u=\max_{\partial\Omega}u$.

Weak Maximum Principle with $c\leq 0$. Assume $u,v\in C^{2,1}(\Omega_T)\cap C\left(\bar{\Omega}_T\right)$ and $c(x,t)\leq 0$ in Ω_T

- Max. If $Lu+c(x,t)u\geq 0$ in Ω_T, then $\max_{\bar{\Omega}_T}u\leq\max_{\partial_p\Omega_T}u^+$.
- Min. If $Lu+c(x,t)u\leq 0$ in Ω_T, then $\min_{\bar{\Omega}_T}u\geq\min_{\partial_p\Omega_T}u^-$.
- Comparison. If $Lu+c(x,t)u\geq Lv+c(x,t)v$ in Ω_T and $u\leq v$ on $\partial_p\Omega_T$, then $u\leq v$ in Ω_T.

Proof. Again it suffices to handle the first case. And we can just assume $\max_{\bar{\Omega}_T}u>0$.We repeat the above argument for no c term. Note $\max_{\bar{\Omega}_T}(u-\varepsilon t)>0$ for small enough ε. This maximum cannot attain inside Ω_T. Otherwise we have two inconsistent inequalities

$$L(u-\varepsilon t)+c(u-\varepsilon t)\geq\varepsilon-c\varepsilon t>0$$

and also at the interior maximum point

$$L(u-\varepsilon t)+c(u-\varepsilon t)\leq c(u-\varepsilon t)\leq 0.$$

Thus the positive maximum can only happen at the parabolic boundary

$$\max_{\bar{\Omega}_T}(u-\varepsilon t)=\max_{\partial_p\Omega_T}(u-\varepsilon t).$$

Let ε go to 0, we have

$$\max_{\bar{\Omega}_T}u=\max_{\partial_p\Omega_T}u=\max_{\partial_p\Omega_T}u^+.$$

C-eg with $c > 0$.

$$u = \cos x e^t$$
$$u'' + 2u - \partial_t u = 0 \text{ in } \Omega_T = (-\pi/2, \pi/2) \times (0, 1]$$
$$\max_{\bar{\Omega}_1} u = e \not< 1 = \max_{\partial_p \Omega_1} u.$$

C-eg for necessity of the negative part in $\min_{\partial\Omega} u^-$.

$$u = chx\ e^{-t}$$
$$u'' - 2u - \partial_t u = 0 \ \text{ in } \Omega = (-9, 9) \times (0, 1]$$
$$\min_{\bar{\Omega}_1} u = e^{-1} \not> 1 = \min_{\partial_p \Omega_1} u.$$

And $-chxe^{-t}$ gives us a counterexample for necessity of the positive part in $\max_{\partial_p \Omega_T} u^+$.

Strong Maximum Principle (Nirenberg). Assume $u \in C^{2,1}(\Omega_T) \cap C(\bar{\Omega}_T)$ and Ω_T is connected.

• Max. If $Lu \geq 0$ in Ω_T and u attains its global maximum at an interior point (x_M, t_M), then u is constant in $\Omega_{t_M} = \Omega \times [0, t_M]$.

• Min. If $Lu \leq 0$ in Ω and u attains its global minimum at an interior point (x_m, t_m), then u is constant $\Omega_{t_m} = \Omega \times [0, t_m]$.

It suffices to prove only one case, say minimum one; the maximum case follows by considering $-u$. The inconsistency: if "strict" minimum occurs, then either space gradient (Step2.1) or time derivative (Step2.2) at the point is not zero. Nirenberg constructed a Hopf like barrier to achieved.

Proof. Set-up.

Suppose u attains its global minimum $m = \min_{\bar{\Omega}_T} u$ at an interior point (x_m, t_m). Set

$$\Omega_{t_m}^m = \{(x, t) \in \Omega_{t_m} : u(x, t) = m\}.$$

If $u(x, t)$ is not constant m, then $\Omega_{t_m} \backslash \Omega_{t_m}^m = \{(x, t) \in \Omega_{t_m} : u(x, t) > m\}$ is non-empty. Next we choose an (n+1)-dim ball touching the boundary of $\Omega_{t_m}^m$ and its lower part $t \leq t_m$ inside Ω_{t_m}. There exists an interior point $Y = (y, s) \in \Omega_{t_m} \backslash \Omega_{t_m}^m$ such that $\rho = dist\left(Y, \partial\Omega_{t_m}^m\right) < dist\left(Y, \partial_p \Omega_{t_m}\right)$. Enlarge the (n+1)-dim ball centered at Y until it hits the boundary $\partial\Omega_{t_m}^m$, say at (x_0, t_0), before hitting $\partial_p \Omega_{t_m}$. We also assume the final ball hits the boundary only at (x_0, t_0), say, by moving the center and shrinking the radius a bit. The final radius is ρ.

figure

For simplicity of notation, we assume $(y, s) = (0, 0)$. What we have now is

$$B_\rho(0, 0) \cap \{t \leq t_m\} \subset \Omega_{t_m} \backslash \Omega_{t_m}^m \quad \text{and} \quad (x_0, t_0) \in \partial B_\rho(0, 0) \cap \Omega_{t_m}^m.$$

We need to handle two cases separately: Case $x_0 \neq 0$ and Case $x_0 = 0$.

Step2.1 Case $x_0 \neq 0$ (**Hopf boundary lemma**).

We take another ball $B_{\rho'}(x_0, t_0)$ centered at (x_0, t_0) with radius $\rho' = |x_0 - 0|/2$.

We construct a subsolution v such that $u \underset{=\text{at } (x_0,t_0)}{\geq} v$ near (x_0,t_0) and $v_r(x_0,t_0) < 0$. This forces $u_r(x_0,t_0) < 0$ or $D_x u(x_0,t_0) \neq 0$, a contradiction to interior minimum $u(x_0,t_0)$ at (x_0,t_0).

First take

$$w(x) = e^{-K\left(|x|^2+t^2\right)} - e^{-K\rho^2}$$

with large constant K to be determined later. We compute

$$\begin{aligned} Lw &= e^{-K\left(|x|^2+t^2\right)}\left[\sum a_{ij}\left(4K^2x_ix_j - 2K\delta_{ij}\right) + \sum b_i\left(-2Kx_i\right) + 2Kt\right] \\ &> 0 \quad \text{in } B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\} \end{aligned}$$

provided K is taken large enough.

Notice $u > m$ inside $B_\rho(0,0)$. Then we can take small enough ε so that $v(x) = \varepsilon w(x) + u(x_0)$ stays below $u(x)$ on the parabolic boundary of the intersection

$$B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\},$$

that is

$$\partial B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\}$$

and

$$B_\rho(0,0) \cap \partial B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\}$$

By the construction

$$Lu \leq 0 \leq Lv \quad \text{in } B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\}.$$

From the comparison principle (weak maximum principle valid even in this non straight lateral boundary case)

$$u(x,t) \geq v(x,t) \quad \text{in } B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\}.$$

Remember $u(x_0,t) = v(x_0,t_0)$, taking radial derivatives, we get

$$-u_r(x_0,t_0) \geq -v_r(x_0,t_0) = 2\varepsilon K\left|x_0 - 0\right| e^{-K\rho^2} > 0.$$

This contradicts the fact that $D_x u(x_0,t_0) = 0$ at minimum interior point (x_0,t_0).

Therefore, $u(x) \equiv m$ in $\Omega \times [0,t_m]$.

Step2.2 Case $x_0 = 0$ (**Time-like Hopf boundary lemma**).

We construct a different subsolution v such that $u \underset{=\text{at } (x_0,t_0)}{\geq} v$ near (x_0,t_0) and $v_t(x_0,t_0) < 0$. This forces $u_t(x_0,t_0) < 0$, in turn, $Lu > 0$ at the minimum point (x_0,t_0), a contradiction to the equation $Lu \leq 0$.

First take

$$w(x,t) = -\left|x\right|^2 - K(t - t_0)$$

for a large constant K to be determined later. This function w vanishes on the paraboloid $P_K = \left\{(x,t) : |x|^2 + K(t-t_0) \ = 0\right\}$ which is inside $B_\rho(0,0) \cap \{t \le t_m\}$, now just $B_\rho(0,0)$ when t is close to t_0.We compute

$$\begin{aligned} Lw &= \sum a_{ij} \cdot 4 + \sum b_i(-2x_i) + K \\ &> 0 \quad \text{in } B_\rho(0,0) \end{aligned}$$

provided K is taken large enough.

Notice $u > m$ inside $B_\rho(0,0)$. Then we can take small enough ε so that $v(x,t) = \varepsilon w(x,t) + m$ stays below $u(x,t)$ on the boundary $\partial B_\rho(0,0)$ portion under the paraboloid P_k. Certainly $v(x,t)$ is below $u(x,t)$ on the paraboloid P_K intersecting $B_\rho(0,0)$, as v vanishes and $u \ge m$ there. By the construction

$$Lu \le 0 < Lv \ \text{ in } PB = B_\rho(0,0) \cap \left\{|x|^2 + K(t-t_0) < 0\right\}.$$

It follows that $\max(v-u)$ cannot happen at interior points of PB. Otherwise, $L(v-u) \le 0$ or $Lv \le Lu$ there. As $v - u \le 0$ on the boundary of PB, we can only have

$$u(x,t) \ge v(x,t) \quad \text{in } B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \le t_0\}.$$

Remember $u(x_0,t) = v(x_0,t_0)$, taking time derivatives, we get

$$u_t(x_0,t_0) \le v_t(x_0,t_0) = -K < 0.$$

It follows that $Lu \ge K > 0$ at (x_0,t_0). This contradicts the equation $Lu \le 0$.

Therefore, $u(x) \equiv m$ in $\Omega \times [0,t_m]$.

Remark. Note that we can only draw the constant conclusion below t_m. Otherwise, Step2.2 fails for ball $B_\rho(0,0)$ touching (x_0,t_0) above (for example, no comparison principle). More effectively, recall the heat kernel $\Phi = t^{-1/2}e^{-x^2/4t}$ vanishes in $B_1(2) \times (-1,0]$, but become positive for positive time.

Strong maximum principle with $c \le 0$. Assume $u \in C^2(\Omega_T) \cap C\left(\bar{\Omega}_T\right)$.

• Max. If $Lu + cu \ge 0$ in Ω_T and u attains its nonnegative global maximum at an interior point (x_M, t_M), then u is constant in $\Omega_{t_M} = \Omega \times [0,t_M]$.

• Min. If $Lu + cu \le 0$ in Ω and u attains its nonpositive global minimum at an interior point (x_m, t_m), then u is constant in $\Omega_{t_m} = \Omega \times [0,t_{Mm}]$.

Proof. Again it suffices to justify only one case, say the minimum one.

Now

$$m = \min_{\bar{\Omega}} u \le 0.$$

We go over the same two steps as in proof of the strong maximum principle without c term. The difference is in Step2.1, for $v = \varepsilon\left(e^{-K\left(|x|^2+t^2\right)} - e^{-K\rho^2}\right) + m$

$$\begin{aligned} & Lv + cv \\ & = e^{-K\left(|x|^2+t^2\right)}\left[\sum a_{ij}\left(4K^2x_ix_j - 2K\delta_{ij}\right) + \sum b_i(-2Kx_i) + c\right] \\ & + cm \\ & \ge e^{-K\left(|x|^2+t^2\right)}\left[\sum a_{ij}\left(4K^2x_ix_j - 2K\delta_{ij}\right) + \sum b_i(-2Kx_i) + c\right] \\ & > 0 \quad \text{in } B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \le t_0\} \end{aligned}$$

provided K is taken large enough. Then

$$\begin{aligned} Lu + cu \leq 0 \leq Lv + cv \quad \text{in } B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\}, \\ u(x,t) \geq v(x,t) \quad \text{on } \partial_p \left[B_\rho(0,0) \cap B_{\rho'}(x_0,t_0) \cap \{t \leq t_0\}\right]. \end{aligned}$$

The remaining argument goes through.

Step2.2 can be adjusted similarly.

Strong maximum principle with no sign restriction on c but with one sign restriction on solution. Assume $u \in C^{2,1}(\Omega_T) \cap C\left(\bar{\Omega}_T\right)$ and the arbitrary sign of $c(x,t)$ in Ω_T.

• Max. If $Lu + cu \geq 0$ in Ω and $\max\limits_{\bar{\Omega}} u = 0$ attains at an interior point (x_0, t_0), then $u \equiv 0$ in $\Omega \times [0, t_0]$.

• Min. If $Lu + cu \leq 0$ in Ω and $\min\limits_{\bar{\Omega}} u = 0$ attains at an interior point (x_0, t_0), then $u \equiv 0$ in $\Omega \times [0, t_0]$.

Proof. Again, we only need to handle one case, say the minimum case. Note that $u \geq 0$, then splitting $c = c^+ - c^-$, we have

$$Lu - c^- u \leq -c^+ u \leq 0.$$

We are back to the above strong maximum principle with $c \leq 0$.

Remember C-eg to $c \leq 0$.

$$\begin{aligned} u &= \cos x \\ u'' + u - \partial_t u &= 0 \ \text{ in } \ (-\pi/2, \pi/2) \times (0, 10] \\ u(0,1) &\ \text{ is an interior global maximum.} \end{aligned}$$

C-eg to nonpositive minimum.

$$\begin{aligned} u &= chx \\ u'' - u - \partial_t u &= 0 \ \text{ in } R^1 \times R^1 \\ \text{positive } u(0,0) &= 1 \text{ is an interior global minimum.} \end{aligned}$$