Elliptic PDE Fall 2013 HW1

1. For $f \in C_0^\infty(R^n)$, show that a solution to $\triangle u = f$ is given by

$$u(x) = c_n \int \frac{f(y)}{|x-y|^{n-2}} dy$$

for some constant c_n.

2. Let $r = |x|$. Let $P_k(x)$ be a homogeneous polynomial of order k. Show that $P_k(D) r^{2-n}$ is homogeneous of order $2-n-k$ and can be written in the form

$$P_k(D) \frac{1}{r^{n-2}} = \frac{H_k(x)}{r^{n-2+2k}}.$$

3. Show that not only are those H_ks derived the previous problem harmonic polynomial of degree k, but all harmonic polynomial of degree k are an H_k. The H_ks form a linear space, find its dimension.

4. Show that if $u(x)$ is harmonic, then so is $|x|^{2-n} u\left(x/|x|^2\right)$.

5. Suppose v is a real harmonic function. Show that if

$$|v(x)| \leq C\left(|x|^k + 1\right),$$

then $v = H_k$, a harmonic polynomial of degree k.