Elliptic PDE Fall 2013 HW4

1. Let Ω be a strongly convex and bounded C^2 domain, let $\varphi(x) = \varphi(x', x_n) \in C^2(\partial\Omega)$. For any point on the boundary, say $(0,0)$. Show that there exists a large positive M such that

$$L^+(x) = \varphi(0,0) + \varphi_{x'}(0,0)\, x' + M x_n \geq \varphi(x) \quad \text{on } \partial\Omega$$

and

$$L^-(x) = \varphi(0,0) + \varphi_{x'}(0,0)\, x' - M x_n \geq \varphi(x) \quad \text{on } \partial\Omega.$$

2. Let

$$u(x,y,z) = \int_0^1 \frac{t}{|(x-t, y, z)|} dt.$$

Show that $\triangle u = 0$ in $R^3 \backslash [0,1]$ and compute the integral for u explicitly.

3. Let Γ be the fundamental solution for the Laplacian on R^n. Let $f \in C_0^\infty(R^n)$. Then the convolution $\Gamma * f$ satisfies $\triangle \Gamma * f = f$. Show that

$$D_{ij} u(x) = \int_{R^n} \Gamma_{ij}(y) f(x-y)\, dy = \lim_{\varepsilon \to 0} \int_{R^n \backslash B_\varepsilon(x)} c_n \frac{y_i y_j - \delta_{ij} n^{-1} |y|^2}{|y|^{n+2}} f(x-y)\, dy + \frac{\delta_{ij}}{n} f(x).$$