

## Lecture 1 Introduction

- equations
- source for equations
- explicit solutions
- reason for better behavior (than waves)
- nonlinear theory
  - De Giorgi-Nash
  - Krylov-Safonov

The equations

$$u \text{ first derivatives } Du \text{ and double derivatives } D^2u \sim \begin{bmatrix} \lambda_1 \\ & \ddots \\ & & \lambda_n \end{bmatrix}$$

**Algebraically**

$$\begin{aligned} \text{Laplace } \Delta u = \sigma_1 &= \lambda_1 + \cdots + \lambda_n = c \\ \sigma_k &= \lambda_1 \cdots \lambda_k + \cdots = c \\ \text{M-A } \det D^2u = \sigma_n &= \lambda_1 \cdots \lambda_n = c \\ \ln \lambda_1 + \cdots + \ln \lambda_n &= C \\ \arctan \lambda_1 + \cdots + \arctan \lambda_n &= C \quad (\text{take tan, then algebraic}) \\ \text{Also } \ln \det(\partial \bar{\partial} u) &= 0. \end{aligned}$$

*figure*

elliptic  $\Leftrightarrow f(\lambda)$  monotonic

**Geometrically**

$$\begin{aligned} \lambda \dashrightarrow \kappa &= (\kappa_1, \dots, \kappa_n), \text{ principle curvatures of hypersurface } (x, u(x)). \\ \sigma_k(\kappa) &= c, \\ \text{in particular, mean curvature } H &= \kappa_1 + \cdots + \kappa_n = \operatorname{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right), \\ \text{and Gauss curvature } K &= \kappa_1 \cdots \kappa_n = \frac{\det D^2u}{(1+|Du|^2)^{\frac{n}{2}+1}} \end{aligned}$$

Tangent way: Re-represent graph  $(x, u(x))$  at  $(x_0, u(x_0)) = (0, 0)$  over its tangent plane,  $\bar{u}(\bar{x}) = 0 + 0 \cdot \bar{x} + \frac{1}{2}(\kappa_1 \bar{x}_1^2 + \cdots + \kappa_n \bar{x}_n^2)$ .

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Assume  $Du \stackrel{0}{=} (0, \dots, 0, u_n)$  and  $u_n = \tan \theta$ , then

$$\bar{u}(\bar{x}) = [u(x) - \tan \theta x_n] \cos \theta$$

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and

$$(x_1, \dots, x_{n-1}, x_n) = (\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{x}_n \cos \theta).$$

Now at 0

$$\begin{aligned} D_{\bar{x}}^2 \bar{u}(\bar{x}) &= \cos \theta \begin{bmatrix} u_{11} & u_{1,n-1} & u_{1n} \cos \theta \\ u_{n-1,1} & u_{n-1,n-1} + u_{nn} \cos^2 \theta & u_{nn} \cos^2 \theta \\ u_{n1} \cos \theta & u_{n-1,1} \cos \theta & u_{nn} \cos^2 \theta \end{bmatrix} \\ &= \cos \theta \begin{bmatrix} 1 & & \\ & 1 & \\ & & \cos \theta \end{bmatrix} D^2 u \begin{bmatrix} 1 & & \\ & 1 & \\ & & \cos \theta \end{bmatrix}. \end{aligned}$$

Hence  $H = \cos \theta [u_{11} + \dots + u_{n-1,n-1} + u_{nn} \cos^2 \theta] \stackrel{Du=(0, \dots, 0, u_n)}{=} \operatorname{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right)$  and

$$K = \cos^n \theta \cos^2 \theta \det D^2 u = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n}{2}+1}}.$$

$$\text{Second fundamental way: } II = \frac{D^2 u}{\sqrt{1+|Du|^2}}$$

$$g = I + (Du)^T (Du) \quad g^{-1} = I - \frac{1}{1+|Du|^2} (Du)^T (Du)$$

$$g^{-1} II = B \quad \text{or} \quad g^{-1/2} II g^{-1/2} \sim \begin{bmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_n \end{bmatrix}$$

$$H = \kappa_1 + \dots + \kappa_n = \frac{1}{\sqrt{1+|Du|^2}} \left( \Delta u - \frac{u_i u_j}{1+|Du|^2} u_{ij} \right) = \operatorname{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right)$$

$$K = \kappa_1 \cdots \kappa_n = \frac{1}{(1+|Du|^2)(1+|Du|^2)^{n/2}} \det D^2 u = \frac{1}{n} \partial_i \left[ \frac{\partial \sigma_n(B)}{\partial B_{ij}} ? \frac{\partial_j u}{\sqrt{1+|Du|^2}} \right]$$

$$\sigma_k(\kappa) = \frac{1}{k} \partial_i \left[ \frac{\partial \sigma_k(B)}{\partial B_{ij}} ? \frac{\partial_j u}{\sqrt{1+|Du|^2}} \right]$$

### Probability

$$\frac{2}{3} \lambda_{\max} + \frac{1}{3} \lambda_{\min} = 0$$

$$u = \varphi(x) \text{ on } \partial \Omega$$

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Random walk, when hits boundary, the pay off is  $\varphi(x)$ .

Let  $u(x)$  be the maximal expectation of pay off, starting from interior point  $x \in \Omega$ , with directional probability  $\frac{1}{3} \leq p \leq \frac{2}{3}$ . Say we are in 2d case.

$$u(x) = p_h \left[ \frac{u(x + \varepsilon e_h) + u(x - \varepsilon e_h)}{2} \right] + p_v \left[ \frac{u(x + \varepsilon e_v) + u(x - \varepsilon e_v)}{2} \right]$$

$$\Rightarrow 0 = p_h u_{hh} + p_v u_{vv}$$

Now  $\frac{1}{3} \leq \begin{bmatrix} p_h & \\ & p_v \end{bmatrix} \leq \frac{2}{3}$ , take max

$$\frac{2}{3} \lambda_{\max} + \frac{1}{3} \lambda_{\min} = \sup_{\substack{\frac{1}{3} \leq (a_{ij}) \leq \frac{2}{3} \\ tr=1}} a_{ij} D_{ij} u = 0$$

When compare with solution to

$$\begin{cases} \sum a_{ij} D_{ij} w = 0 & \text{in } \Omega \\ w = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

$$\sum a_{ij} D_{ij} w = 0 = \frac{2}{3} \lambda_{\max} + \frac{1}{3} \lambda_{\min} (u) \geq a_{ij} D_{ij} u$$

$$w = u \quad \text{on } \partial\Omega$$

$\Rightarrow u \geq w$  by the maximum principle.

### Other sources for equations.

o Fluid mechanics

vector field  $\vec{V}$

incompressible  $\operatorname{div}(\vec{V}) = 0$

irrotational  $\operatorname{curl} \vec{V} = 0 \iff D\vec{V} = (D\vec{V})^T \implies \vec{V} = Du$

$\Rightarrow \Delta u = 0$

o Variational

$E[u] = \int_{\Omega} F(Du) dx$

$\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned} \frac{d}{dt} \int F(Du + tD\varphi) dx|_{t=0} &= \int \sum F_{p_i}(Du) \frac{\partial \varphi}{\partial x_i} dx \\ &= \int - \sum \frac{\partial}{\partial x_i} [F_{p_i}(Du)] \varphi d \end{aligned}$$

$$\sum \frac{\partial}{\partial x_i} [F_{p_i}(Du)] = 0.$$

eg1.  $F(Du) = |Du|^2$  Energy  $F_{p_i} = 2Du \dashrightarrow \Delta u = 0$ .

eg2.  $F(Du) = \sqrt{1 + |Du|^2}$  Area  $F_{p_i} = \frac{Du}{\sqrt{1+|Du|^2}} \dashrightarrow \operatorname{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right) = 0$ .

eg3.  $E[u] = \int \sigma_{k-1}(\kappa) \sqrt{1 + |Du|^2} dx$ , E-L equation  $\sigma_k(\kappa) = 0$  (Reilly).

RMK. One obvious thing

1d principle curvature of curve  $(x, f(x))$

$$\kappa = \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} = \left( \frac{f_x}{\sqrt{1+f_x^2}} \right)_x$$

$$\text{also } \int \kappa ds = \int \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} \sqrt{1+f_x^2} dx = \int (\arctan f_x)_x dx = \arctan f_x|_{\partial}$$

Q. In 2d similar thing should happen to the total Gauss curvature

$$\int \sigma_n(\kappa) \sqrt{1 + |Du|^2} dx?$$

More variationals

$$\text{Eg } \sigma_k : E[u] = -\frac{1}{k+1} \int u \sigma_k(D^2u) dx + \int u dx$$

E-L equation  $\sigma_k(D^2u) = 1$ . This can be derived using the following divergence structure.

$$\begin{aligned} k\sigma_k &= D_\lambda \sigma_k \cdot \lambda = \frac{\partial \sigma_k(D^2u)}{\partial m_{ij}} D_{ij}u \\ &= \frac{\partial}{\partial x_i} \left[ \frac{\partial \sigma_k(D^2u)}{\partial m_{ij}} \partial_{x_j} u \right] + \underbrace{\frac{\partial}{\partial x_i} \left[ \frac{\partial \sigma_k(D^2u)}{\partial m_{ij}} \right]}_0 \partial_{x_j} u. \end{aligned}$$

$$\text{Eg Slag: } A[DU] = \int \sqrt{\det(I + (DU)^T DU)} dx, U : \Omega \rightarrow R^n.$$

Insist **minimizer** irrotational, i.e.  $U = Du$ , then E-L

$$D \sum \arctan \lambda_i = 0 \Leftrightarrow \sum \arctan \lambda_i = c.$$

$$A[DU] = \int \sqrt{\det(I - (DU)^T DU)} dx, U : \Omega \rightarrow R^n.$$

Insist **maximizer** irrotational, i.e.  $U = Du$ , then E-L

$$D \sum \ln \frac{1 + \lambda_i}{1 - \lambda_i} = 0 \Leftrightarrow \sum \ln \frac{1 + \lambda_i}{1 - \lambda_i} = c \dashrightarrow \sum \ln \bar{\lambda}_i = c.$$

*figure?*

Explicit solutions

$$\circ H = 0$$

catenoid:  $|(x, y)| = \cosh z$

helicoid:  $z = \arctan \frac{y}{x}$

Scherk's surface:  $z = \ln \frac{\cos y}{\cos x}$

$$\circ H_k = \text{const.}$$

unit sphere

o  $\Delta u = 0$

complex analysis in even d:  $u = \operatorname{Re} z^k, z^{-k}, e^z, z_1^3 e^{z_2}, \dots$

algebraic n-d  $u = \sigma_k(x_1, \dots, x_n)$

radial

$$\Delta u = \partial_r^2 u + \frac{n-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^{n-1}} u$$

$$u_{rr} + \frac{n-1}{r} u_r = 0$$

$$r^{n-1} u_{rr} + (n-1) r^{n-2} u_r = 0 \text{ or } (r^{n-1} u_r)_r = 0$$

$$u_r = \frac{c}{r^{n-1}}$$

$$u = \frac{c}{r^{n-2}}, \ln |(x_1, x_2)|, \text{ or } |x_1|$$

o  $\arctan \lambda_1 + \dots + \arctan \lambda_3 = \pi/2$  or  $\sigma_2 = 1$

$$u = |(x_1, x_2)| \cosh x_3$$

o  $\arctan \lambda_1 + \dots + \arctan \lambda_3 = 0$  or  $\Delta u = \det D^2 u$

$$u = x_3 \sinh^{-1} |(x_1, x_2)|$$

o  $\arctan \lambda_1 + \arctan \lambda_2 = \pi/2$  or  $\det D^2 u = 1$

$$u = \int^x du = \int^x u_1 dx_1 + u_2 dx_2$$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(t - Dh(t)) = \frac{1}{\sqrt{2}}(t_1 - 3t_1^2 + 3t_2^2, t_2 + 6t_1 t_2) \\ Du(x) = \frac{1}{\sqrt{2}}(t + Dh(t)) = \frac{1}{\sqrt{2}}(t_1 + 3t_1^2 - 3t_2^2, t_2 - 6t_1 t_2) \end{cases}.$$

$$v = \int^x dv = \int^x v_1 dx_1 + v_2 dx_2$$

$$\begin{cases} x = (t_1, -h_2) = (t_1, 6t_1 t_2) \\ Dv(x) = (h_1, t_2) = (3t_1^2 - 3t_2^2, t_2) \end{cases}$$

Exercise: Show that  $u$  and  $v$  are indeed solutions. Express  $u$  and  $v$  explicitly in terms of  $x$ . ( $v = x_1^3 + \frac{x_2^2}{12x_1}$ )

Reasons for better behavior (than wave equations)

R<sub>0</sub> harmonic

R<sub>1</sub> Energy minimizer

R<sub>2</sub> Comparison principle

two solutions cannot touch each other

*figure*

$$\Delta u = 0$$

$$D^2 u_1 > D^2 u_2 \Rightarrow 0 = \Delta u_1 - \Delta u_2 > 0 \rightarrow \leftarrow$$

two solutions can cross each other

*figure*

In contrast to  $w_{tt} - w_{xx} = 0, w_1 = x^2 + t^2, w_2 = 0$ .

R<sub>3</sub> Fourier transform

$$\Delta u = f \in C_0^\infty(R^n)$$

$$\widehat{\Delta u} = \widehat{f} \Rightarrow -|\xi|^2 \widehat{u} = \widehat{f}, \text{ or } \widehat{u} = -\frac{\widehat{f}}{|\xi|^2}$$

Exercise: Verify

$$u = c_n \int \frac{1}{|x-y|^{n-2}} f(y) dy.$$

$$\widehat{D_{ij}u} = -\xi_i \xi_j \widehat{u} = \underbrace{\frac{\xi_i \xi_j}{|\xi|^2} \widehat{f}}_{\text{bounded}}$$

$$f \in L^2 \Leftrightarrow \widehat{f} \in L^2 \Rightarrow \widehat{D_{ij}u} \in L^2 \Rightarrow D_{ij}u \in L^2$$

Also  $f \in C^\alpha \Rightarrow D^2u \in C^\alpha$ .

Outlook:

Linear theory

$$\sum a_{ij}(x) D_{ij}u = f(x)$$

$a_{ij}(x) \in C^\alpha$   $C^\alpha$  invariant space Schauder

$a_{ij}(x) \in C^0/VMO$   $L^p$  invariant space Calderon-Zygmund

Nonlinear theory

o Quasilinear equations

$$\sum F_{p_ip_j}(Du) D_{ij}u = 0$$

want  $Du \in C^\alpha$ , in order to apply Schauder.

Divergence equation

$$\begin{aligned} \partial_e : \sum \frac{\partial}{\partial x_i} [F_{p_i}(Du)] &= 0 \\ \sum \frac{\partial}{\partial x_i} \left[ F_{p_ip_j}(Du) \frac{\partial}{\partial x_j} u_e \right] &= 0 \end{aligned}$$

De Giorgi-Nash  $u_e \in C^\alpha$ .

o Fully nonlinear equations

$$F(D^2u) = 0$$

want  $D^2u \in C^\alpha$  in order to run Schauder, then continuity method.

$$\begin{aligned} \partial_e : \sum F_{u_{ij}} D_{ij}u_e &= 0 \\ \partial_e^2 : \sum F_{u_{ij}} D_{ij}u_{ee} + \sum F_{u_{ij}u_{kl}} D_{ij}u_e D_{kl}u_e &= 0 \end{aligned}$$

Assume convexity to drop cubit derivatives. Thus need to study

Nondivergence equation

$$\sum a_{ij}(D^2u) D_{ij}w = 0$$

Krylov-Safonov  $w \in C^\alpha$ .