Lecture 2 Harmonic functions

○ invariance
○ mean value
  maximum principle,
  (higher order) derivative estimates,
  Harnack
○ weak formulations
mean value
weak/Weyl
viscosity

Invariance for Harmonic functions, solutions to $\triangle u = 0$
· $u(x+x_0)$
· $u(Rx)$
· $u(tx)$
RMK. Equations don't know/care which coordinates they are in.
· $u+v$, au, where $\triangle v = 0$
· $\int u(x-y)\varphi(y)\,dy$
·$\frac{u(x+\varepsilon e)-u(x)}{\varepsilon} \to D_e u$, so is $D^k u$
·$\frac{u(R\varepsilon x)-u(x)}{\varepsilon} \to D_\theta u = x_i u_j - x_j u_i$
· $\frac{u((1+\varepsilon)x)-u(x)}{\varepsilon} \to Du(x)\cdot x = ru_r$, so are $r\partial_r(ru_r) = ru_r + \underline{r^2 u_{rr}}$, $r^3 u_{rrr}, \cdots$
· $|x|^{2-n} u\left(\frac{x}{|x|^2}\right)$ Kelvin transformation

RNK. "Kelvin" transformation for the heat equation $u_t - \triangle u = 0$, $\frac{1}{t^{n/2}} e^{-\frac{|x|^2}{4t}} u\left(\frac{x}{t}, \frac{-1}{t}\right)$.
More harmonic functions.
eg1.

$$D_1 r^{2-n} = (2-n)\, r^{1-n}\frac{x_1}{r} = (2-n)\, r^{-n} x_1 = (2-n)\,\frac{x_1}{r^n}$$
$$D_{11} r^{2-n} = (2-n)\left[-nr^{-n-1}\frac{x_1}{r}x_1 + r^{-n}\right] = (2-n)\,\frac{r^2 - nx_1^2}{r^{n+2}}$$
$$D_{12} r^{2-n} = (2-n)\,\frac{-nx_1x_2}{r^{n+2}}$$

Let $P_k(x)$ be any homogeneous polynomial of degree k, $P_k(D)\, r^{2-n} = \frac{H_k(x)}{r^{n-2-2k}}$. For example, $\sigma_k(D)\, r^{2-n} = \frac{\sigma_k(x)}{r^{n-2-2k}}$. Note $H_k \neq P_k$ in general, but $H_k(x) = r^{2-n}\frac{H_k\left(\frac{x}{r^2}\right)}{\left|\frac{x}{r^2}\right|^{n-2-2k}}$ is the Kelvin transform of harmonic function $P_k(D)\, r^{2-n}$, thus harmonic.

Exercise: $H_k(x)$ are ALL harmonic polynomials of degree k.

eg2. Harmonic function

$$|x-x_0|^{2-n} - |x|^{2-n}\left|\frac{x}{|x|^2} - x_0\right|^{2-n} \overset{|x|=1}{=} |x-x_0|^{2-n} - |x-x_0|^{2-n} = 0,$$

[0]October 15, 2013

is Green's function (up to a multiple) for the unit ball.

<u>Mean value equality</u>

Recall the divergence formula (the fundamental theorem of calculus)

$$\int_\Omega div\left(\vec{V}\right)dx = \int_{\partial\Omega}\left\langle \vec{V},\gamma\right\rangle dA.$$

$\vec{V} = Du$, then $0 = \int_{\partial\Omega} u_\gamma dA$.
$\vec{V} = vDu$, then $\int_\Omega \langle Dv, Du\rangle + v \bigtriangleup u = \int_{\partial\Omega} vu_\gamma dA$.
$\vec{V} = uDv$, then $\int_\Omega \langle Du, Dv\rangle + u \bigtriangleup v = \int_{\partial\Omega} uv_\gamma dA$.

$$\int_\Omega v \bigtriangleup u - u \bigtriangleup v = \int_{\partial\Omega} vu_\gamma - uv_\gamma dA.$$

Mean value case. Now $\triangle u = 0$ in B_1, take $v = |x|^{2-n}$, $\Omega = B_1 \backslash B_\varepsilon$,

$$B_1\backslash B_\varepsilon \ figure$$

we then have $0 = \int_{\partial\Omega} vu_\gamma - uv_\gamma dA$, or

$$\overbrace{\int_{\partial(B_1\backslash B_\varepsilon)} vu_\gamma dA}^{0} = \int_{\partial(B_1\backslash B_\varepsilon)} uv_\gamma dA = \int_{\partial B_1} u\frac{(2-n)}{r^{n-1}}dA - \int_{\partial B_\varepsilon} u\frac{(2-n)}{r^{n-1}}dA. \qquad (*)$$

We get $\displaystyle\int_{\partial B_1} udA = \int_{\partial B_\varepsilon} u\frac{1}{\varepsilon^{n-1}}dA \overset{\varepsilon\to 0}{\to} |\partial B_1|\, u(0)$. So $u(0) = \frac{1}{|\partial B_1|}\int_{\partial B_1} udA$.

RMK. In hindsight one just takes $v = \dfrac{-1}{(n-2)\,|\partial B_1|}\dfrac{1}{|x|^{n-2}} \overset{\text{def}}{=} \Gamma$.

Also

$$u(0) = \frac{1}{|\partial B_r|}\int_{\partial B_r} udA.$$

Take a weight function $|\partial B_r|$, $u(0)\,|B_1| = \int_0^1 u(0)\,|\partial B_r|\,dr = \int_0^1\int_{\partial B_r} udAdr = \int_{B_1} udx$. So $u(0) = \frac{1}{|B_1|}\int_{B_1(0)} udx$.

Also

$$u(0) = \frac{1}{|B_r|}\int_{B_r(0)} udx.$$

RMK1. Tracing the sign of $\triangle u$, one gets mean value inequalities for superharmonic functions $\triangle u \le 0 : u(0) \le ⨍ u$ and subharmonic functions $\triangle u \ge 0 : u(0) \le ⨍ u$.

RMK2. "$\cdots$ all the women are strong, all the men are good-looking, and all the children are above average." –A Prairie Home Companion with Garrison Keillor.

RMK3. Also for $u \in C_0^\infty(\mathbb{R}^n)$

$$u(x) = \int_{\mathbb{R}^n} \frac{-1}{(n-2)\,n\,|B_1|} \frac{1}{|x-y|^{n-2}} \triangle u(y)\,dy = \Gamma * u.$$

Green case. Still $\triangle u = 0$ in B_1, but

$$v = G(x, x_0) = \frac{-1}{(n-2)\,|\partial B_1|} \left(|x - x_0|^{2-n} - |x|^{2-n} \left| \frac{x}{|x|^2} - x_0 \right|^{2-n} \right),$$

$\Omega = B_1 \backslash B_\varepsilon(x_0)$.

$\mathrm{B}_1 \backslash \mathrm{B}_\varepsilon(x_0)$ figure

Taking limits on two ends of (*), we get

$$u(x_0) = \int_{\partial B_1} \frac{\partial G(x, x_0)}{\partial \gamma_x} u(x)\,dA = \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{1 - |x_0|^2}{|x - x_0|^n} u(x)\,dA.$$

Note $u(x) = \int_{\partial B_1} \frac{\partial G(y,x)}{\partial \gamma_y} \varphi(y)\,dA_y$, as sum of harmonic functions $\frac{1-|x|^2}{|y-x|^n}$, is harmonic–smooth, analytic in terms regularity–for $\varphi \in C^0, L^1, \cdots$.

Application 1. Strong maximum principle (No toughing).

$$\triangle u_1 = \triangle u_2 = 0$$
$$u_1 \geq u_2, \text{ ``}='' \text{ at } 0$$

then

$$0 = u_1(0) - u_2(0) = \frac{1}{|B_r|} \int_{B_r} (u_1 - u_2)\,dx \geq 0.$$

It follows that $u_1 \equiv u_2$.

Application 2. Smooth effect and derivative test.

Take radial weight $\varphi(y) = \varphi(|y|) \in C_0^\infty(R^n)$ such that $1 = \int \varphi(y)\,dy = \int_0^\infty \varphi(r)\,|\partial B_r|\,dr$. Then

$$\begin{aligned}\int_{R^n} u(y)\,\varphi(x-y)\,dy &= \int_0^\infty \int_{\partial B_r(x)} u(y)\,\varphi(x-y)\,dA dr \\ &= \int_0^\infty u(x)\,\varphi(r)\,|\partial B_r|\,dr = u(x) \int \varphi(y)\,dy \\ &= u(x).\end{aligned}$$

Consequence $u(x) = \int_{R^n} u(y)\,\varphi(x-y)\,dy$ is smooth for continuous initial $u(y)$, and

$$D^k u(0) = \int u(y)\,D_x^k \varphi(x-y)\,dy = (-1)^k \int u(y)\,D_y^k \varphi(x-y)\,dy.$$

Thus

$$\left|D^k u(0)\right| \leq C(k,n,\varphi)\,\|u\|_{L^1(B_1)}.$$

Scaled version

$$\left|D^k u(0)\right| \leq \begin{cases} \frac{C(k,n,\varphi)\|u\|_{L^1(B_R)}}{R^{n+k}} \\ \frac{C(k,n,\varphi)\|u\|_{L^\infty(B_R)}}{R^k} \end{cases}.$$

That is the larger the domain, the flatter the harmonic graph.

Application 3. Harnack inequalities–a quantitative version of the strong maximum principle.

eg. Consider positive harmonic functions r^{2-n}, $x_1 r^{-n}$ on $\{x_1 > 0\}$.

$r^{2-n}, x_1 r^{-n}$ graph *figure*

eg. In general for $\triangle \mathrm{u} = 0$, $u > 0$ in $B_1(0)$, we have

$$u(x) = \frac{1}{\left|B_{1-|x|}\right|}\int_{B_{1-|x|}(x)} u dx \leq \frac{1}{\left|B_{1-|x|}\right|}\int_{B_1(0)} u dx = \frac{|B_1|}{\left|B_{1-|x|}\right|}u(0) = \frac{1}{(1-|x|)^n}u(0).$$

RNK. As those two examples suggest, from estimating the kernal of Poisson representation, we have a sharper comparison

$$\frac{(1-|x|)}{2^{n-1}}u(0) \leq u(x) \leq \frac{2}{(1-|x|)^{n-1}}u(0).$$

Harnack. Suppose $\triangle \mathrm{u} = 0$, $u > 0$ in $B_r(x_0)$. Then we have

$$\sup_{B_{r/4}(x_0)} u \leq 3^n \inf_{B_{r/4}(x_0)} u.$$

In fact

4 circle figure $\mathrm{B}_1, B_{1/4}, B_{1/4}(x_{\max}), B_{3/4}(x_{\min})$

$$\begin{aligned}\max_{B_{1/4}} u = u(x_{\max}) &= \frac{1}{\left|B_{1/4}\right|}\int_{B_{1/4}(x_{\max})} u dx \\ &\leq \frac{1}{\left|B_{1/4}\right|}\int_{B_{3/4}(x_{\min})} u dx \\ &= 3^n u(x_{\min}) = 3^n \min_{B_{1/4}} u.\end{aligned}$$

Consequences $\cdots$, for example one sided Liouville for entire harmonic functions.

RMK. Harnack inequality is in fact a quantitative version of the strong maximum principle. It measures how much the minimum leaps when moving inside, or flipping around how much the maximum drops when moving inside. For example, to move inside $B_{1/4}$ from B_1,

$$\min_{B_{1/4}} (u - m_1) \geq 3^{-n} \max_{B_{1/4}} (u - m_1)$$

or

$$m_{1/4} \geq m_1 + 3^{-n} \left(M_{1/4} - m_1 \right).$$

The flip version is

$$\min_{B_{1/4}} (M_1 - u) \geq 3^{-n} \max_{B_{1/4}} (M_1 - u)$$

or

$$M_{1/4} \geq M_1 - 3^{-n} \left(M_1 - m_{1/4} \right).$$

(This should be Moser's observation: subtracting the leap from the drop, one has oscillation decay of the "harmonic" function.)

Weak formulation for Laplace equation: $\triangle u = 0$.

Mean value formulation.

Suppose $u \in L^1$ satisfy $u(x) = \fint_{B_r(x)} u(y)\, dy$ for all x and r.

Exercise. Then u is continuous, since

$$u(x) - u(x_0) = \fint_{B_1(x)} u(y)\, dy - \fint_{B_1(x_0)} u(y)\, dy \overset{x \to x_0}{\to} 0.$$

2 minor overlap circle*figure*

In turn, we have $u(x) = \fint_{\partial B_r(x)} u(y)\, dy$. In fact

$$\frac{d}{dr}: \; r^n |B_1| u(x_0) = \int_{B_r(x_0)} u(y)\, dy$$
$$n r^{n-1} |B_1| u(x_0) = \int_{\partial B_r(x_0)} u(y)\, dy$$
$$|\partial B_r| u(x_0) = \int_{\partial B_r(x_0)} u(y)\, dy.$$

Then

$$u(x) = \int_{R^n} \varphi(x - y) u(y)\, dy \in C^\infty$$

for $\varphi(x) = \varphi(|x|)$ with $\int_{R^n} \varphi(|x|)\, dx = 1$. Let us check $\triangle u = 0$.

$$\int_{\partial B_\varepsilon(0)} u dA = \int_{\partial B_\varepsilon(0)} u(0) + Du(0) \cdot x + \frac{1}{2} \underbrace{D_{ij} u(0)\, x_i x_j}_{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2} + \varepsilon^3 dA$$

$$|\partial B_\varepsilon|\, u(0) = |\partial B_\varepsilon|\, u(0) + 0 + \frac{1}{2}\left(\lambda_1 \frac{\varepsilon^2}{n} + \cdots + \lambda_n \frac{\varepsilon^2}{n}\right) |\partial B_\varepsilon| + O\left(\varepsilon^3\right) |\partial B_\varepsilon|$$

$$\Rightarrow \frac{1}{2n} \triangle u(0) = 0.$$

Integration by parts formulation.

For $u \in C^0/L^1$/distribution $\int u \triangle \varphi = 0$ for any $\varphi \in C_0^\infty$. How to move to mean value formulation?

Q. How to find $\varphi \in C_0^\infty$ such that

$$\triangle \varphi = \frac{1}{|B_2|} \chi_{B_2} - \frac{1}{|B_1|} \chi_{B_1} ?$$

$C^{1,1}$ approach. $\varphi \sim$ "$\frac{|x|^2}{2n|B_2|}\chi_{B_2}$" $-$ "$\frac{|x|^2}{2n|B_1|}\chi_{B_1}$".

Analytic way. We just look for those radial ones by solving

$$\varphi_{rr} + \frac{n-1}{r} \varphi_r = \frac{1}{|B_1|} \chi_{B_1} \quad \text{or} \quad \frac{1}{|B_2|} \chi_{B_2}.$$

For $r \leq 1$

$$\varphi = \frac{1}{|B_1|} \frac{r^2}{2n} \chi_{[0,1]} + c_1.$$

For $r > 1$

$$\varphi = c_2 r^{2-n} + c_3.$$

After $\mathrm{C}^{1,1}$ matching at $r = 1$, we have

$$\varphi_1 = \begin{cases} \frac{1}{|B_1|} \frac{r^2}{2n} \chi_{B_1} - \frac{1}{|B_1| 2n} - \frac{1}{|B_1|(n-2)n} & \text{for } |x| \leq 1 \\ \frac{-1}{|B_1|(n-2)n} \frac{1}{r^{n-2}} & \text{for } |x| > 1 \end{cases}.$$

Similarly

$$\varphi_2 = \begin{cases} \frac{1}{|B_2|} \frac{r^2}{2n} \chi_{B_2} - \frac{2^2}{|B_2| 2n} - \frac{1}{|B_1|(n-2)n} & \text{for } |x| \leq 1 \\ \frac{-1}{|B_1|(n-2)n} \frac{1}{r^{n-2}} & \text{for } |x| > 1 \end{cases}.$$

"Incidentally" the gradient matching coefficient c_2 leads exactly the coefficient for the fundamental solution $\Gamma = \frac{-1}{|B_1|(n-2)n} \frac{1}{|x|^{n-2}}$.

Geometric way (Caffarelli).

quadratics drop down to fundamental*figure*

This requires $\varphi_2 = \frac{|x|^2}{2n|B_2|} - A$ to touch r^{2-n}, in fact $\frac{-1}{?r^{n-2}}$ at $|x| = 2$. We have a system $\frac{2^2}{2n|B_2|} - A = \frac{-1}{?2^{n-2}}$ and $\frac{2\cdot 2}{2n|B_2|} = \frac{(n-2)}{?2^{n-1}}$ which implies $? = n\,(n-2)\,|B_1|$ and $A = \frac{2(n-1)}{n(n-2)|B_2|}$. Similarly we get $\varphi_1 = \frac{|x|^2}{2n|B_1|} - A'$ touching $\frac{-1}{?r^{n-2}} = \frac{-1}{n(n-2)|B_1|r^{n-2}}$ at $|x| = 1$. Thus $\varphi = \varphi_2 - \varphi_1 \in C_0^{1,1}$ answers the above question.

$$\text{For } u \in L^1,\ \int u \bigtriangleup \varphi = 0 \Rightarrow \fint_{B_2} u = \fint_{B_1} u.$$

Therefore (exercise)

$$u\,(x) = \lim_{r\to 0} \fint_{B_r(x)} u \ \text{ a.e. at Lebesgue point of } L^1\ u.$$

Cor. (Weyl) $u \in L^1/C^0$ satisfying $\int u \bigtriangleup \varphi = 0$ for any $\varphi \in C_0^\infty$. Then $u \in C^\infty$ and $\bigtriangleup u = 0$.

Warning:

$$\int \frac{1}{|x|^{n-2}} \bigtriangleup \varphi = c_n \varphi\,(0) \neq 0\ !$$

C^∞ approach (Weyl)

Work for $u \in$distribution

$\psi\,(x) = \psi\,(|x|) \in C_0^\infty$ with $\int \psi = 1$

$\psi_\varepsilon\,(x) = \frac{1}{\varepsilon^n}\psi\left(\frac{x}{\varepsilon}\right)$

$\Gamma * \psi_\varepsilon$ graph figure

Step 1. $\varphi_\varepsilon = \Gamma * \psi_\varepsilon = \left\{ \begin{array}{c} \Gamma \quad \text{for} \quad |x| \geq \varepsilon \\ \text{smooth for } \ |x| \leq \varepsilon \end{array} \right.$. Recall $\Gamma = \frac{-1}{(n-2)|\partial B_1|}\frac{1}{|x|^{n-2}}$.

Step 2. $\bigtriangleup \Gamma * \psi = \psi$.

Step 3. $\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1} \in C_0^\infty$

$$\int_{\mathbb{R}^n} u \bigtriangleup (\varphi_{\varepsilon_2} - \varphi_{\varepsilon_1}) = 0 \Rightarrow \int_{\mathbb{R}^n} u\psi_{\varepsilon_2} = \int_{\mathbb{R}^n} u\psi_{\varepsilon_1}$$

- $u * \psi_\varepsilon$ is independent of ε
- $u * \psi_\varepsilon \in C^\infty$ (Review distribution theory, try it.)
- $u * \psi_\varepsilon = u$ as a distribution (Exercise).

Pointwise (viscosity) formulation.

Definition: $u \in C^0$ is a viscosity solution to $\bigtriangleup u = 0$, if for any quadratic $P \underset{(\leq)}{\geq} u$ near an interior point x_0 and "$=$" at x_0, then $\bigtriangleup P \underset{(\leq)}{\geq} 0$.

RMK. If there is no quadratic touching u from above or blow at x_0, then one checks nothing. No touching, no checking!

RMK. We can replace those quadratics by equivalent C^2/C^∞ testing functions. Certainly C^2 harmonic functions satisfy this definition. We do have C^0 but non C^2 solutions to (fully nonlinear) elliptic equations such as Monge-Ampere/Special Lagrangian equations.

We verify C^0 harmonic functions in the viscosity sense are in fact smooth and satisfy the "harmonic" equation by Poisson representation formula. Note explicitly representation for solutions to nonlinear equations are NOT available in general.

Let

$$h = \int_{\partial B_1} P\left(x, y\right) u\left(y\right)|_{\partial B_1} dA_y$$

$\cdot$ $h = u$ on ∂B_1.
$\cdot$ $\triangle h = 0$ in B_1.

Now if $u > h$ somewhere at $x_0 \in \overset{0}{B}_1$, say $\left(u - h\right)\left(x_0\right) = \max_{B_1}\left(u - h\right) > 0$

u,h graph *figure*

$h + \max \geq u$ in B_1, "$=$" at x_0.

Also $h + \max' - \varepsilon\left|x\right|^2 \geq u$, "$=$" at $x_0' \in \overset{0}{B}_1$, yes we can replace.
But $\triangle$left$= -2n\varepsilon < 0$. This contradiction shows $u \leq h$.

Similarly, if $u < h$ somewhere at $x_0 \in \overset{0}{B}_1$, say $\left(u - h\right)\left(x_0\right) = \min_{B_1}\left(u - h\right) < 0$

u,h graph *figure*

$h + \min \leq u$ in B_1, "$=$" at x_0.

Also $h + \min' + \varepsilon\left|x\right|^2 < u$, "$=$" at $x_0' \in \overset{0}{B}_1$, yes we can replace.
But $\triangle$left$= 2n\varepsilon > 0$. This contradiction shows $u \geq h$.
Thus $u \equiv h$.