Lecture 3 Solvability for $\triangle$ in Ω

$\circ \begin{cases} \triangle u = 0 \text{ in } B_1 \\ u = \varphi \text{ on } \partial B_1 \end{cases}$ Poisson formula

$\circ$ Perron method

$\circ$ Boundary values/Barriers/"Harnack"

Poisson formula

Recall for

$$\triangle u = 0 \qquad (\triangle (u - \Gamma * RHS) = 0)$$
$$u(0) = \frac{1}{|\partial B_1|} \int_{\partial B_1} u(y)\, dA_y.$$

Question: $u(x) = \int_{\partial B_1} u(y)\, dA_y(x)$?

Recall in the derivation of the mean value equality, we use

$$\int_\Omega v \triangle u - u \triangle v = \int_{\partial\Omega} v u_\gamma - u v_\gamma dA.$$

Now take $\Omega = B_1(0) \setminus B_\varepsilon(x)$ and

$$\begin{aligned} v(x,y) = G(x,y) &= c_n \left[\frac{1}{|x-y|^{n-2}} - \frac{1}{|y|^{n-2} |x-\bar{y}|^{n-2}} \right] \overset{|y|=1}{=} 0 \text{ with } \bar{y} = \frac{y}{|y|^2} \\ &= c_n \left[\frac{1}{|y-x|^{n-2}} - \frac{1}{|x|^{n-2} |y-\bar{x}|^{n-2}} \right], \end{aligned}$$

as $|y|\,|x-\bar{y}| = |x|\,|y-\bar{x}|$, which can be seen from similarity of $\triangle ox\bar{y}$ and $\triangle oy\bar{x}$ or just squaring both sides.

We continue for harmonic function u,

$$\begin{aligned} \int_{\partial B_1} u(y)\, D_y G(x,y) \cdot \gamma_y\, dA_y &= \lim_{\varepsilon \to 0} \left[\int_{\partial B_\varepsilon} u(y)\, D_y G(x,y) \cdot \gamma_y - G(x,y)\, u_{\gamma_y}\, dA_y \right] \\ &= \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon} u(y)\, D_y G(x,y) \cdot \gamma_y\, dA_y \\ &= -(n-2)\, c_n\, |\partial B_1|\; u(x) = u(x), \end{aligned}$$

if c_n is chosen as $\dfrac{-1}{(n-2)\,|\partial B_1|}$.

Using the above symmetry of Green's function,

$$\begin{aligned} D_y G(x,y) &= c_n \left[\frac{-(n-2)(y-x)}{|y-x|^n} - \frac{-(n-2)(y-\bar{x})}{|x|^{n-2} |y-\bar{x}|^n} \right] \\ &= -(n-2)\, c_n \left[\frac{(y-x)}{|y-x|^n} - \frac{|x|^2 (y-\bar{x})}{|x|^n |y-\bar{x}|^n} \right] \\ &\underset{|y||x-\bar{y}|=|x||y-\bar{x}|}{\overset{|y|=1}{=}} \frac{1}{|\partial B_1|} \frac{\left(1-|x|^2\right) y}{|y-x|^n}, \end{aligned}$$

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then

$$D_y G(x,y) \cdot \frac{y}{|y|} = \frac{1}{|\partial B_1|} \frac{\left(1-|x|^2\right)}{|y-x|^n}.$$

Thus

$$u(x) = \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{\left(1-|x|^2\right)}{|y-x|^n} u(y)\, dA_y.$$

Note that as a linear combinations of harmonic functions $D_y G(x,y)$ for $|y|=1$, the kernel $\dfrac{1-|x|^2}{|y-x|^n}$ is still harmonic in terms of x. This can also be checked easily (assume $y=(1,0\cdots,0)$ or change variable $x-y$ to x).

Exercise (Inverse problem): Given

$$u(x) = \frac{1}{|\partial B_1|} \int_{\partial B_1} \frac{\left(1-|x|^2\right)}{|y-x|^n} \varphi(y)\, dA_y \quad \text{for } \varphi \in C^0(\partial B_1),$$

then $\triangle u(x) = 0$ and $\lim\limits_{x\to y} u(x) = \varphi(y)$.

RMK. If $u(x) = \int_{B_1} G(x,y) f(y)\, dy$, then $\begin{cases} \triangle u = f(x) \text{ in } B_1 \\ u = 0 \text{ on } \partial B_1 \end{cases}$. We have the following asymptotic behavior–obvious in 1-d. Here $d(y)$ denotes the distance to the boundary.

If $f(y) \sim d^{-2}(y)$, then $u(x) \sim \displaystyle\int d(y) \frac{1}{d^2(y)} = \infty.$

If $f(y) \sim d^{-2+\beta}(y)$, then $u(x) \sim \displaystyle\int_0^{d(x)} d(y) \frac{1}{d^{2-\beta}(y)} \sim d^\beta(x).$

RMK. From the Poisson formula, we see $u(x)$ is analytic. In fact $\dfrac{1-|x|^2}{|y-x|^n}$ is analytic in terms of x for $|y|=1$. That is

$$\begin{aligned} |y-x|^2 &= |y|^2 - 2y\cdot x + |x|^2 \overset{|y|=1}{=} 1 - 2y\cdot x + |x|^2 \\ \frac{1}{|y-x|^n} &= \left(1-2y\cdot x+|x|^2\right)^{-n/2} = 1 - \frac{n}{2}\left(|x|^2 - 2y\cdot x\right) + \cdots. \end{aligned}$$

Then integrate, we find $u(x)$ is analytic.

RMK. For $\begin{cases} \triangle u = 0 \text{ in } B_R \\ u = \varphi \text{ on } \partial B_R \end{cases}$, after scaling we have

$$u(x) = \frac{1}{|\partial B_1|} \int_{\partial B_R} \frac{R^2 - |x|^2}{R\ |y-x|^n} \varphi(y)\, dA_y.$$

<u>Perron method</u>

Preliminary.

Def. u is a super (sub) harmonic function in the classical sense for $u \in C^2$ if $\triangle u \underset{(\geq)}{\leq} 0$; in the viscosity sense for $u \in C^0$ if for any quadratic $P \underset{(\geq)}{\leq} u$ near an interior point x_0 and "$=$" at x_0, then $\triangle P \underset{(\geq)}{\leq} 0$; in the distribution/IBP sense for $u \in L^1$ if $\int u \triangle \varphi \underset{(\geq)}{\leq} 0$ for all $\varphi \in C_0^{\infty/1,1}(R^n)$ and $\varphi \geq 0$.

Proposition. For $u \in C^0$, $\triangle u \leq 0$ in the distribution/IBP sense $\Leftrightarrow$ $\triangle u \leq 0$ in the mean value inequality sense $u(x) \geq \fint_{B_\rho(x)} u(y)\,dy$ for all x and ρ $\Leftrightarrow$ $\triangle u \leq 0$ in the viscosity sense.

Proof.

IBP$\Rightarrow$MVI.

If we take $\varphi = \Gamma_\rho - \Gamma_\varepsilon$, as before with Γ_ρ being the $C^{1,1}$ modification of the fundamental solution Γ by $\frac{|x|^2}{2n}\chi_{B_\rho} - A$ in B_ρ, then

$$\frac{1}{|B_\rho|}\int_{B_\rho} u \leq \frac{1}{|B_\varepsilon|}\int_{B_\varepsilon} u \to u(0) \text{ as } \varepsilon \to 0.$$

MVI$\Leftarrow$IBP.

Take $\psi(x) = \psi(|x|) \in C_0^\infty(R^n)$ s.t. $\psi \geq 0$ and $\int \psi = 1$. First

$$\begin{aligned} u * \psi(x) &= \int_{R^n} u(y)\,\psi(x-y)\,dy = \int_{R^n} u(y)\,\psi(y-x)\,dy \\ &= \int_{R^n} u(z+x)\,\psi(z)\,dz. \end{aligned}$$

Then

$$\begin{aligned} \fint_{B_\rho(x)} u * \psi(y)\,dy &= \fint_{B_\rho(x)} \int_{R^n} u(z+y)\,\psi(z)\,dz\ dy \\ &= \int_{R^n} \fint_{B_\rho(x)} u\left(\overset{+x}{z} + \overset{-x}{y}\right) dy\ \psi(z)\,dz \\ &\overset{\psi \geq 0}{\leq} \int_{R^n} u(z+x)\ \psi(z)\,dz \\ &= u * \psi(x). \end{aligned}$$

Recall the argument from $\fint_{B_\rho(x)} u(y)\,dy = u(0)$ for all ρ leading to $\triangle u(0) = 0$ for $u \in C^2$. Similarly for $u * \psi \in C^\infty$ with $\fint_{B_\rho(x)} u * \psi(y)\,dy \leq u * \psi(x)$ for all ρ and x we show

$$\triangle u * \psi(x) \leq 0.$$

It follows that for all $\varphi \in C_0^\infty(R^n)$ and $\varphi \geq 0$

$$\int u * \psi\ \ \triangle \varphi \leq 0$$

and also

$$\int u * \psi_\varepsilon\ \ \triangle \varphi \leq 0.$$

Let ε go to 0, we have $\int u \bigtriangleup \varphi \leq 0$ for all $\varphi \in C_0^\infty(R^n)$ and $\varphi \geq 0$.

MVI$\Rightarrow$Viscosity.

For any quadratic P touching u from below at, say 0, we have $P(0) \geq u(0) \geq \fint_{B_\rho(0)} u \geq \fint_{B_\rho(0)} P$ for all small ρ. That is

$$P(0) \geq \fint_{B_\rho(0)} P = P(0) + \frac{\rho^2}{2(n+2)} \bigtriangleup P.$$

Thus

$$\triangle P \leq 0.$$

Viscosity$\Rightarrow$MVI.

For any x, say 0 and all $r \leq \rho$, let h be the harmonic function by Poisson representation s. t. $\triangle h = 0$ in B_r and $h = u$ on ∂B_r. Using the "shifting" argument for equivalency of viscosity harmonicity and classical harmonicity, we see the comparison $u(x) \geq h(x)$ in B_r, in particular

$$u(0) \geq h(0) \geq \frac{1}{|\partial B_r|} \int_{\partial B_r} u.$$

Integrate we get

$$u(0) \geq \frac{1}{|B_\rho|} \int_{B_\rho} u.$$

RMK. Certainly the subharmonic version of the above proposition is also true.

• Local property RMK. As super/sub harmonicity in the viscosity sense is a local property–only requires a local touching, so is super/sub harmonicity in the distribution/IBP (the compact support set of the test function could be small) or mean value sense (the valid radius could be less than any small fixed number)

We also need and are ready for the following three properties.

• Comparison Theorem.

a) u and v are C^0 viscosity super and sub solutions respectively to $\triangle w = 0$ in Ω.

b) $u \geq u$ on $\partial\Omega$.

THEN $u \geq v$ in Ω.

Proof. It is straightforward if we use the mean value inequality formulation.

RMK. If we only use the viscosity formulation (which is only available for a large class of general fully nonlinear elliptic equations), and if u or v is C^2, then the proof is the same as the "shifting" argument for equivalency of viscosity harmonicity and classical harmonicity–just replace the C^2 harmonic function h with the C^2 sub/super solution. If both u and v are only continuous, the comparison was achieved for general fully nonlinear equations in late 1980s.

• "Minimum"" or "Maximum choice property.

If u_1 and u_2 are both C^0 continuous super or sub viscosity solutions to $\triangle w = 0$, then so is continuous $u_1 \wedge u_2 = \min\{u_1, u_2\}$ or $u_1 \vee u_2 = \max\{u_1, u_2\}$.

Proof. For any quadratic P touches $u_1 \wedge u_2$ from below near an interior point, it automatically lower-touches u_1 or u_2 (both if $u_1 = u_1$ at the point) near this point.

As both u_1 and u_2 are viscosity super solution, then $\triangle P \le 0$. The subsolution part is just an upper touching argument.

RMK.The mean value proof is similarly easy, the distribution/IBP way is not so obvious

• Replacement property.

Let $u \in C^0(\Omega)$ be a super (sub) solution to $\triangle w = 0$. For any interior ball, say centered at 0, $B_r \in \Omega$, let h be harmonic inside B_r and shares u value on the boundary, that is

$$\begin{cases} \triangle h = 0 & \text{in } B_r \\ h = u & \text{on } \partial B_r \end{cases}.$$

Then the the harmonic replacement of $u : v = \begin{cases} u & \text{outside } B_r \\ h & \text{inside } B_r \end{cases}$ is also super (sub) harmonic.

Proof. By the comparison theorem, $v \ge h$ in B_r. For any quadratic P touches v from below near an interior point x_0 of Ω, if x_0 is inside B_r, then P lower-touches h near x_0; if x_0 is on the boundary or outside of B_r, then P lower-touched u nearby. In either case, we get $\triangle P \le 0$. (The sub case is similar.)

Preliminary example. In 1-d the harmonic/linear function $u = \sup\limits_{\substack{v'' \ge 0 \\ v(a)=A \\ v(b)=B}} v$.

convex functions wrapping up the linear one

Perron method for $\begin{cases} \triangle u = 0 & \text{in } \Omega \quad \text{(nice domain)} \\ u = \varphi \text{ on } \partial\Omega \end{cases}$. Set

$$\mathcal{S} = \left\{ u : \; u \in C^0\left(\bar{\Omega}\right), \; \triangle u \ge 0 \text{ in } \Omega, \; u \le \varphi \; \text{ on } \partial\Omega \right\}.$$

Then

$$w(x) = \sup_{u \in \mathcal{S}} u(x)$$

is the desired harmonic solution.

Part I. $\triangle w = 0$ in Ω

Step1. $\mathcal{S}$ is nonempty and w is well defined.

Choose M such that $|\varphi| \le M$ on $\partial\Omega$, then constant function $-M \in S$. And $w \ge -M$, otherwise,$w = \sup\limits_{-M \vee u \in S} -M \vee u$. Also $w \le M$ as $u \le M$ for all $u \in S$.

Step2. Convergence to w.

Step2.1 Fix $x_0 \in \Omega$, say $x_0 = 0$, there exist a sequence of u_k in S such that $\lim\limits_{k\to\infty} u_k(0) = u(0)$. Let $\bar{U}_k$ be the harmonic lift of u_k in $B_r(0) \subset \Omega$. Then by the replacement property $U_k \in S$ and by the comparison

$$w \ge U_k \ge u_k$$
$$w(0) = \lim_{k\to\infty} U_k(0).$$

By the a priori C^2, C^3, $\cdots$ interior estimate estimate, we can extract a subsequence, still denoted by U_k such that

$$U_k \to v \text{ in } C^{2,\alpha} \text{ norm, in any fixed subdomain of } B_r.$$

Then $\triangle v = 0$ in classical sense and $v \leq w$ in B_ρ and $v(0) = w(0)$.

Step2.2 Claim $v = w$ in $B_r(0)$. Suppose there is x_1 s.t. $v(x_1) < w(x_1)$. From the definition of w, there exists $u \in S$ s.t.

$$w(x_1) \geq u(x_1) > v(x_1).$$

Let $\bar{U}_k$be the harmonic lift of $u \vee U_k$ in B_r. Repeat Step2.1, we have

$$\begin{aligned} \bar{U}_k &\to \bar{v} \text{ in } C^{2,\alpha} \text{ norm, in any fixed subdomain of } B_r \\ \triangle \bar{v} &= 0 \\ w &\geq \bar{v} \geq v \\ \bar{v}(0) = v(0) &= w(0) \quad \text{and} \quad \bar{v}(x_1) > v(x_1). \end{aligned}$$

By the strong maximum principle, $\bar{v}$ must coincide with v, It contradicts $\bar{v}(x_1) > v(x_1)$.

Part II When $\lim\limits_{x \to y} u(x) = \varphi(x)$?

eg1/Exercise Let Ω be a C^2 strongly convex domain (principle curvatures $\kappa \geq \eta > 0$) and $\varphi \in C^2$. Locally near every boundary point, $\partial\Omega$ is written as $x_n = g(x') \approx \frac{1}{2}\left(\kappa_1 x_1^2 + \cdots + \kappa_{n-1} x_{n-1}^2\right)$

$$\varphi(x', g(x')) = \varphi(0,0) + \varphi_{x'}(0,0)\, x' + \varphi_{x_n}(0,0)\, g(x') + O\left(|x'|^2\right) >$$

Let

$$B^{\pm}(x', x_n) = \varphi(0,0) + \varphi_{x'}(0,0)\, x' \pm N x_n.$$

figure φ between two linear functions on the boundary

Then

$$\begin{cases} \triangle B^{\pm} = 0 \quad \text{in } \Omega \\ B^- \leq \varphi \leq B^+ \quad \text{on } \partial\Omega \text{ for large enough } N \end{cases}.$$

It follows that Perron solution satisfies

$$B^- \leq u \leq B^+.$$

Hence $\lim\limits_{x \to 0} u(x) = \varphi(0)$.

eg2. 2-d For any domain missing the negative x-axis, at locally. Let

$$B(x) = -\operatorname{Re}\frac{1}{\log z} = \frac{-\log r}{\log^2 r + \theta^2} \geq 0 \quad \text{and} \quad \to 0 \quad \text{as } x \to 0.$$

Then $\triangle B=0$.

figure domain missing the negative x-axis

Claim: Perron solution satisfies $\lim_{x\to 0} u\left(x\right)=\varphi\left(0\right).$

Indeed for any $\varepsilon>0$, there exists δ s.t. $\varphi\left(0\right)+\varepsilon\geq\varphi\left(x\right)\geq\varphi\left(0\right)-\varepsilon$ on $B_{\delta}\left(0\right)\cap\partial\Omega$.

Lower bound. There exists large N s.t. $\tilde{B}^{-}=-N\ B+\varphi\left(0\right)-\varepsilon\leq\varphi$ on $\partial\Omega$. Also harmonic function $\tilde{B}^{-}\in S$. Hence

$$u=\sup_{v\in S} v\geq\tilde{B}^{-}.$$

Thus there exists δ_1 s.t. in $B_{\delta_1}\left(0\right)\cap\Omega$

$$u\geq\varphi\left(0\right)-2\varepsilon.$$

Upper bound. There exists large N s.t. harmonic function $\tilde{B}^{+}=N\ B+\varphi\left(0\right)+\varepsilon\geq\varphi\geq v$ on $\partial\Omega$ for all $v\in S$. By the comparison

$$\tilde{B}^{+}\geq v \text{ in } \Omega.$$

Thus there exists δ_2 s.t. in $B_{\delta_2}\left(0\right)\cap\Omega$

$$u\leq\varphi\left(0\right)+2\varepsilon.$$

RMK. For u taking the continuous boundary value φ at x_0, for the above argument, only need

$$\begin{cases}\triangle B=0\\ B\geq 0\ \ \text{“}=\text{''} \text{ only at } x_0\\ B\in C^0\left(\bar{\Omega}\right)\end{cases}.$$

eg3. Counterexample. Lebesgue spine 3-d. Set $\rho=\sqrt{y^2+z^2}$

figure $\rho=e^{-\frac{1}{kx}}$ graph w/ $k=1,2,k$

Set $X=\left(x,y,z\right)$, integrate harmonic fundamental solution $\frac{1}{|X|}$ along x-axis

$$\begin{aligned}u\left(X\right)&=\int_0^1\frac{t}{\sqrt{\left(x-t\right)^2+y^2+z^2}}dt\\&=\left|X-\left(1,0,0\right)\right|-\left|X\right|+x\log\frac{1-x+\sqrt{\left(x-1\right)^2+\rho^2}}{-x+\sqrt{\left(x-1\right)^2+\rho^2}}\\&=\text{nice }-2x\log\rho.\end{aligned}$$

Thus

$$\lim_{X\to 0} u\left(X\right)=\begin{matrix}0\\2/k\\2\end{matrix}.$$

doesn't exist.

eg4. Let $\Omega = B_1 \backslash (0)$

$$\begin{cases} \triangle u = 0 \ \text{ in } \Omega \\ u = 1 \text{ on } \partial B_1 \\ u(0) = 2 \end{cases}$$

has no solution. As $\lim_{x\to 0} u(x)$ equals 1 for $u \equiv 1$ and ∞ for $u = \frac{1}{r^{n-2}}$.

Proposition. If $\partial\Omega$ satisfies the exterior sphere condition at x_0 (one $C^{1,1}$ condition) then Perron solution u takes the continuous boundary value $\varphi(x_0)$, that is, $\lim_{x\to x_0} u(x) = \varphi(x_0)$.

figure exterior sphere condition

figure graph of $\dfrac{1}{\delta^{n-2}} - \dfrac{1}{|x-x_1|^{n-2}}$.

Proof. Set $B = \dfrac{1}{\delta^{n-2}} - \dfrac{1}{|x-x_1|^{n-2}}$. Then $B \geq 0$ and $= 0$ at x_0; and also $\triangle B = 0$.

Theorem. If $\partial\Omega \cap B_R(x_0) = \{x_n = L(x')\}$ where $L(x')$ is Lipshittz with $|DL| \leq N$. Then the Perron solution u takes the continuous boundary value $\varphi(x_0)$.

figure Lip domain

Barrier way. Construct barrier directly in 2-d $r^\alpha \cos \alpha\theta$, implicitly in 3-d and above $r^\alpha f(\theta)$, where θ is the angle from the Lip cone axis.

Mean value way. $\cdots$ More powerful/general but long.