

## Lecture 4 Schauder for Laplace

- o Definitions and examples
- o Estimates

For  $f \in C_0^\infty(R^n)$ , set

$$u(x) = \Gamma * f(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y) dy \stackrel{\text{def}}{=} \lim_{\varepsilon \rightarrow 0} \int_{R^n \setminus B_\varepsilon(x)} \Gamma(x-y) f(y) dy.$$

Recall  $\Delta u = f$

$$\begin{aligned} \Delta u(x) &= D\Gamma * Df \\ &= \lim_{\varepsilon \rightarrow 0} \int_{R^n \setminus B_\varepsilon(x)} D\Gamma(x-y) \cdot Df(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(x)} \Gamma_\gamma f dA = \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(x)} f dA = f(x). \end{aligned}$$

Further

$$\begin{aligned} D_{ij}u &= \Gamma_i * f_j \\ &= \lim_{\varepsilon \rightarrow 0} \int_{R^n \setminus B_\varepsilon(x)} \Gamma_i(x-y) \cdot f_j(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(x)} \Gamma_i \gamma_j f dA - \int_{R^n \setminus B_\varepsilon(x)} \Gamma_{ij}(x-y) f(y) dy \end{aligned}$$

say  $x = 0$ ,

$$\begin{aligned} D_{ij}u(0) &= \lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon} \frac{1}{|\partial B_1|} \frac{y_i}{|y|^n} \frac{y_j}{|y|} f dA + \int_{R^n \setminus B_\varepsilon(x)} c_n \frac{y_i y_j - n^{-1} \delta_{ij} |y|^2}{|y|^{n+2}} f(y) dy \\ &= \frac{\delta_{ij}}{n} f(0) + \lim_{\varepsilon \rightarrow 0} \int_{R^n \setminus B_\varepsilon(x)} c_n \frac{y_i y_j - n^{-1} \delta_{ij} |y|^2}{|y|^{n+2}} f(y) dy. \end{aligned}$$

RMK. As one can see the limit  $\lim_{\varepsilon \rightarrow 0}$  is uniform (from the calculation in the proof of the following Schauder), we can switch the order of differentiation and integral.

Q. How to make sense of those singular integrals  $\Gamma_{ij} * f$ ?

Eg 1.  $f(x) = \chi_\Omega \in L^\infty(R^2)$ , where  $\Omega = \{x_1 x_2 \geq 0\} \cap B_1$ .

figure of  $\Omega$

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<sup>0</sup>November 2, 2016

$$\begin{aligned} u_{12}(0) &= C_2 \int_{\Omega} \frac{y_1 y_2}{|y|^4} f(-y) dy \\ &\sim \int \int \frac{\frac{1}{2} \sin 2\theta}{r^2} r dr d\theta \sim \int \frac{1}{r} dr = \infty. \end{aligned}$$

RMK. One can smooth out  $f$  to get smooth counterexamples for  $W^{2,\infty}$  estimates.

Eg 2.  $f(x) = \frac{x_1 x_2}{|x|^2} \frac{1}{|\ln|x||^{1/2}} \in C^0$ . figure?

$$\begin{aligned} u_{12}(0) &= C_2 \int \frac{y_1 y_2}{|y|^4} \frac{y_1 y_2}{|y|^2} \frac{1}{|\ln|y||^{1/2}} dy \\ &\sim \int \frac{1}{r} \frac{1}{|\ln r|^{1/2}} dr = \infty. \end{aligned}$$

Eg 3.

$$\begin{aligned} u &= h(x) (\ln r)^{1/3} \quad \text{with } h(x) = \operatorname{Im} z^k \\ |D^k u(0)| &= \infty \\ \Delta u &= O\left(r^{k-2} \frac{1}{(\ln r)^{2/3}}\right) \in C^0, C^{k-2}. \end{aligned}$$

Message.  $\Delta u = f \in C^0$  denote imply  $D^2 u \in L^\infty$ .

For general right hand side  $\Delta w = f$  in  $B_1$ , what happens to  $w$  in  $B_{1/2}$ . Let  $\eta \in C_0^\infty(B_1)$  be a cut-off function with  $\eta = 1$  on  $B_{3/4}$ . Let  $u = \Gamma * (\eta f)$ , then  $\Delta(w - u) = 0$  in  $B_{3/4}$ . That is  $w = (w - u) + u$  as a sum of harmonic function and Newtonian potential.

Theorem. Given  $u = \Gamma * f$  with  $f \in C_0^\alpha(B_2)$  and  $0 < \alpha < 1$ . THEN  $\|D^2 u\|_{C^\alpha(B_1)} \leq C(n) \|f\|_{C_\alpha(B_2)}$ .

Proof.  $L^\infty$  bound.

$$\begin{aligned} D_{ij} u(x) &= \int_{R^n \setminus B_3} f(x-y) \frac{y_i y_j}{|y|^{n+2}} dy \quad x \in B_1 \\ &= \int_{B_3} [f(x-y) - f(x)] \frac{y_i y_j}{|y|^{n+2}} dy \\ &\leq \|f\|_{C^\alpha} \int_{B_3} |y|^\alpha \frac{1}{|y|^n} dy \leq C(n) \|f\|_{C^\alpha}. \end{aligned}$$

$$D_{ii} u(x) = \int_{R^n \setminus B_3} f(x-y) \frac{y_i^2 - n^{-1} |y|^2}{|y|^{n+2}} dy + \frac{1}{n} f(x)$$

Similarly,

$$\|D_{ii} u\|_{C^\alpha(B_1)} \leq C(n) \|f\|_{C^\alpha}.$$

$C^\alpha$  bound.

All we need about the singular kernel  $K = \Gamma_{ij}$  or  $\Gamma_{ii}$  are

- $\int_{\partial B_1} K = 0$ ,
- $K$  homogeneous degree  $-n$ ,

- $DK$  homogeneous degree  $-(n+1)$ .

Let  $h = |x_1 - x_2|$ .

$$\begin{aligned} & \int_{B_3} [f(x_1 - y) - f(x_2 - y)] K(y) dy \\ &= \int_{\substack{|y| \leq h \\ I}} + \int_{\substack{|y| > h \\ II}} \end{aligned}$$

$$\begin{aligned} I &= \int_{|y| \leq h} [f(x_1 - y) - f(x_1)] K(y) dy - \int_{|y| \leq h} [f(x_2 - y) - f(x_2)] K(y) dy \\ &\leq \|f\|_{C^\alpha} \int_{|y| \leq h} |y|^\alpha \frac{1}{|y|^n} dy + \text{similar} \\ &\leq C(n) \|f\|_{C^\alpha} \int_0^h r^{\alpha-1} dr = \frac{C(n)}{\alpha} \|f\|_{C^\alpha} h^\alpha. \end{aligned}$$

Let  $\bar{x} = \frac{x_1+x_2}{2}$

figure circle  $B_{3h/2}(\bar{x})$ ,  $B_h(x_1)$ ,  $B_h(x_2)$

$$\begin{aligned} II &= \int_{|y| > h} [f(x_1 - y) - f(\bar{x})] K(y) dy - \int_{|y| > h} [f(x_2 - y) - f(\bar{x})] K(y) dy \\ &= \int_{|x_1 - y| > h} [f(y) - f(\bar{x})] K(x_1 - y) dy - \int_{|x_2 - y| > h} [f(y) - f(\bar{x})] K(x_2 - y) dy \\ &= \int_{\substack{|\bar{x} - y| \geq \frac{3}{2}h \\ II_1}} + \int_{\substack{|\bar{x} - y| < \frac{3}{2}h / |x_1 - y| > h \\ II_c}} - \int_{\substack{|\bar{x} - y| \geq \frac{3}{2}h \\ II_2}} - \int_{\substack{|\bar{x} - y| < \frac{3}{2}h / |x_2 - y| > h \\ II_c}} \end{aligned}$$

$$\begin{aligned} II_1 - II_2 &= \int_{|\bar{x} - y| \geq \frac{3}{2}h} [f(y) - f(\bar{x})] [K(x_1 - y) - K(x_2 - y)] dy \\ &\leq C(n) \|f\|_{C^\alpha} \int_{3 > |\bar{x} - y| \geq \frac{3}{2}h} |y - \bar{x}|^\alpha \frac{|x_1 - x_2|}{|x_* - y|^{n+1}} dy. \end{aligned}$$

As mean value  $x_*$  is between  $x_1 \& x_2$  and  $|\bar{x} - y| \geq 3h/2$ , then  $|x_* - y| \sim |y - \bar{x}|$ .

$$\begin{aligned} II_1 - II_2 &\leq C(n) \|f\|_{C^\alpha} h \int_{3h/2}^3 r^{\alpha-2} dr \\ &\leq C(n) \|f\|_{C^\alpha} h^\alpha. \end{aligned}$$

Let  $\Omega = \{|\bar{x} - y| < \frac{3}{2}h\} / \{|x_1 - y| > h\}$

$$\begin{aligned} II_c &= \int_{\Omega} [f(y) - f(\bar{x})] K(x_1 - y) dy \\ &\leq \|f\|_{C^\alpha} \left(\frac{3h}{2}\right)^\alpha \int_{h < |x_1 - y| < 2h} \frac{1}{|x_1 - y|^n} dy \\ &\leq C(n) \|f\|_{C^\alpha} \left(\frac{3h}{2}\right)^\alpha \int_h^{2h} \frac{1}{r} dr \\ &= C(n) \|f\|_{C^\alpha} h^\alpha. \end{aligned}$$

Collecting all the terms, we arrive at the desired estimate.

**RMK.** Higher order elliptic equations such as  $\Delta^2 u = f$ . Take a homogeneous order  $2k$  polynomial  $P$ , consider equation

$$P(D)u = f.$$

Fourier transform

$$\begin{aligned} P(i\xi)\hat{u} &= \hat{f} \text{ then} \\ \hat{u} &= \frac{1}{P(i\xi)}\hat{f}. \end{aligned}$$

Need  $P(i\xi) = i^{2k}P(\xi) \neq 0$  for  $\xi \neq 0$ , so order even (called elliptic). Now

$$\widehat{D^{2k}u} = (i\xi)^{2k} \hat{u} = \underbrace{\frac{\xi^{2k}}{P(\xi)}}_h \hat{f}$$

and then

$$D^{2k}u = \left(\frac{\xi^{2k}}{P(\xi)}\hat{f}\right)^\vee = \left(\frac{\xi^{2k}}{P(\xi)}\right)^\vee * f.$$

The kernel

$$K(x) = \int_{R^n} h(\xi) e^{-ix\xi} d\xi$$

satisfies

- $K(tx) = \int_{R^n} h(t\xi) e^{-ix \cdot t\xi} \frac{dt\xi}{t^n} = \frac{1}{t^n} K(x)$
- $DK$  homogeneous order  $-(n+1)$
- Cancellation  $\int_{\partial B_1} K(x) dx = \int_{\partial B_1} \int_{R^n} h(\xi) e^{-ix\xi} d\xi dx = \int_{R^n} \int_{\partial B_1} h(\xi) e^{-ix\xi} dx d\xi = 0$ .

So Schauder estimates also holds for higher order elliptic equations

$$\|D^{2k}u\|_{C^\alpha(B_{1/2})} \leq C(n, \alpha) \left( \|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)} \right).$$

Application to  $\Delta u = f$  in  $B_1$ . What happens to  $u$  in  $B_{1/2}$ ?

Let  $\eta \in C_0^\infty(B_1)$  be a cut-off function with  $\eta = 1$  on  $B_{3/4}$ .

$$\begin{aligned}\Delta \Gamma * (f\eta) &= f\eta \\ \Delta(u - \Gamma * (f\eta)) &= 0 \text{ in } B_{3/4}.\end{aligned}$$

Then by interior estimates for harmonic functions

$$\begin{aligned}\|u - \Gamma * (f\eta)\|_{C^{2,\alpha}(B_{1/2})} &\leq C(n) \|u - \Gamma * (f\eta)\|_{L^\infty(B_{3/4})} \\ &\leq C(n) \left[ \|u\|_{L^\infty(B_{3/4})} + \|\Gamma * (f\eta)\|_{L^\infty(B_{3/4})} \right] \\ &\leq C(n) \left[ \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} \right].\end{aligned}$$

$$\begin{aligned}\|u\|_{C^{2,\alpha}(B_{1/2})} &\leq \|u - \Gamma * (f\eta)\|_{C^{2,\alpha}(B_{1/2})} + \|\Gamma * (f\eta)\|_{C^{2,\alpha}(B_{1/2})} \\ &\leq C(n) \left[ \|u\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} \right] + C(n, \alpha) \|f\|_{C^\alpha(B_1)} \\ &= C(n, \alpha) \left[ \|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)} \right],\end{aligned}$$

where in the second inequality we used

$$\|\Gamma * (f\eta)\|_{Lip(B_{1/2})} \leq C(n) \|f\|_{L^\infty(B_1)}$$

as  $D(\Gamma * g) = D\Gamma * g$  and homogeneous order  $-(n-1)$  function  $D\Gamma$  is locally integrable.

Scaling version.

Given  $\Delta u = f$  in  $B_R$ . Let  $v(y) = u(Ry)$ ,  $g(y) = R^2 f(Ry)$ , then

$$\Delta v(y) = R^2 \Delta u(Ry) = R^2 f(Ry) = g(y) \text{ in } B_1.$$

By the above just proved,

$$\|v\|_{C^{2,\alpha}(B_{1/2})} \leq C(n, \alpha) \left[ \|v\|_{L^\infty(B_1)} + \|g\|_{C^\alpha(B_1)} \right].$$

In terms of  $u$

$$\begin{aligned}Dv(y) &= R Du(Ry) \\ D^2v(y) &= R^2 D^2u(Ry) \\ \frac{D^2v(y) - D^2v(z)}{|y-z|^\alpha} &= R^{2+\alpha} \frac{D^2u(Ry) - D^2u(Rz)}{|Ry-Rz|^\alpha},\end{aligned}$$

thus we have

$$\begin{aligned}\|u\|_{L^\infty(B_R)} + R \|Du\|_{L^\infty(B_R)} + R^2 \|D^2u\|_{L^\infty(B_R)} + R^{2+\alpha} [D^2u]_{C^\alpha(B_R)} \\ \leq C(n, \alpha) \left\{ \|u\|_{L^\infty(B_R)} + R^2 \|f\|_{L^\infty(B_R)} + R^{2+\alpha} [f]_{C^\alpha(B_R)} \right\}\end{aligned}$$

and also  $R^{-\beta}$  times both sides of the inequality (for weighted norm in continuity method).