Lecture 5 Weighted norm and solvability

∘ weighted norm and a priori estimates
∘ continuity method

Motivation.

$$a_{ij}(x) D_{ij}u = f \quad \text{with } C^{\alpha} \text{ coefficients } \lambda I \le (a_{ij}) \le \lambda^{-1} I$$
$$a_{ij}(0) D_{ij}u = [a_{ij}(0) - a_{ij}(x)] D_{ij}u + f$$

From Schauder (Lecture 4)

$$\|u\|_{c^{2,\alpha}(B_{1/2})} \le C\left[\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(B_1)} + \left\|[a_{ij}(0) \underset{\varepsilon}{-} a_{ij}(x)] D_{ij}u\right\|_{C^\alpha(B_1)}\right].$$

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$$\|[a_{ij}(0) - a_{ij}(x)] D_{ij}u\|_{L^\infty} \le \|a_{ij}(0) - a_{ij}(x)\|_{L^\infty} \ \|D_{ij}u\|_{L^\infty}$$

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$$\begin{aligned} & \frac{[a_{ij}(0) - a_{ij}(x)] D_{ij}u(x) - [a_{ij}(0) - a_{ij}(y)] D_{ij}u(y)}{|x-y|^\alpha} \\ & = \frac{[a_{ij}(y) - a_{ij}(x)] D_{ij}u(x)}{|x-y|^\alpha} + \frac{[a_{ij}(0) - a_{ij}(y)][D_{ij}u(x) - D_{ij}u(y)]}{|x-y|^\alpha} \\ & \le [a]_{C^\alpha} \left|D^2u(x)\right| + |a(0) - a(y)| \left[D^2u\right]_{C^\alpha}. \end{aligned}$$

Weighted norm

Let Ω be a bounded domain and $d = dist(x, \partial\Omega)$, set

$$\|u\|_{C^{2,\alpha}_w(\Omega)} = \sup_{x\in\Omega} \underbrace{\left[\|u\|_{L^\infty(B_d(x))} + d\|Du\|_{L^\infty(B_d(x))} + d^2\left\|D^2u\right\|_{L^\infty(B_d(x))} + d^{2+\alpha}\left[D^2u\right]_{C^\alpha(B_d(x))}\right]}_{\widetilde{\|u\|}_{B_d(x)}}$$

$$\|f\|_{C^{2+\alpha}_w(\Omega)} = \sup_{x\in\Omega}\left[d^2\|f\|_{L^\infty(B_d(x))} + d^{2+\alpha}[f]_{C^\alpha(B_d(x))}\right]$$

<u>Theorem.</u> For any solution $u \in C^{2,\alpha}(\Omega)$ to $a_{ij}(x) D_{ij}u = f$, we have a priori estimate

$$\|u\|_{C^{2,\alpha}_w(\Omega)} \le C\left(n, \lambda, \alpha, [a]_{C^\alpha(\Omega)}\right)\left[\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{2+\alpha}_w(\Omega)}\right].$$

Proof. 1st.

$$\begin{aligned} \|u\|_{C^{2,\alpha}_w(\Omega)} & \approx C(\varepsilon) \sup_{\substack{x\in\Omega \\ d\le\min\{dist(x),\varepsilon\}}} \widetilde{\|u\|}_{B_d(x)} \\ & \approx C(2)\ C(\varepsilon) \sup_{\substack{x\in\Omega \\ d\le\min\{dist(x),\varepsilon\}}} \widetilde{\|u\|}_{B_{d/2}(x)} \end{aligned}$$

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2nd. By interior Schauder

$$\widetilde{\|u\|}_{B_{d/2}(x)} \leq C(n,\alpha)\left\{\|u\|_{L^\infty(B_d(x))} + d^2\|f\|_{L^\infty(B_d(x))} + d^{2+\alpha}\ [f]_{C^\alpha(B_d(x))} + \text{perturbation}\right\}$$

and

$$\begin{aligned}
&\text{perturbation} \\
&= d^2\|[a_{ij}(0) - a_{ij}(x)]\, D_{ij}u\|_{L^\infty(B_d(x))} + d^{2+\alpha}\ \left[(a_{ij}(0) - a_{ij}(x))\, D_{ij}u\right]_{C^\alpha(B_d(x))} \\
&\leq \begin{array}{l} d^2 [a]_{C^\alpha(B_d(x))}\, d^\alpha \|D^2u\|_{L^\infty(B_d(x))} \\ +d^{2+\alpha}\ \left([a]_{C^\alpha(B_d(x))}\|D^2u\|_{L^\infty(B_d(x))} + [a]_{C^\alpha(B_d(x))}\, d^\alpha\, [D^2u]_{C^\alpha(B_d(x))}\right) \end{array} \\
&\leq 2\,[a]_{C^\alpha(B_d(x))}\, d^\alpha \left\{d^2\left\|D^2u\right\|_{L^\infty(B_d(x))} + d^{2+\alpha}\left[D^2u\right]_{C^\alpha(B_d(x))}\right\}.
\end{aligned}$$

Take sup

$$\begin{aligned}
&\sup_{\substack{x\in\Omega \\ d\leq \min\{dist(x),\varepsilon\}}} \widetilde{\|u\|}_{B_{d/2}(x)} \\
&\leq \sup_{x\in\Omega} C(n,\alpha)\left\{\|u\|_{L^\infty(B_d(x))} + d^2\|f\|_{L^\infty(B_d(x))} + d^{2+\alpha}\ [f]_{C^\alpha(B_d(x))} + \text{perturbation}\right\} \\
&\leq \sup_{x\in\Omega} C(n,\alpha)\left\{\|u\|_{L^\infty(B_d(x))} + d^2\|f\|_{L^\infty(B_d(x))} + d^{2+\alpha}\ [f]_{C^\alpha(B_d(x))}\right\} \\
&+ 2\,[a]_{C^\alpha(B_d(x))}\,\varepsilon^\alpha\ C(n,\alpha) \sup_{\substack{x\in\Omega \\ d\leq \min\{dist(x),\varepsilon\}}} \left\{d^2\left\|D^2u\right\|_{L^\infty(B_d(x))} + d^{2+\alpha}\left[D^2u\right]_{C^\alpha(B_d(x))}\right\} \\
&\leq \sup_{x\in\Omega} C(n,\alpha)\left\{\|u\|_{L^\infty(B_d(x))} + d^2\|f\|_{L^\infty(B_d(x))} + d^{2+\alpha}\ [f]_{C^\alpha(B_d(x))}\right\} \\
&+ 2\,[a]_{C^\alpha(B_d(x))}\,\varepsilon^\alpha\ C(n,\alpha)\ \ C(2) \sup_{\substack{x\in\Omega \\ d\leq \min\{dist(x),\varepsilon\}}} \left\{\begin{array}{c} d^2\|D^2u\|_{L^\infty\left(B_{d/2}(x)\right)} \\ +d^{2+\alpha}\,[D^2u]_{C^\alpha\left(B_{d/2}(x)\right)} \end{array}\right\} \\
&\leq \sup_{x\in\Omega} C(n,\alpha)\left\{\|u\|_{L^\infty(B_d(x))} + d^2\|f\|_{L^\infty(B_d(x))} + d^{2+\alpha}\ [f]_{C^\alpha(B_d(x))}\right\} \\
&+ \frac{1}{2} \sup_{\substack{x\in\Omega \\ d\leq \min\{dist(x),\varepsilon\}}} \widetilde{\|u\|}_{B_{d/2}(x)},
\end{aligned}$$

provided we take ε depending on $[a]_{C^\alpha(B_d(x))}$, $C(n,\alpha)$, and $C(2)$ so small so that

$$2\,[a]_{C^\alpha(B_d(x))}\,\varepsilon^\alpha\ C(2)\ \ C(n,\alpha) \leq \frac{1}{2},$$

Thus

$$\sup_{\substack{x\in\Omega \\ d\leq \min\{dist(x),\varepsilon\}}} \widetilde{\|u\|}_{B_{d/2}(x)} \leq C(n,\alpha)\left[\|u\|_{L^\infty(\Omega)} + \|f\|_{C_w^{2+\alpha}(\Omega)}\right].$$

And finally

$$\|u\|_{C_w^{2,\alpha}(\Omega)} \leq C\left(n,\lambda,\alpha,[a]_{C^\alpha(\Omega)}\right)\left[\|u\|_{L^\infty(\Omega)} + \|f\|_{C_w^{2+\alpha}(\Omega)}\right].$$

RMK. Lower order terms $b_i(x) D_i u + c(x) u$ (no sign restriction on c for this estimate) with Hölder coefficient b and c can be handled via interpolation (tedious work):

$$\|Du\|_{C^\alpha} \le C(\varepsilon) \|u\|_{L^\infty} + \varepsilon \left\|D^2 u\right\|_{L^\infty}.$$

The workable form of this theorem is to add another $\beta < \alpha$ weight (for free). Set

$$\|u\|_{C^{2,\alpha}_{w_\beta}(\Omega)} =$$

$$\sup_{x\in\Omega} d^{-\beta} \left[\|u\|_{L^\infty(B_d(x))} + d\|Du\|_{L^\infty(B_d(x))} + d^2 \left\|D^2 u\right\|_{L^\infty(B_d(x))} + d^{2+\alpha} \left[D^2 u\right]_{C^\alpha(B_d(x))}\right]$$

$$\|f\|_{C^{2+\alpha}_w(\Omega)} = \sup_{x\in\Omega} d^{-\beta} \left[d^2 \|f\|_{L^\infty(B_d(x))} + d^{2+\alpha} [f]_{C^\alpha(B_d(x))}\right]$$

$$\|u\|_{L^\infty_{W_\beta}(\Omega)} = \sup_{x\in\Omega} d^{-\beta} \|u\|_{L^\infty(B_d(x))}$$

<u>Theorem.</u> For any solution $u \in C^{2,\alpha}(\Omega)$ to $a_{ij}(x) D_{ij} u = f$ and $0 < \beta < \alpha$, we have a priori estimate

$$\|u\|_{C^{2,\alpha}_{w_\beta}(\Omega)} \le C\left(n, \lambda, \alpha, [a]_{C^\alpha(\Omega)}, \beta\right) \left[\|u\|_{L^\infty_{w_\beta}(\Omega)} + \|f\|_{C^{2+\alpha}_{w_\beta}(\Omega)}\right].$$

Proof. Just multiply both sides of Schauder estimate in each ball $B_d(x)$ by $d^{-\beta}$, then take SUP, repeat the above argument. As $\beta < \alpha$, that 1/2-coefficient can still be created for the left hand side to swallow the perturbed terms.

Continuity method to solve (Bernstein 1910s)

$$\begin{cases} Lu = \sum a_{ij}(x) D_{ij} u + b_i(x) D_i u - c(x) u = f(x) & \text{in } \Omega \text{ (nice, } C^{1,1} \text{ domain)} \\ u = \varphi \text{ on } \partial\Omega \end{cases}$$

where

- $\lambda I \le (a_{ij}) \le \lambda^{-1} I$
- $\|a\|_{C^\alpha}, \|b\|_{C^\alpha}, \|c(x)\|_{C^\alpha} \le \Lambda$
- $c(x) \ge 0$
- $f \in C^\alpha(\Omega)$

eg. $u_{xx} + \left(1 + \frac{y}{100}\right) u_{yy} - u = f(x)$ in $B_1(0)$ and $u = \varphi$ on $\partial B_1(0)$ has a (unique) solution in $C^{2,\alpha}(\Omega) \cap C^0\left(\bar\Omega\right)$.

Step1. In Lecture 3, we have uniquely solved in $C^{2,\alpha}(\Omega) \cap C^0\left(\bar\Omega\right)$

$$\begin{cases} \triangle u = f \text{ or } 0 & \text{in } \Omega \text{ nice} \\ u|_\Omega = g & \text{continuous} \end{cases}.$$

Step2. Link to $\triangle$ and assume 0 boundary data $g = 0$

Set $L^t = tL + (1-t)\triangle = [ta_{ij}(x) + (1-t)\delta_{ij}] D_{ij} + tb_i(x) D_i + tc(x)$

$$\begin{cases} L^t u = f & \text{in } \Omega \\ u|_\Omega = g \end{cases}$$

Ellipticity and Hölder continuity for L^t is the same as L. We can solve L^0 equation, for t near 0

$$\begin{cases} L^0 u = (L^0 - L^t) u + f & \text{in } \Omega \\ u|_\Omega = g \end{cases}.$$

We solve this equation via fixed point procedure in Banach space. We check the needed contraction property for the map $v \to u$ via the solution to $L^0 u = (L^0 - L^t) v + f$ (from Step1.)

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$$\begin{aligned} \|u\|_{C^{2,\alpha}_{w_\beta}(\Omega)} &\le C(\lambda, \Lambda, n)\left[\|u\|_{L^\infty_{W_\beta}(\Omega)} + \left\|\left(L^0 - L^t\right) v + f\right\|_{C^{2+\alpha}_{w_\beta}(\Omega)}\right] \\ &\le C(\lambda, \Lambda, n)\left[\|v\|_{C^{2+\alpha}_{w_\beta}(\Omega)} + \|f\|_{C^{2+\alpha}_{w_\beta}(\Omega)}\right] \end{aligned}$$

provided one has

$$\sup_{x\in\Omega} d^{-\beta}|u(x)| \le C(\beta) \sup_{x\in\Omega} d^{2-\beta}\left|\tilde{f}(x)\right|$$

for solutions to $\begin{cases} Lu = \tilde{f} & \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$. This can be achieved by constructing barriers on the exterior $C^{1,1}$ domain. It is "easier" to do so for Ω being a ball. SKIP.

RMK. One has to go through this extra β-weight step, as we saw even in 1-d case, the above estimate is invalid for $\beta = 0$.

•• For

$$\begin{cases} L^0(u_1 - u_2) = (L^0 - L^t)(v_1 - v_2) & \text{in } \Omega \\ u_1 - u_2 = 0 \text{ on } \partial\Omega \end{cases}$$

from the workable weighted Schauder estimate and skipped weighted L^∞ estimate, we have

$$\begin{aligned} \|u_1 - u_2\|_{C^{2,\alpha}_{w_\beta}(\Omega)} &\le C\ \left[\|u_1 - u_2\|_{L^\infty_{W_\beta}(\Omega)} + \left\|\left(L^0 - L^t\right)(v_1 - v_2)\right\|_{C^{2+\alpha}_{w_\beta}(\Omega)}\right] \\ &\le C\ [(t-0)]\,\|(v_1 - v_2)\|_{C^{2+\alpha}_{w_\beta}(\Omega)} + [(t-0)]\,\|(v_1 - v_2)\|_{C^{2+\alpha}_{w_\beta}(\Omega)} \\ &\le C(\lambda, \Lambda, n, \alpha, \beta)\,[(t-0)]\,\|(v_1 - v_2)\|_{C^{2+\alpha}_{w_\beta}(\Omega)}. \end{aligned}$$

For t close to 0, the map $v \to u$ is a contraction one from complete space $C^{2+\alpha}_{w_\beta}(\Omega)$ to itself. Thus the equation $\begin{cases} L^t u = f & \text{in } \Omega \\ u|_\Omega = 0 & \text{on } \partial\Omega \end{cases}$ has a solution for t near 0. Similarly, if $L^{t_0}u = f$ has a solution, $L^t u = f$ also has a solution for t near t_0.

Thus the solvable set t is open.

Next the solvable set t is closed.

Let $t_k \to t_*$ with $\begin{cases} L^{t_k} u_k = f & \text{in } \Omega \\ u_k|_\Omega = 0 & \text{on } \partial\Omega \end{cases}$, by the workable weighted Schauder and Arscoli-Arzela, there is a subsequence, still denote by u_k such that $u_k \to u_*$ in $C^{2,\alpha-\varepsilon}_{w_\beta}(\Omega)$ and $u_* \in C^{2,\alpha}_{w_\beta}(\Omega)$ satisfies $\begin{cases} L^{t_*} u_* = f & \text{in } \Omega \\ u_*|_\Omega = 0 & \text{on } \partial\Omega \end{cases}$.

Therefore the solvable set t is $[0,1]$.

Step3. General boundary data g

Extend $g \in C^0(\partial\Omega)$ to $g \in C^0(\bar{\Omega})$, approximate the extension by $g_\varepsilon \in C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$ with $\|g_\varepsilon\|_{C^{2,\alpha}_W(\Omega)} \overset{\text{subtle}}{\leq} C$, ε-free. Now repeat Step2 to solve

$$\begin{cases} L(u_\varepsilon - g_\varepsilon) = f - Lg_\varepsilon \in C^\alpha(\Omega) \\ u_\varepsilon - g_\varepsilon = 0 \;\; \text{on } \partial\Omega \end{cases}$$

for $u_\varepsilon \in C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$ with $C^{2,\alpha}_W(\Omega) \cap C(\bar{\Omega})$ norm independent of ε. Thus we get a subconvergence sequence (still denoted by u_ε)

$$u_\varepsilon \to u \text{ in } C^{2,\alpha-}_W(\Omega) \cap C(\bar{\Omega}), \text{ and } u \in C^{2,\alpha}_W(\Omega) \cap C(\bar{\Omega})$$

satisfying

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}.$$

Alternative w/o the subtle inequality. Approximate the extension g by $g_\varepsilon \in C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$ such that

$$g_\varepsilon \to g \text{ in } C^0(\partial\Omega)$$

and

$$\|g_\varepsilon\|_{C^{2,\alpha}(\Omega')} \leq C_{\Omega'} \text{ independent of } \varepsilon \text{ for any } \Omega' \subset\subset \Omega.$$

By Schauder interior estimates, the solution $u_\varepsilon \in C^{2,\alpha}(\Omega) \cap C^0(\bar{\Omega})$ satisfies

$$\|u_\varepsilon\|_{C^{2,\alpha}(\Omega')} \leq C_{\Omega'} \text{ independent of } \varepsilon.$$

Thus we get a subconvergence sequence (still denoted by u_ε) such that

$$u_\varepsilon \to u \text{ in } C^{2,\alpha-}(\Omega') \cap C(\bar{\Omega}), \text{ and } u \in C^{2,\alpha}(\Omega) \cap C(\bar{\Omega})$$

with

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}.$$