

Lecture 6 Boundary and global Schauder

- o Poisson formula on the upper half space & estimates
- o Schauder estimate on the upper half ball via weighted norm
- o Straighten boundary
- o Global estimates

Poisson formula

Consider

$$\begin{cases} \Delta u = f & x_n > 0 \quad C^\alpha(\overline{R_+^n}) \\ u = g & x_n = 0 \quad C^{2,\alpha} \end{cases}$$

Make an even extension of f to R^n , $\bar{f} \in C^\alpha(R^n)$, then

$$\begin{cases} \Delta(u - \Gamma * \bar{f}) = 0 \\ u - \Gamma * \bar{f} = g & x_n = 0 \quad C^{2,\alpha} \end{cases}.$$

So we just need to study the case $f = 0$.

RMK. Case $g = 0$, general f is plain and simple.

Case general g , $f = 0$ is fun and informative, but a bit involved.

Representation of u

2-d. Consider holomorphic function

$$\begin{aligned} u + iv &= \frac{1}{2\pi i} \int_{R^1} \frac{1}{\xi - z} g(\xi) d\xi \\ &= \frac{1}{2\pi i} \int_{R^1} \frac{\xi - x + iy}{(\xi - x)^2 + y^2} g(\xi) d\xi \end{aligned}$$

then

$$\begin{aligned} u &= \frac{1}{2\pi} \int_{R^1} \frac{y}{(\xi - x)^2 + y^2} g(\xi) d\xi \\ v &= \frac{1}{2\pi} \int_{R^1} \frac{x - \xi}{(\xi - x)^2 + y^2} g(\xi) d\xi = \frac{1}{2\pi} \int_{R^1} \frac{\xi}{\xi^2 + y^2} g(x - \xi) d\xi. \end{aligned}$$

The direct calculation of u_y and u_{yx} is a mess, we go the conjugate way

$$\begin{aligned} u_y &= -v_x = -\frac{1}{2\pi} \int_{R^1} \frac{\xi}{\xi^2 + y^2} g_x(x - \xi) d\xi \\ u_{yx} &= -\frac{1}{2\pi} \int_{R^1} \frac{\xi}{\xi^2 + y^2} g_{xx}(x - \xi) d\xi. \end{aligned}$$

eg. Harmonic function $\operatorname{Re} 1/z$ satisfies

$$\begin{cases} \Delta \frac{x}{x^2 + y^2} = 0 \\ \frac{x}{x^2 + y^2} = \frac{x}{|x|^2} \quad y = 0 \end{cases}$$

⁰October 27, 2016

and

$$\frac{x}{x^2 + y^2} = \frac{1}{2\pi} \int_{R^1} \frac{y}{(\xi - x)^2 + y^2} \frac{\xi}{|\xi|^2} d\xi.$$

n-d. For $\begin{cases} \Delta u = 0 & y > 0 \\ u = g & y = 0 \end{cases}$ with $(x, y) \in R^n$

$$u = \frac{1}{c_n} \int_{R^{n-1}} \frac{y}{|(\xi - x, y)|^n} g(\xi) d\xi.$$

We derive this formula in the following.

$$G = \frac{-1}{(n-2)|\partial B_1|} \left[\frac{1}{|(x, y) - (\xi, \eta)|^{n-2}} - \frac{1}{|(x, y) - (\xi, -\eta)|^{n-2}} \right] \stackrel{\eta=0}{=} 0$$

$$G_{-\eta} = \frac{1}{|\partial B_1|} \left[\frac{y - \eta}{|(x, y) - (\xi, \eta)|^n} - \frac{y + \eta}{|(x, y) - (\xi, -\eta)|^n} \right] \stackrel{\eta=0}{=} \frac{2}{|\partial B_1|} \frac{y}{|(x - \xi, y)|^n}.$$

RMK. Curiously Kelvin(y) = $\frac{1}{|X|^{n-2}} \frac{y}{|X|^2} = \frac{y}{|X|^n}$, where $X = (x, y)$.

Heuristically by Green's identify, we get

$$u = \frac{1}{c_n} \int_{R^{n-1}} \frac{y}{|(\xi - x, y)|^n} g(\xi) d\xi$$

$$= \frac{1}{c_n} \int_{R^{n-1}} \frac{y}{|(\xi, y)|^n} g(x - \xi) d\xi.$$

Rigorously one can check (Exercise) harmonic function u take boundary value $g \in C^0(R^{n-1}) \cap L^\infty(R^n)$.

To compute u_y and u_{yx} , we push y to g by changing variable $\xi = y\bar{\xi}$

$$u = \frac{1}{c_n} \int_{R^{n-1}} \frac{1}{|(\bar{\xi}, 1)|^n} g(x - y\bar{\xi}) d\bar{\xi}.$$

Then

$$u_y = \frac{1}{c_n} \int_{R^{n-1}} \frac{1}{|(\bar{\xi}, 1)|^n} (-\bar{\xi}) \cdot D_x g(x - y\bar{\xi}) d\bar{\xi}$$

$$\stackrel{\xi=y\bar{\xi}}{=} \frac{1}{c_n} \int_{R^{n-1}} \frac{1}{|(\xi, y)|^n} (-\xi) \cdot D_x g(x - \xi) d\xi$$

$$= \frac{1}{c_n} \int_{R^{n-1}} \frac{-\xi}{|(\xi, y)|^n} \cdot D_x g(x - \xi) d\xi$$

and

$$u_{yx} = \frac{1}{c_n} \int_{R^{n-1}} \frac{-\xi}{|(\xi, y)|^n} \cdot D_{xx} g(x - \xi) d\xi = \frac{-\xi}{|(\xi, y)|^n} * D_{xx} g.$$

Observe (Exercise) as in 2-d, harmonic functions of (ξ, y)

$$\begin{aligned} \frac{\xi}{|(\xi, y)|^n} &= \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \frac{2}{|\partial B_1|} \int_{(B_R \setminus B_\varepsilon) \cap R^{n-1}} \frac{y}{|(\xi - x, y)|^n} \frac{x}{|x|^n} dx \\ &= P_y * \frac{x}{|x|^n} \quad \text{with } P_y(x) = \frac{2}{|\partial B_1|} \frac{y}{|(x, y)|^n}. \end{aligned}$$

Thus we have obtained Dirichlet to Neumann map

$$u_y = -P_y * \frac{x}{|x|^n} * D_x g$$

and

$$u_{yx} = -P_y * \frac{x}{|x|^n} * D_{xx} g.$$

Note that as a function on R^{n-1} , $\frac{x}{|x|^n}$ is

- homogeneous order $n - 1$ and $D_{\frac{x}{|x|^n}}$ is homogeneous order n
- $\int_{B_R} \frac{x}{|x|^n} dA = 0$.

By Newtonian estimates in Lecture 4, we have

$$\left\| \frac{x}{|x|^n} * D_{xx} g \right\|_{C^\alpha(B_{1/2})} \leq C(n) \|D^2 g\|_{C^\alpha(B_1)}.$$

Next the Poisson kernel $P_y(x)$ is a nice one, approximating $\delta_{R^{n-1}}$ as $y \rightarrow 0^+$. Then for $\bar{g} \in C^0(R^{n-1})$

$$\|P_y * \bar{g}\|_{L^\infty(R^{n-1})} \leq C(n) \|\bar{g}\|_{L^\infty(R^{n-1})}$$

and

$$\|P_y * \bar{g}\|_{C^\alpha(R^{n-1})} \leq C(n) \|\bar{g}\|_{C^\alpha(R^{n-1})}.$$

Finally $u_{xx} = P_y * D_{xx} g$ and $u_{yy} = -(u_{x_1 x_1} + \dots + u_{x_{n-1} x_{n-1}})$. We arrive at Theorem. For $u = P * g$ with $g \in C_0^{2,\alpha}(B_1 \cap R^{n-1})$, we have

$$\|u\|_{C^{2,\alpha}(\overline{R_+^n})} \leq C(n) \|g\|_{C_0^{2,\alpha}(B_1 \cap R^{n-1})}.$$

Now $\begin{cases} \widehat{\Delta(u - P * g)} = 0 & \text{in } R_+^n \\ u - P * g = 0 & \text{on } x_n = 0 \end{cases}$

Claim. Odd extension $\bar{h} = \begin{cases} h & x_n \geq 0 \\ -h(x', -x_n) & \text{is harmonic in } R^n. \end{cases}$

In fact (by Poisson formula or Perron for general domain/equation) let H be the solution of $\begin{cases} \Delta H = 0 & \text{in } B_1 \\ H = \bar{h} & \text{on } \partial B_1 \end{cases}$. We have

$$\begin{cases} \Delta [H(x', x_n) + H(x', -x_n)] = 0 & \text{in } B_1 \\ H(x', x_n) + H(x', -x_n) = 0 & \text{on } \partial B_1 \end{cases}.$$

By maximum principle, $H(x', x_n) + H(x', -x_n) = 0$ in B_1 , in particular $H(x', 0) = 0$. Further

$$\begin{cases} \Delta H = 0 = \Delta \bar{h} & \text{in } B_1^+ \\ H = \bar{h} & \text{on } \partial B_1^+ \end{cases}.$$

Again by maximum principle, $H = \bar{h}$ in B_1^+ . Symmetrically equality happens on the lower half. Thus

$$\bar{h} = H \text{ is harmonic in } B_1.$$

Summary. Let u satisfy $\begin{cases} \Delta u = f & \in C^\alpha(B_1) \\ u = g & \in C^{2,\alpha}(B_1 \cap R^{n-1}) \end{cases}$. Then

$$u = h + P * g + \Gamma * f$$

and

$$\|u\|_{C^{2,\alpha}(B_{1/2}^+)} \leq C(n) \left[\|u\|_{L^\infty(B_1^+)} + \|g\|_{C^{2,\alpha}(B_1 \cap R^{n-1})} + \|f\|_{C^\alpha(B_1^+)} \right].$$

Counterexamples.

Eg 0'. $u_x = g' \in L^\infty$, but $u_y \notin L^\infty$

$$u = \operatorname{Im} z \log z = x\theta + y \ln r$$

$$u(x, 0) = x\theta = \begin{cases} 0 & x \geq 0 \\ \pi x & x < 0 \end{cases}$$

$$u_x(x, 0) = \theta \in L^\infty$$

$$u_y = \partial_y(x\theta + y \ln r) \text{ little tedious, instead}$$

$$u_y = \partial_y \operatorname{Im} z \log z = \operatorname{Im} \partial_y(z \log z)$$

$$= \operatorname{Im} \frac{1}{i} (\bar{\partial} - \partial)(z \log z) = \operatorname{Im} i\partial(z \log z) \quad \text{here } \begin{cases} \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \\ \partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \end{cases}$$

$$= \operatorname{Re} \partial(z \log z) = \operatorname{Re}(\log z + 1)$$

$$= \ln r + 1 \notin L^\infty.$$

Eg 0'-. $u_x = g' \in C^0$, but $u_y \notin L^\infty$

$$u = \operatorname{Im} z (\log z)^{1/3} = r (\ln^2 r + \theta^2)^{1/6} \sin \left(\frac{1}{3} \arctan \frac{\theta}{\ln r} + \theta \right)$$

$$u(x, 0) = \begin{cases} 0 & x \geq 0 \\ -|x| (\ln^2 |x| + \pi^2)^{1/6} \sin \left(\frac{1}{3} \arctan \frac{\pi}{\ln r} \right) & x < 0 \end{cases} \in C^1, \text{ of course near } 0!$$

$$u_y = \operatorname{Re} \partial \left(z \log^{\frac{1}{3}} z \right) = \operatorname{Re} \left[\log^{\frac{1}{3}} z + \frac{1}{3} (\log z)^{-2/3} \right]$$

$$= \underbrace{(\ln^2 r + \theta^2)^{1/6} \cos \left(\frac{1}{3} \arctan \frac{\theta}{\ln r} \right)}_{\notin L^\infty} + \underbrace{\frac{1}{3} (\ln^2 r + \theta^2)^{-1/3} \cos \left(-\frac{2}{3} \arctan \frac{\theta}{\ln r} \right)}_{\in C^0}.$$

Eg 0'' $u_{xx}(x, 0) = g'' \in L^\infty$, but $u_{xy} \notin L^\infty$.

$$u = \operatorname{Im} z^2 (\log z) = 2xy \ln r + (x^2 - y^2) \theta$$

$$u(x, 0) = \begin{cases} 0 & x \geq 0 \\ \pi x^2 & x < 0 \end{cases} \in C^{1,1}$$

$$u_{xx}(x, 0) \in L^\infty$$

$$u_{xy} = \partial_{xy} \operatorname{Im} z^2 (\log z) = \operatorname{Im} \frac{1}{i} (\bar{\partial}^2 - \partial^2) z^2 (\log z)$$

$$= \operatorname{Re} \partial^2 [z^2 (\log z)] = \operatorname{Re} (\log z + 1) = \ln r + 1 \notin L^\infty .$$

$$u_{yy} = -2\theta \in L^\infty$$

Schauder estimate on the upper half ball

Consider

$$\begin{cases} \sum a_{ij}(x) D_{ij} u + b_i(x) D_i u + c(x) u = f(x) & B_1^+ \\ u = g \text{ on } x_n = 0 \end{cases}$$

where

- $\lambda I \leq (a_{ij}) \leq \lambda^{-1} I$
- $\|a\|_{C^\alpha}, \|b\|_{C^\alpha}, \|c(x)\|_{C^\alpha} \leq \Lambda$, no sign restriction for $c(x)$ yet.
- $f \in C^\alpha(B_1^+)$.

We want

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}^+})} \leq C(n, \lambda, \Lambda) \left[\|u\|_{L^\infty(B_1^+)} + \|f\|_{C^\alpha(\overline{B_1^+})} + \|g\|_{C^{2,\alpha}(B_1 \cap \{x_n=0\})} \right].$$

As in the interior case, we freeze coefficients

$$\begin{cases} \sum a_{ij}(x_0) D_{ij} u = [a_{ij}(x_0) - a_{ij}(x)] D_{ij} u - b_i D_i u - c u + f & \text{in } B_\delta(x_0) \cap R_+^n \\ u = g \text{ on } x_n = 0 \text{ if } B_\delta(x_0) \cap \{x_n = 0\} \neq \emptyset \end{cases},$$

where $\delta(x) = 1 - |x|$. Apply the above boundary estimate (summary, scaled version) and previous interior estimate for Laplace, we have (recall scaled norm $\widetilde{\|u\|}$ was defined in Lec5)

$$\widetilde{\|u\|}_{B_{\delta/2}^+} \leq C(\lambda, n) \left[\begin{array}{l} \|u\|_{L^\infty(B_\delta^+)} + \widetilde{\|g\|}_{B_\delta^+ \cap \{x_n=0\}} + \delta^2 \|f\|_{L^\infty(B_\delta^+)} + \delta^{2+\alpha} [f]_{C^\alpha(B_\delta^+)} \\ \quad + \text{perturbation} \end{array} \right]$$

where perturbation

$$\begin{aligned} &= \delta^2 \|a_{ij}(x_0) - a_{ij}(x)\|_{L^\infty(B_\delta^+)} \|D^2 u\|_{L^\infty(B_\delta^+)} \\ &+ \delta^{2+\alpha} \left\{ \|a_{ij}(x_0) - a_{ij}(x)\|_{L^\infty(B_\delta^+)} [D^2 u]_{C^\alpha(B_\delta^+)} + [a_{ij}(x_0) - a_{ij}(x)]_{C^\alpha(B_\delta^+)} \|D^2 u\|_{L^\infty(B_\delta^+)} \right\} \\ &+ \delta^2 \left\{ \|bDu\|_{L^\infty(B_\delta^+)} + \|cu\|_{L^\infty(B_\delta^+)} \right\} + \delta^{2+\alpha} \left\{ [bDu]_{C^\alpha(B_\delta^+)} + [cu]_{C^\alpha(B_\delta^+)} \right\} \\ &\leq \delta^\alpha \Lambda \delta^2 \|D^2 u\|_{L^\infty(B_\delta^+)} + \delta^\alpha \Lambda \delta^{2+\alpha} [D^2 u]_{C^\alpha(B_\delta^+)} + \delta^\alpha \Lambda \delta^2 \|D^2 u\|_{L^\infty(B_\delta^+)} \\ &+ \delta \Lambda \left\{ \delta \|Du\|_{L^\infty(B_\delta^+)} + \delta \|u\|_{L^\infty(B_\delta^+)} \right\} \\ &+ \delta \Lambda \delta^2 \|D^2 u\|_{L^\infty(B_\delta^+)} + \delta^{1+\alpha} \Lambda \delta \|Du\|_{L^\infty(B_\delta^+)} + \delta \Lambda \delta^2 \|Du\|_{L^\infty(B_\delta^+)} + \delta^{2+\alpha} \Lambda \|u\|_{L^\infty(B_\delta^+)}. \end{aligned}$$

Take SUP of both sides for weighted norm

$$\|u\|_{C_w^{2,\alpha}(B_1^+)} = \sup_{x \in B_1^+} \widetilde{\|u\|}_{B_{\delta(x)}^+}.$$

As before, we may and do restrict δ less than a small number so that $C(\lambda, n) \Lambda \delta^\alpha < 1/2$, then we obtain

$$\|u\|_{C_w^{2,\alpha}(B_1^+)} \leq C(n, \lambda, \Lambda) \left[\|u\|_{L^\infty(B_1^+)} + \|g\|_{C_W^{2,\alpha}(B_1 \cap \{x_n=0\})} + \|f\|_{C_W^{\alpha+\alpha}(\overline{B_1^+})} \right].$$

Note for $x \in B_{1/2}^+$ with $\delta(x) \geq 1/2$, then what we wanted follows

$$\|u\|_{C^{2,\alpha}(\overline{B_{1/2}^+})} \leq C(n, \lambda, \Lambda) \left[\|u\|_{L^\infty(B_1^+)} + \|f\|_{C^\alpha(\overline{B_1^+})} + \|g\|_{C^{2,\alpha}(B_1 \cap \{x_n=0\})} \right].$$

Straighten the boundary

figure curved boundary to upper half space

$$\{x_n \geq \varphi(x')\} \underset{y=\Phi(x)=(x', x_n - \varphi(x'))}{\overset{x=\Psi(y)}{\rightleftarrows}} \{y_n \geq 0\}$$

$$\begin{aligned} v(y) &= u(x(y)) = u(\Psi(y)) \text{ and } u(x) = v(y(x)) = v(\Phi(x)). \\ D_x u &= D_y v D_x \Phi, D_{xx} u = D_y v D_{xx} \Phi + (D_x \Phi)' D_{yy} v D_x \Phi \end{aligned}$$

$$(A_{ij}(y)) = D_x \Phi(a_{ij})(D_x \Phi)', B_i(y) = b_i D_x \Phi + (a_{ij}) D_{xx} \Phi, C(y) = c(\Psi(y))$$

$$F(y) = f(\Psi(y)), G(y) = g(\Psi(y))$$

The equation in y-coordinate system is

$$\begin{cases} \sum A_{ij}(y) D_{ij} v + B_i(y) D_i v + C(y) v = F(y) & y_n > 0 \\ v(y) = G(y) \text{ on } y_n = 0 \end{cases}$$

with

$$\lambda I \leq (A_{ij}(y)) \leq \lambda^{-1} \|D_x \Phi\|_{L^\infty} I$$

$$\|A\|_{C^\alpha}, \|B\|_{C^\alpha}, \|C\|_{C^\alpha} \leq \Lambda C(\|\Phi\|_{C^{2,\alpha}})$$

and

$$\begin{aligned} \|u\|_{C^{2,\alpha}} &\leq C(\|\Phi\|_{C^{2,\alpha}}) \|v\|_{C^{2,\alpha}} \\ \|G\|_{C^{2,\alpha}} &\leq C(\|\Phi\|_{C^{2,\alpha}}) \|g\|_{C^{2,\alpha}} \\ \|F\|_{C^\alpha} &\leq C(\|\Phi\|_{C^\alpha}) \|f\|_{C^\alpha}. \end{aligned}$$

By the estimate on B_+^1 , we obtain

$$\|u\|_{C^{2,\alpha}(B_{1/2} \cap \Omega)} \leq C(\Phi, \lambda, \Lambda, n) \left[\|u\|_{L^\infty(B_1 \cap \Omega)} + \|g\|_{C^{2,\alpha}(B_1 \cap \partial\Omega)} + \|f\|_{C^\alpha(B_1 \cap \Omega)} \right].$$

Through finite covering, we arrive at the

Global estimate:

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\|\partial\Omega\|_{C^{2,\alpha}}, \lambda, \Lambda, n) \left[\|u\|_{L^\infty(\Omega)} + \|g\|_{C^{2,\alpha}(\partial\Omega)} + \|f\|_{C^\alpha(\bar{\Omega})} \right].$$

Further, for $C(x) \leq 0$, we have

$$\|u\|_{L^\infty(\Omega)} \leq \|g\|_{L^\infty(\partial\Omega)} + C(\lambda, \text{diam}\Omega) \|f\|_{L^\infty(\Omega)}.$$

L^∞ estimate

$$\begin{cases} Lu = \sum a_{ij}(x) D_{ij} u + b_i(x) D_i u - c(x) u = f(x) & \text{in } \Omega \\ u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \\ c(x) \geq 0 \end{cases}$$

Then

$$\|u\|_{L^\infty(\Omega)} \leq \sup_{\partial\Omega} |u| + C(\lambda, \text{diam}\Omega, \|b\|_{L^\infty(\Omega)}) \|f\|_{L^\infty(\Omega)}.$$

Proof. W.L.G, we assume $\Omega \subset \{0 \leq x_1 \leq d\}$.

Upper bound

Take concave enough barrier

$$W = (e^{Kd} - e^{Kx_1}) \|f\|_{L^\infty(\Omega)} + \sup_{\partial\Omega} |u|$$

then $W \geq u$ on $\partial\Omega$ and

$$\begin{aligned} LW &= - (a_{11}K^2 + b_1K) e^{Kx_1} \|f\|_{L^\infty(\Omega)} - c(x) W \\ &\stackrel{c \geq 0}{\leq} - (a_{11}K^2 + b_1K) e^{Kx_1} \|f\|_{L^\infty(\Omega)} \\ &\leq - \|f\|_{L^\infty(\Omega)} \end{aligned}$$

for large $K = K(\lambda, \|b\|_{L^\infty(\Omega)})$. By the comparison principle, $W \geq u$ in Ω .

Lower bound barrier $-W$.

Thus the L^∞ estimate.

Existence

Theorem.

$$\begin{cases} Lu = \sum a_{ij}(x) D_{ij}u + b_i(x) D_iu - c(x) u = f(x) \in C^\alpha(\Omega) \quad \text{and } c(x) \geq 0 \\ u = g \in C^{2,\alpha}(\partial\Omega) \end{cases}$$

has a unique solution u with

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(\|\Omega\|_{C^{2,\alpha}}, \lambda, \Lambda, n) \left[\|g\|_{C^{2,\alpha}(\partial\Omega)} + \|f\|_{C^\alpha(\bar{\Omega})} \right].$$

Continuity method under global estimate.

$$L^t = (1-t) \Delta + tL$$

$$\text{Step1. } \begin{cases} L^0 u = f \\ u = g \end{cases} \text{ uniquely solvable.}$$

By Perron $C^0(\bar{\Omega})$ solution u exists, by Laplace regularity $u \in C^{2,\alpha}(\bar{\Omega})$.

$$\text{Step2. Openness. Suppose } t_0 \text{ in solvable set. } \begin{cases} L^{t_0}u = (L^{t_0} - L^t) \overset{v}{\not\sim} u + f \\ u = g \end{cases}$$

By global Schauder, the map: $v \rightarrow u$, $C^{2,\alpha}(\bar{\Omega}) \rightarrow C^{2,\alpha}(\bar{\Omega})$ for t near t_0 is a contraction, and then has a fixed point u as the solution.

Step3. Closedness. Suppose t sequence in the solvable set. By global Schauder
 $\begin{cases} L^t u_t = f \quad \text{as } t \rightarrow t_*, u_t \xrightarrow{C^{2,\alpha-\varepsilon}(\bar{\Omega})} u_{t_*} \in C^{2,\alpha}(\bar{\Omega}) \\ u_t = g \end{cases}$ Still

$$\begin{cases} L^{t_*} u_{t_*} = f \quad \text{in } \Omega \\ u_{t_*} = g \quad \text{on } \partial\Omega \end{cases}$$

Thus 1 in solvable set, that is $\begin{cases} L^1 u = f \quad \text{in } \Omega \\ u = g \quad \text{on } \partial\Omega \end{cases}$ has a solution. The uniqueness and estimates are already derived.