## Lecture 7 Calderon-Zygmund for $\triangle \|$ Schauder for $\triangle$

- egs
- $L^{2}$ weak 2-2
- Weak 1-1
- interpolation to $L^{p}$
- $L^{p}$ for $a_{i j} D_{i j} \mathrm{w} / a_{i j} \in C^{0}$
- Bdry/Global $L^{p}$
- $L^{2}$ for $\sum_{i, j}^{2} a_{i j} D_{i j} \mathrm{w} / a_{i j} \in L^{\infty}$ \&elliptic

Question: $\triangle u=f \in L^{p} \stackrel{?}{\Rightarrow} D^{2} u \in L^{p}$
eg1. $p=2 \widehat{D_{i j} u}=-\xi_{i} \xi_{j} \widehat{u}=\frac{\xi_{i} \xi_{j}}{|\xi|^{2}} \widehat{f}$ implies $\left\|D_{i j} u\right\|_{L^{2}} \leq\|\widehat{f}\|_{L^{2}}=\|f\|_{L^{2}}$.
eg2. $p=\infty$ Let $f=\chi_{\left\{x_{1} x_{2}>0\right\} \cap B_{1}}, u=\Gamma * f$, recall $D_{12} u \notin L^{\infty}$.
eg3. $p=1$ Let $f_{\varepsilon}=\varphi_{\varepsilon}(x)=\varphi(x / \varepsilon) / \varepsilon^{n}$ with $\varphi(x)=\varphi(|x|) \in C_{0}^{\infty}\left(R^{n}\right)$ and $\int \varphi=1$. Then the convolution with fundamental solution

$$
u_{\varepsilon}=\Gamma * f_{\varepsilon}=\left\{\begin{array}{ll}
\Gamma & |x| \geq \varepsilon \\
\sim & |x|<\varepsilon
\end{array} .\right.
$$

Then

$$
D_{i j} u_{\varepsilon}=\frac{x_{i} x_{j}}{|x|^{n+2}} \text { for }|x| \geq \varepsilon
$$

It follows

$$
C\left\|f_{\varepsilon}\right\|_{L^{1}} \not \equiv\left\|D_{i j} u_{\varepsilon}\right\|_{L^{1}\left(B_{1}\right)}>\int_{B_{1} / B_{\varepsilon}} \frac{\left|x_{i} x_{j}\right|}{|x|^{n+2}}>\int_{B_{1} / B_{\varepsilon}} \frac{1}{|x|^{n}}>\ln \varepsilon \rightarrow \infty .
$$

Recall for $\varphi \in C_{0}^{\infty}\left(R^{n}\right), u=\Gamma * \varphi$

$$
D_{i j} u=\left\{\begin{array}{lc}
c_{n} \int_{R^{n}} \frac{y_{i} y_{j}}{|y|^{n+2}} \varphi(x-y) d y & i \neq j \\
c_{n} \int_{R^{n}} \frac{y_{i}^{2}-n^{-1}|y|^{2}}{|y|^{n+2}} \varphi(x-y) d y+\frac{1}{n} \varphi(x) \quad i \neq j
\end{array} .\right.
$$

Next the above convolution $u=\Gamma * f$ and $D_{i j} u$ also make sense for $f \in L^{p}$. Indeed by Young

$$
\|\Gamma * f\|_{p} \leq\|\Gamma\|_{1}\|f\|_{p} .
$$

We define $D_{i j} u$ by approximating $f$ with $f_{\varepsilon} \in C_{0}^{\infty}\left(R^{n}\right)$ in $L^{p}$, provided the singular integrals have the right $L^{p}$ estimates. For example, $p=2$ is fine as in eg1 $\left\|D_{i j} u\right\|_{L^{2}} \leq$ $\|\widehat{f}\|_{L^{2}}=\|f\|_{L^{2}}$.

Theorem. Let $u=\Gamma * f$ and $f \in L^{p}(\Omega)$ with $\Omega$ bounded and also $1<p<\infty$. Then we have
a) $\left\|D^{2} u\right\|_{L^{2}\left(R^{n}\right)} \leq\|f\|_{L^{2}(\Omega)}$
b) $\left|\left\{\left|D^{2} u\right|>t\right\} \cap \Omega\right| \leq \frac{C}{t}\|f\|_{L^{1}(\Omega)}$ (weak 1-1).
c) $\left\|D^{2} u\right\|_{L^{p}(\Omega)} \leq C_{p}\|f\|_{L^{p}(\Omega)}$

[^0]Proof. a)
First way. Fourier transform.
Second way. We approximate $f \in L^{2}(\Omega)$ by $f \in C_{0}^{\infty}\left(R^{n}\right)$, then as before $\triangle u=$ $\triangle \Gamma * f=f$. By Green's identity

$$
\begin{aligned}
\int_{B_{R}} \Delta u \Delta u & =\int_{\partial B_{R}} \Delta u u_{\gamma}-\int_{B_{R}} \nabla \Delta u \cdot \nabla u \\
& =\overbrace{\int_{\partial B_{R}} f u_{\gamma}-\int_{B_{R}} \Delta u_{k} u_{k}} \\
& =-\int_{\partial B_{R}} \partial_{\gamma} u_{k} u_{k}+\int_{B_{R}} \nabla u_{k} \cdot \nabla u_{k} \\
& =\sim \frac{1}{R^{n}} \frac{1}{R^{n-1}} R^{n-1}+\int_{B_{R}}\left|D^{2} u\right|^{2} \\
& \rightarrow \int_{R^{n}}\left|D^{2} u\right|^{2} \quad \text { as } R \rightarrow \infty,
\end{aligned}
$$

where we used the fact for $f$ with bounded support

$$
\begin{aligned}
u_{k}(x) & =\int_{\Omega} \frac{x_{k}-y_{k}}{|x-y|^{n}} f(y) d y \sim \frac{1}{R^{n-1}} \\
u_{k j}(x) & =\int_{\Omega} \frac{\left(x_{k}-y_{k}\right)\left(x_{j}-y_{j}\right)}{|x-y|^{n+2}} f(y) d y \sim \frac{1}{R^{n}} \\
u_{i i} & =\text { similar. }
\end{aligned}
$$

b)

Idea. W.L.O.G. Assume $f \geq 0$.
Warm up:

$$
\begin{aligned}
\int_{\Omega} f d x & =\int_{0}^{\infty} \underbrace{|\{f>t\}|}_{\sim 1 / t} d t \\
|\{f>t\}| & \leq \int_{\Omega} \frac{f}{t}=\frac{1}{t}\|f\|_{L^{1}(\Omega)} \\
|\{f>t\}| & \leq \int_{\Omega} \frac{f^{p}}{t^{p}}=\frac{1}{t^{p}}\|f\|_{L^{p}(\Omega)}^{p}
\end{aligned}
$$

We split $f$ in two parts: $f$ itself where $f \leq t$ and its average where $f>t$.
figure f itself where $f \leq t$ and its average where $f>t$
In $\{f \leq t\}$, the distribution of $K * f=D_{i j} \Gamma * f$ can be controlled by $L^{2}$ estimate of $L^{2}$ lower part in a);

In $\{f>t\}$, the distribution of $K * f$ can be taken care of by the cancellation effect of the average and the integrability for the gradient of singular kernel $D K$ near $\infty$.

Now Calderon-Zygmund decomposition.
Step0. Fix $t$, choose large enough cube $Q_{0}$ (possibly larger than $\Omega$ ) s.t.

$$
\frac{1}{\left|Q_{0}\right|} \int_{Q_{0}} f<t
$$

Step1. Split $Q_{0}$ in half along all axis directions and check $f_{Q} f$ on each subcube $Q$ :
figure cubes collected and cubes splitting

$$
\begin{aligned}
& \text { If } f_{Q} f \geq t \text {, collect } Q \\
& \text { if } f_{Q} f<t \text {, keep splitting. }
\end{aligned}
$$

For the collection of cubes $Q_{j}$ and their predecessor $\tilde{Q}_{j}$ we know

$$
t \leq \frac{1}{\left|Q_{j}\right|} \int_{Q_{j}} f \leq \frac{2^{n}}{2^{n}\left|Q_{j}\right|} \int_{\tilde{Q}_{j}} f=2^{n} f_{\tilde{Q}_{j}} f<2^{n} t
$$

Set (good) set $G=\Omega / \cup_{j} Q_{j}$. For a.e. $x \in G$, there exist a sequence of cubes $Q$ shrinking to $x$ such that $f_{Q} f<t$, thus by Lebesgue $f(x) \leq t$ a.e. in $G$. Then (good) function

$$
g(x)=\left\{\begin{array}{cc}
f(x) & x \in G \\
f_{Q_{j}} f & x \in Q_{j}
\end{array} \leq 2^{n} t \quad \text { a.e. in } \Omega .\right.
$$

Its square integral

$$
\int_{\Omega} g^{2} \leq 2^{n} t \int_{\Omega} g=2^{n} t \int f=2^{n} t\|f\|_{L^{1}} .
$$

Step2. Let (bad) function $b=f-g=\left\{\begin{array}{l}0 x \in G \\ f-f_{Q_{j}} f \quad x \in Q_{j}\end{array}\right.$. Then

$$
K * f=K *(g+b)=K * g+K * b
$$

and

$$
\{K * f>t\} \subset\left\{K * g>\frac{t}{2}\right\} \cup\left\{K * b>\frac{t}{2}\right\} .
$$

From a) and Step1

$$
\left|\left\{K * g>\frac{t}{2}\right\}\right| \leq \frac{1}{\left(\frac{t}{2}\right)^{2}}\|K * g\|_{L^{2}(\Omega)}^{2} \leq \frac{1}{\left(\frac{t}{2}\right)^{2}}\|g\|_{L^{2}(\Omega)}^{2} \leq \frac{2^{n+1}}{t}\|f\|_{L^{1}(\Omega)}
$$

Next we cover the latter set (go with $2 Q_{j}$ for easy calculation of following improper integrals)

$$
\left\{K * b>\frac{t}{2}\right\} \subset \bigcup_{j} 2 Q_{j} \cup\left(\left\{K * b>\frac{t}{2}\right\} \backslash \bigcup_{j} 2 Q_{j}\right)
$$

For $\bigcup_{j} 2 Q_{j}$, summing the collection criterion

$$
\sum_{j}:\left|Q_{j}\right|<\frac{1}{t} \int_{Q_{j}} f
$$

we get

$$
\left|\bigcup_{j} 2 Q_{j}\right| \leq 2^{n}\left|\bigcup_{j} Q_{j}\right|<\frac{2^{n}}{t}\|f\|_{L^{1}(\Omega)}
$$

For $x \in \Omega \backslash 2 Q_{j}$, the convolution

$$
\begin{aligned}
K * b(x) & =\int_{\Omega} K(x-y) b(y) d y=\int_{Q_{j}} K(x-y) b(y) d y \\
& =\int_{Q_{j}} K(x-y) b(y) d y \\
& =\int_{Q_{j}}\left[K(x-y)-K\left(x-y_{j}\right)\right] b(y) d y \text { as } \int_{Q_{j}\left(y_{j}\right)} b=0 \\
& \leq \int_{Q_{j}} \frac{C_{n} d i a m Q_{j}}{\left|x-y_{j}\right|^{n+1}} b(y) d y \\
& \leq \frac{C_{n} d i a m Q_{j}}{\left|x-y_{j}\right|^{n+1}} 2\|f\|_{L^{1}\left(Q_{j}\right)} \quad \text { since }\left|\int_{Q_{j}} b(y) d y\right| \leq 2 \int_{Q_{j}}|f| d y
\end{aligned}
$$

Integrate
figure: 2 concentric cubes nesting a concentric sphere

$$
\begin{aligned}
\int_{\Omega \backslash \cup_{j} 2 Q_{j}}|K * b(x)| d x & \leq \sum_{j} \int_{\Omega \backslash 2 Q_{j}} \frac{C_{n} \operatorname{diam} Q_{j}}{\left|x-y_{j}\right|^{n+1}} 2\|f\|_{L^{1}\left(Q_{j}\right)} d x \\
& \leq \sum_{j} 2\|f\|_{L^{1}\left(Q_{j}\right)} \int_{\left|x-y_{j}\right| \geq \operatorname{diam} Q_{j} / 2} \frac{C_{n} \operatorname{diam} Q_{j}}{\left|x-y_{j}\right|^{n+1}} d x \\
& =\left.\sum_{j} 2\|f\|_{L^{1}\left(Q_{j}\right)} \frac{C_{n} \operatorname{diam} Q_{j}}{r}\right|_{\infty} ^{\operatorname{diam} Q_{j} / 2} \\
& =\sum_{j} C_{n}\|f\|_{L^{1}\left(Q_{j}\right)} \leq C_{n}\|f\|_{L^{1}(\Omega)}
\end{aligned}
$$

It follows that

$$
\left|\left\{K * b>\frac{t}{2}\right\} \backslash \bigcup_{j} 2 Q_{j}\right| \leq \frac{C_{n}}{t / 2}\|f\|_{L^{1}(\Omega)}
$$

and then

$$
\left|\left\{K * b>\frac{t}{2}\right\}\right| \leq \frac{C_{n}}{t}\|f\|_{L^{1}(\Omega)}
$$

Therefore from this and the distribution estimate for $K * g$ in the beginning, we obtain

$$
\begin{aligned}
|\{K * f>t\}| & \leq\left|\left\{K * g>\frac{t}{2}\right\}\right|+\left|\left\{K * b>\frac{t}{2}\right\}\right| \\
& \leq \frac{C_{n}}{t}\|f\|_{L^{1}(\Omega)}
\end{aligned}
$$

c) Case $1<p<2$.

Again, w.l.o.g. assume $L^{p} f \geq 0$ a.e. First

$$
\begin{aligned}
\|K * f\|_{L^{p}(\Omega)}^{p} & =\int_{\Omega}|K * f|^{p} d x=\int_{0}^{\infty}\left|\left\{|K * f|^{p}>s\right\}\right| d s \\
& \stackrel{s=t^{p}}{=} \int_{0}^{\infty}|\{|K * f|>t\}| p t^{p-1} d t .
\end{aligned}
$$

Split $f$ into two parts along $t$ line

$$
\text { figure above } \mathrm{t} f_{1} \text { below } t f_{2}
$$

$$
f_{1}=\left\{\begin{array}{ll}
f & f \geq t \\
0 & f<t
\end{array} \quad \text { and } f_{2}=\left\{\begin{array}{ll}
0 & f \geq t \\
f & f<t
\end{array} .\right.\right.
$$

By weak $1-1 \mathrm{~b}$ ) and $L^{2}$ estimate a), we have

$$
\begin{aligned}
|\{|K * f|>t\}| & \leq\left|\left\{\left|K * f_{1}\right|>\frac{t}{2}\right\}\right|+\left|\left\{\left|K * f_{2}\right|>\frac{t}{2}\right\}\right| \\
& \leq \frac{2 C_{1}}{t}\left\|f_{1}\right\|_{L^{1}}+\frac{2 C_{2}}{t^{2}}\left\|f_{2}\right\|_{L^{2}}^{2} \\
& =\frac{2 C_{1}}{t} \int_{\{f \geq t\}} f d x+\frac{2 C_{2}}{t^{2}} \int_{\{f<t\}} f^{2} d x .
\end{aligned}
$$

Integrating, we have

$$
\begin{aligned}
\int_{0}^{\infty}\left(\frac{2 C_{1}}{t} \int_{\{f \geq t\}} f d x\right) p t^{p-1} d t & =2 C_{1} p \int_{\Omega} f \int_{0}^{f} t^{p-2} d t d x=2 C_{1} p \int_{\Omega} f \frac{f^{p-1}}{p-1} d x \\
& =\frac{2 C_{1} p}{p-1} \int_{\Omega} f^{p} d x
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{0}^{\infty}\left(\frac{2 C_{2}}{t^{2}} \int_{\{f<t\}} f^{2} d x\right) p t^{p-1} d t & =2 C_{2} p \int_{\Omega} f^{2} \int_{f}^{\infty} t^{p-3} d t d x=2 C_{2} p \int_{\Omega} f^{2} \frac{f^{p-2}}{2-p} d x \\
& =\frac{2 C_{2} p}{2-p} \int_{\Omega} f^{p} d x
\end{aligned}
$$

Thus

$$
\|K * f\|_{L^{p}(\Omega)}^{p} \leq p\left(\frac{2 C_{1}}{p-1}+\frac{4 C_{2}}{2-p}\right)\|f\|_{L^{p}(\Omega)}^{p}
$$

RMK. The coefficient $C(p) \sim \frac{1}{p-1}$ blows up for $p$ near 1 as expected. But $C(p) \sim$ $\frac{1}{2-p}$ for $p$ near 2 is bad.

Question/Exercise. Make $C(p)$ bounded near 2. Hint: as above interpolate between $L^{1}$ and $L^{4}$.

Case $2<p<\infty$
Let $g \in L^{p^{\prime}}(\Omega)$ with $\frac{1}{p^{\prime}}+\frac{1}{p}=1$. Let us do a duality argument,

$$
\begin{aligned}
\int_{\Omega} K * f(x) g(x) d x & =\int_{\Omega} \int_{\Omega} K(x-y) f(y) d y g(x) d x \\
& =\int_{\Omega} \int_{\Omega} K(x-y) g(x) d x f(y) d y \\
& =\int_{\Omega} \int_{\Omega} \bar{K}(y-x) g(x) d x f(y) d y \quad \text { with } \bar{K}(x)=K(-x) \\
& \leq\|\bar{K} * g\|_{L^{p^{\prime}}}\|f\|_{L^{p}} \\
& \leq C\left(p^{\prime}\right)\|g\|_{L^{p^{p}}}\|f\|_{L^{p}} .
\end{aligned}
$$

Thus

$$
\|K * f\|_{L^{p}(\Omega)} \leq C\left(p^{\prime}\right)\|f\|_{L^{p}}
$$

RMK. To get bounded $C(p)$ for $p$ near 2 , one can also do a strong interpolation between say $L^{1.5}$ and $L^{4}$ by Riesz convexity theorem: Given $\|K * f\|_{p} \leq C_{p}\|f\|_{p}$ and $\|K * f\|_{q} \leq C_{q}\|f\|_{q}$. Then

$$
\|K * f\|_{r} \leq C_{p}^{\alpha} C_{q}^{1-\alpha}\|f\|_{r} \quad \text { with } \frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q}
$$

Or the coefficient $C_{r}$ is $\log$ convex in terms of $\frac{1}{r}$.
Model equation: $\triangle u=f \in L^{p}$ in $B_{1}$ :

$$
\triangle(u-\Gamma * f)=0
$$

then the the interior estimate for harmonic function and C-Z, we have

$$
\begin{aligned}
\|u-\Gamma * f\|_{W^{2, p}\left(B_{1 / 2}\right)} & \leq C_{n}\|u-\Gamma * f\|_{L^{\infty}\left(B_{3 / 4}\right)} \leq C_{n}\|u-\Gamma * f\|_{L^{p}\left(B_{1}\right)} \\
& \leq C_{n}\|u\|_{L^{p}\left(B_{1}\right)}+\|\Gamma * f\|_{L^{p}\left(B_{1}\right)} \\
& \leq C(n, p)\left(\|u\|_{L^{p}\left(B_{1}\right)}+\|f\|_{L^{p}\left(B_{1}\right)}\right) .
\end{aligned}
$$

Note that $\frac{1}{|x|^{n-2}}$ and $\frac{1}{|x|^{n-1}}$ are in $L^{1}(\Omega)$, by Young's inequality

$$
\begin{aligned}
\|\Gamma * f\|_{L^{p}(\Omega)} & \leq\|\Gamma\|_{L 1(\Omega)}\|f\|_{L^{p}(\Omega)} \leq C\|f\|_{L^{p}(\Omega)} \\
\|D \Gamma * f\|_{L^{p}(\Omega)} & \leq\|D \Gamma\|_{L 1(\Omega)}\|f\|_{L^{p}(\Omega)} \leq C\|f\|_{L^{p}(\Omega)}
\end{aligned}
$$

Therefore we obtain

$$
\|u\|_{W^{2, p}\left(B_{1 / 2}\right)} \leq C(n, p)\left(\|u\|_{L^{p}\left(B_{1}\right)}+\|f\|_{L^{p}\left(B_{1}\right)}\right) .
$$

Perturbation.

$$
\sum a_{i j}(0) D_{i j} u=\sum\left[a_{i j}(0)-a_{i j}(x)\right] D_{i j} u-b_{i}(x) D_{i} u-c(x) u+f
$$

we have (heuristically)

$$
\left\|D^{2} u\right\|_{L^{p}\left(B_{1 / 2}\right)} \leq\|u\|_{L^{p}\left(B_{1}\right)}+\|f\|_{L^{p}\left(B_{1}\right)}+\varepsilon\| \| D^{2} u\left\|_{L^{p}\left(B_{1}\right)}\right\| .
$$

As for interior Schauder estimate, via weigthted norm, we move $\varepsilon$-term to the left hand side to obtain $W^{2, p}$ estimates for

$$
\sum a_{i j}(x) D_{i j} u+b_{i}(x) D_{i} u+c(x) u=f
$$

with coefficients $a_{i j}(x) \in C^{0} \lambda I \leq\left(a_{i j}\right) \leq \lambda^{-1} I,|b|,|c| \leq \Lambda$

$$
\left\|D^{2} u\right\|_{L^{p}\left(B_{1 / 2}\right)} \leq C\left(\|a\|_{C^{0}}, \lambda, \Lambda, n, p\right)\left[\|u\|_{L^{p}\left(B_{1}\right)}+\|f\|_{L^{p}\left(B_{1}\right)}\right]
$$

For boundary $L^{p}$ estimates on $C^{1,1}$ domains, again it is parallel to boundary Schauder estimates (Poisson convolution is fine, Dirichlet to Neumann singular convolution is handled as in the interior case but with one dimension less, still use weighted norm device ...). Skip.

Instead study 2-d situation: For $u \in W_{0}^{2,2}(\Omega)$ or $u \in C^{2}$ satisfying

$$
\left\{\begin{array}{l}
\sum_{i, j=1}^{2} a_{i j}(x) D_{i j} u=f \text { in } \Omega \text { convex } \\
u=0 \text { on } \partial \Omega
\end{array}\right.
$$

with $L^{\infty}$ coefficients $\lambda I \leq\left(a_{i j}\right) \leq \lambda^{-1} I$. Then

$$
\left\|D^{2} u\right\|_{L^{2}(\Omega)} \leq C(\lambda)\|f\|_{L^{2}(\Omega)}
$$

Proof. At each point, rotate coordinates so that

$$
\left(a_{i j}(x)\right)=\left[\begin{array}{ll}
\lambda_{1} & \\
& \lambda_{2}
\end{array}\right] .
$$

Then the equation becomes

$$
\left(\lambda_{1} u_{11}+\lambda_{2} u_{22}\right)^{2}=f^{2}
$$

or

$$
\lambda_{1}^{2} u_{11}^{2}+\lambda_{2}^{2} u_{22}^{2}+2 \lambda_{1} \lambda_{2} u_{12}^{2}+2 \lambda_{1} \lambda_{2}\left(u_{11} u_{22}-u_{11}^{2}\right)=f^{2} .
$$

Use ellipticity $\lambda \leq \lambda_{i} \leq \lambda^{-1}$, we have

$$
\lambda^{4}\left|D^{2} u(x)\right|^{2}+2 \operatorname{det} D^{2} u(x) \leq \frac{1}{\lambda^{2}} f^{2}(x) .
$$

This coordinate invariant inequality is in fact true everywhere under one same coordinate. Integrate, we have

$$
\int_{\Omega} \lambda^{4}\left|D^{2} u\right|^{2}+\int_{\Omega} 2 \operatorname{det} D^{2} u \leq \int_{\Omega} \frac{1}{\lambda^{2}} f^{2} .
$$

Next we show that the integral of the determinant with $u$ vanishing on the bounded convex boundary is positive. Note

$$
2 \operatorname{det} D^{2} u=(\triangle u)^{2}-\left|D^{2} u\right|^{2} .
$$

Exercise (exactly as in the proof of $W^{2,2}$ estimate in C-Z a), by Green's identity)

$$
\int_{\Omega}(\triangle u)^{2}-\left|D^{2} u\right|^{2}=\int_{\partial \Omega} \triangle u \partial_{\gamma} u-u_{k} \partial_{\gamma} u_{k} .
$$

We finish the proof by proving that the integrand in the boundary integral is positive pointwise. Fix a point on the boundary, say this point at the origin, $x$-axis is tangent to the boundary, and y -axis is along $\gamma$ direction.
figure x -axis is tangent to the boundary, and y -axis is along $\gamma$ direction
As $u=0$ on $\partial \Omega, u_{x}(0)=0$. Also $u_{\gamma}(0)=u_{y}(0)$. We have

$$
\text { integrand } \stackrel{0}{=} \triangle u u_{y}-u_{y} u_{y y}=u_{y} u_{x x} .
$$

W.l.o.g. assume $u_{y}(0)=|D u(0)|>0$, otherwise, consider $-u$. Observe near $0, u$ becomes negative entering the domain $\Omega$ from 0 boundary. By double finite difference to approximate $u_{x x}(0)$, one sees that

$$
u_{x x}(0) \geq 0
$$

Thus the integrand is positive.
Therefore

$$
\int_{\Omega} 2 \operatorname{det} D^{2} u \geq 0
$$

and

$$
\int_{\Omega}\left|D^{2} u\right|^{2} \leq \frac{1}{\lambda^{4}} \int_{\Omega} f^{2}
$$

RMK. In general n-d

$$
\int_{\Omega} \operatorname{det} D^{2} u=\int_{\Omega} d\left(u_{1} \wedge d u_{2} \wedge \cdots \wedge d u_{n}\right)=\int_{\partial \Omega} u_{1} d u_{2} \wedge \cdots \wedge d u_{n}
$$

Is there a sign restriction on the boundary integrand for $C^{2} u$ vanishing on the convex boundary? (Positive in even-dim and negative in odd-dim)


[^0]:    ${ }^{0}$ November 22, 2013

