Lecture 8 Energy method–preliminary: capacity, Poincaré, Sobolev

- egs
- Capacity
- Poincaré/Sobolev

Question: Besides Perron method, can one use energy/variational way to solve Laplace equation $\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = \varphi \text{ nice on } \partial\Omega \end{cases}$? That is, let $E(v) = \int_\Omega |Dv|^2$. If $E(u) = \inf_{v \in S} E(v)$, then $\triangle u = 0$.

Objections:

eg1. Let $u = \operatorname{Im} \log z = \theta$, then $\triangle \theta = 0$ in $\Omega = B_1(1,0)$. Recall

$$Du = \left(\partial_r u, \frac{1}{r} \partial_\theta u \right),$$

then

$$\int_\Omega |D\theta|^2 = \int \int_\Omega \frac{1}{r^2} r dr d\theta = \infty!$$

figure: $B_1(1,0)$ and gradient components ∂_r & $\frac{1}{r}\partial_\theta$

The minor defect is that θ has a jump on $\partial\Omega$.

eg2. Set $u = \sum_{k=1}^{\infty} \frac{1}{k^2} \operatorname{Im} z^{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^2} r^{k^4} \sin k^4 \theta$ (fast enough oscillation on the circle). Then $u \in C^0\left(\bar{B}_1(0)\right)$ and $\triangle u = 0$ in B_1. Let us calculate its Dirichlet energy.

$$Du = \sum_k \frac{1}{k^2} \left(k^4 r^{k^4 - 1} \sin k^4 \theta, k^4 r^{k^4 - 1} \cos k^4 \theta \right)$$

and

$$\begin{aligned} \int_{B_1} |Du|^2 &= \int \int \sum_k k^4 \left(r^{k^4 - 1} \right)^2 \left[\left(\sin k^4 \theta \right)^2 + \left(\cos k^4 \theta \right)^2 \right] r dr d\theta \\ &= 2\pi \sum_k k^4 \frac{1}{2k^4} = \infty! \end{aligned}$$

Moral: Unless $u|_{\partial\Omega}$ is really nice, the energy/variational method cannot capture pointwise info of general continuous $u|_{\partial\Omega}$.

RMK. One remedy would be approximate continuous $u|_{\partial\Omega}$ by nice (say smooth) boundary data in C^0 norm; run variational method to get approximated solutions; by maximum principle, those solutions approach to a unique function on $C^0\left(\bar{B}_1\right)$, by interior estimates for Harmonic functions, the unique limit is harmonic inside B_1.

[0]December 2, 2013

As one application of energy method, let us study capacity. The capacity for the boundary of a domain Ω is defined as

$$Cap\,(\partial\Omega) = \inf_{\substack{v\in C_0^1(R^n)\\ v\geq 1 \text{ on } \Omega}} \int_{R^n} |Dv|^2 .$$

If there exists a minimizer u, then the unique minimizer satisfies

$$\begin{cases} \triangle u = 0 \\ u = 1 \ \text{ on } \partial\Omega \\ u = 0 \ \text{ at } \infty \end{cases} .$$

RMK. The uniqueness follows from convexity of the energy functional as we will see shortly. The fact $u = 1$ on $\partial\Omega$ follows from local energy comparison?, or uniqueness of the minimizer and existence for the above boundary value problem.

eg. 3-d. Let $\Omega = B_R$. For fundamental solution $u = R/r = 1$ on ∂B_R and 0 at ∞, certainly $\triangle u = 0$. Now

$$\int_{R^3\backslash B_R} |Du|^2 = \int_{R^3\backslash B_R} \frac{R^2}{r^4} r^2 dr d\omega = 4\pi R^2 \frac{-1}{r}\bigg|_R^\infty = 4\pi R.$$

Another way

$$\begin{aligned}\int_{R^3\backslash B_R} |Du|^2 &= -\int_{R^3\backslash B_R} u \triangle u + \int_{\partial(R^3\backslash B_R)} u u_\gamma \\ &= \int_{\partial B_R} 1\ \frac{R}{r^2} dA = 4\pi R.\end{aligned}$$

Thus $Cap\,(\partial B_R) = 4\pi R$.

eg. 2-d. $Cap\,(\partial B_1) = 0$. Let $v_k = \begin{cases} 1 & r \leq k \\ 2 - \frac{\log r}{\log k} & k \leq r \leq k^2 \\ 0 & r > k^2 \end{cases}$

figure graph of $v_k(r)$ and even $u_k(r) = v_k(kr)$

$$\int_{R^2\backslash B_1} |Dv_k|^2 = \int_{B_{k^2}\backslash B_k} \frac{1}{\log^2 k} \frac{1}{r^2} r dr d\theta = \frac{2\pi}{\log^2 k}\left(\log k^2 - \log k\right) = \frac{2\pi}{\log k} \to 0 \text{ as } k \to \infty.$$

So $Cap\,(\partial B_1) = 0$. In fact the same estimate shows for any bounded Ω, $Cap\,(\partial\Omega) = 0$.

Existence of minimizer for capacity.

1st way. Perron method.

2nd way. Variational.

Consider Hilbert space $H_0^1(R^n)$ with inner product $\langle u, v\rangle = \int_{R^n} Du \cdot Dv$. Set convex and closed set

$$S = \left\{ v \in H_0^1(R^n) :\ v \geq 1 \text{ on } \Omega \right\}.$$

Theorem. Given a convex (closed) set, say S. Then there exists a unique point $u \in S$ closest to the origin.

Proof. Notice that $\inf_{v \in S} \|v\| = \alpha \geq 0$. There exist a sequence $v_k \in S$ such that $\|v_k\| \to \alpha$. For any $\varepsilon > 0$, there exists large N so that once $k, l \geq N$, we have

$$\begin{aligned} \alpha &\leq \|v_k\| \leq \alpha + \varepsilon \\ \alpha &\leq \|v_l\| \leq \alpha + \varepsilon. \end{aligned}$$

From the parallelogram identity in Hilbert space

$$\|v_k - v_l\|^2 + \|v_k + v_l\|^2 = 2\|v_k\|^2 + 2\|v_l\|^2$$

it follows that

$$\begin{aligned} \|v_k - v_l\|^2 &\leq 2\|v_k\|^2 + 2\|v_l\|^2 - 4\left\|\frac{v_k + v_l}{2}\right\|^2 \\ &\leq 4(\alpha + \varepsilon)^2 - 4\alpha^2 \quad \text{as } \frac{v_k + v_l}{2} \in S \\ &= 8\alpha\varepsilon + 4\varepsilon^2. \end{aligned}$$

Therefore the Cauchy sequence has a limit inside the closed subset S of the complete space H.

The uniqueness also follows from the parallelogram identity.

RMK. For certain non-quadratic convex functional like area one $\int \sqrt{1 + |Dv|^2}$, more complicated argument is needed for the existence and uniqueness of the minimizer.

Poincaré inequality.

Compact support version. Given $u \in C_0^1(\Omega)$, one has

$$\|u\|_{L^2(\Omega)} \leq C(\mathrm{diam}\Omega)\|Du\|_{L^2(\Omega)}.$$

Proof. We integrate the gradient Du along each direction to the boundary and average over all directions:

$$\begin{aligned} u(y) &= \frac{1}{|\partial B_1|}\int_{\partial B_1}\int -u_r(y + r\omega)\,dr\,d\omega \\ &= \frac{1}{|\partial B_1|}\int_{\Omega} Du(x) \cdot \frac{y - x}{|y - x|}\frac{1}{|y - x|^{n-1}}dx \\ &= \frac{1}{|\partial B_1|}Du * \frac{x}{|x|^n}\chi_{B_{2\mathrm{diam}}}. \end{aligned}$$

So by Young's inequality

$$\begin{aligned} \|u\|_{L^p(\Omega)} &\leq \frac{1}{|\partial B_1|}\|Du\|_{L^p(\Omega)}\left\|\frac{x}{|x|^n}\right\|_{L^1(B_{2\mathrm{diam}})} \\ &\leq C(\mathrm{diam}\Omega)\|Du\|_{L^p(\Omega)}. \end{aligned}$$

Average version. Given C^1 function u on convex domain Ω, one has

$$\|u-\bar{u}\|_{L^p(\Omega)} \leq C\left(\mathrm{diam}\Omega\right)\|Du\|_{L^p(\Omega)}.$$

Proof. By convexity any two points in Ω can be joined by a segment inside Ω, then

$$u(y)-u(x)=-\int_0^{|x-y|} u_r(y+r\omega)\,dr \quad \text{with } \omega=\frac{x-y}{|x-y|}.$$

Integrate w.r.t. x,

$$\begin{aligned}
(u(y)-\bar{u})|\Omega| &= -\int_\Omega\int_0^{|x-y|} Du(y+r\omega)\cdot\omega\; dr\; dx \\
&= -\int_{|x-y|\leq d}\int_0^\infty Du(y+r\omega)\cdot\omega\; dr\; d(x-y) \quad \text{extend } u \text{ as 0 outside } \Omega \\
&= -\int_0^d\int_{\partial B_1}\int_0^\infty Du(y+r\omega)\cdot\omega\; dr \quad \rho^{n-1}d\omega d\rho \\
&= -\frac{1}{n}d^n\int_0^\infty\int_{\partial B_1} Du\underbrace{(y+r\omega)}_{z}\cdot\omega\;\frac{r^{n-1}}{r^{n-1}}d\omega\; dr \\
&= -\frac{1}{n}d^n\int_\Omega Du(x)\cdot\frac{z-y}{|z-y|}\frac{1}{|z-y|^{n-1}}dx.
\end{aligned}$$

Then

$$u(y)-\bar{u}=\frac{d^n}{n|\Omega|}Du*\frac{x}{|x|^n}\chi_\Omega.$$

Again by Young's inequality

$$\|u-\bar{u}\|_{L^p(\Omega)} \leq C\left(\mathrm{diam}\Omega\right)\|Du\|_{L^p(\Omega)}.$$

RMK. As the regular kernel $\frac{x}{|x|^n}$ is almost in $L^{n/n-1}$, by the general Young's inequality

$$\|u\|_{L^r} \leq \|Du\|_{L^p}\left\|\frac{x}{|x|^n}\right\|_{L^q} \quad \text{with } \frac{1}{r}=\frac{1}{p}+\frac{1}{q}-1.$$

When $\frac{1}{q}=\frac{n-1}{n}$, then $\frac{1}{r}=\frac{1}{p}-\frac{1}{n}$. Thus we already have $u\in L^s$ for $s<\frac{np}{n-p}$ for $Du\in L^p$.

The borderline case in the compact case in the following

Sobolev inequality. Given $u\in C_0^1(\Omega)$, one has $\|u\|_{L^{n/n-1}(\Omega)} \leq \dfrac{1}{\sqrt{n}}\|Du\|_{L^1(\Omega)}$.

Proof. Step1. As a preparation, we derive Hölder inequality:

$$\|f_1f_2\cdots f_k\|_1 \leq \|f_1\|_{p_1}\cdots\|f_k\|_{p_k} \text{ with } \frac{1}{p_1}+\cdots+\frac{1}{p_k}=1.$$

W.l.o.g. assume $\|f_k\|_{p_k} = 1$ for all k. By convexity of exponential function e^t, we have

$$\begin{aligned} f_1 f_2 \cdots f_k &= \exp\left(\frac{1}{p_1}\ln f_1^{p_1} + \cdots + \frac{1}{p_k}\ln f_k^{p_k}\right) \\ &\le \frac{1}{p_1}\exp\ln f_1^{p_1} + \cdots + \frac{1}{p_k}\exp\ln f_k^{p_k} = \frac{f_1^{p_1}}{p_1} + \cdots + \frac{f_k^{p_k}}{p_k}. \end{aligned}$$

Integrating, we have

$$\int f_1 f_2 \cdots f_k \le \frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1.$$

Step2. 1-d $u(y) = \int_{-\infty}^{y} u_1 dx_1$, then $|u(y)| \le \int_{-\infty}^{\infty} |u_1|\, dx_1 = \int_\Omega |Du|\, dx$.

2-d.

$$u^2(y) = \int_{-\infty}^{y_1} u_1(x_1, y_2)\, dx_1 \int_{-\infty}^{y_2} u_2(y_1, x_2)\, dx_2 \le \int_{-\infty}^{\infty} |u_1(x_1, y_2)|\, dx_1 \int_{-\infty}^{\infty} |u_2(y_1, x_2)|\, dx_2,$$

then

$$\int_{R^1} u^2(y)\, dy_1 \le \int_{R^1} |u_1(x_1, y_2)|\, dx_1 \int_{R^1}\int_{R^1} |u_2(y_1, x_2)|\, dx_2 dy_1$$

and

$$\int_{R^1}\int_{R^1} u^2(y)\, dy_1 dy_2 \le \int_{R^1}\int_{R^1} |u_1(x_1, y_2)|\, dx_1 dy_2 \ \int_{R^1}\int_{R^1} |u_2(y_1, x_2)|\, dx_2 dy_1.$$

It follows that

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\le \left[\|D_1 u\|_{L^1(\Omega)} \|D_2 u\|_{L^1(\Omega)}\right]^{1/2} \\ &\le \frac{\int_\Omega |D_1 u| + |D_2 u|}{2} \le \frac{1}{\sqrt{2}} \int_\Omega |Du|\,. \end{aligned}$$

n-d. First

$$|u(y)|^{n/n-1} \le \left(\int |u_1|\, dx_1\right)^{1/n-1} \cdots \left(\int |u_n|\, dx_n\right)^{1/n-1}$$

where $\int$ means $\int_{R^1}$. Integrating w.r.t. y_1

$$\begin{aligned} \int |u(y)|^{n/n-1} dy_1 &\le \left(\int |u_1|\, dx_1\right)^{\frac{1}{n-1}} \int \left(\int |u_2|\, dx_2\right)^{\frac{1}{n-1}} \cdots \left(\int |u_n|\, dx_n\right)^{\frac{1}{n-1}} \mathbf{dy}_1 \\ &\le \left(\int |u_1|\, dx_1\right)^{\frac{1}{n-1}} \left(\int |u_2|\, dx_2 \mathbf{dy}_1\right)^{\frac{1}{n-1}} \cdots \left(\int |u_n|\, dx_n \mathbf{dy}_1\right)^{\frac{1}{n-1}} \quad \text{by Hölder.} \end{aligned}$$

Continue integration w.r.t. y_2

$$\int\int |u(y)|^{n/n-1} dy_1 dy_2$$

$$\le \left(\int |u_2|\, dx_2 dy_1\right)^{\frac{1}{n-1}} \int \left(\int |u_1|\, dx_1\right)^{\frac{1}{n-1}} \left(\int |u_2|\, dx_2 dy_1\right)^{\frac{1}{n-1}} \cdots \left(\int |u_n|\, dx_n dy_1\right)^{\frac{1}{n-1}} \mathbf{dy}_2$$

$$\overset{\text{Hölder}}{\le} \left(\int |u_2|\, dx_2 dy_1\right)^{\frac{1}{n-1}} \left(\int |u_1|\, dx_1 \mathbf{dy}_2\right)^{\frac{1}{n-1}} \left(\int |u_2|\, dx_2 dy_1 \mathbf{dy}_2\right)^{\frac{1}{n-1}} \cdots \left(\int |u_n|\, dx_n dy_1 \mathbf{dy}_2\right)^{\frac{1}{n-1}}.$$

...

$$\int_\Omega |u(y)|^{n/n-1} dy_1 \cdots dy_n$$

$$\leq \left(\int_\Omega |u_1| \, dx_1 dy_2 \cdots dy_n\right)^{\frac{1}{n-1}} \left(\int_\Omega |u_2| \, dy_1 dx_2 dy_3 \cdots dy_n\right)^{\frac{1}{n-1}} \cdots \left(\int_\Omega |u_n| \, dy_1 \cdots dy_{n-1} dx_n\right)^{\frac{1}{n-1}}.$$

Hence

$$\begin{aligned}\left(\int_\Omega |u(y)|^{n/n-1} dy\right)^{n-1/n} &\leq \left[\int_\Omega |u_1| \, dy \int_\Omega |u_2| \, dy \cdots \int_\Omega |u_n| \, dy\right]^{1/n} \\ &\leq \frac{\int_\Omega |u_1| + \cdots + |u_n|}{n} \leq \frac{1}{\sqrt{n}} \int_\Omega |Du|.\end{aligned}$$

Corollary. For $u \in C_0^1(\Omega)$, one has

$$\|u\|_{L^{np/n-p}(\Omega)} \leq \frac{1}{\sqrt{n}} \frac{(n-1)p}{n-p} \|Du\|_{L^p(\Omega)}.$$

Proof. By the above

$$\begin{aligned}\left[\int_\Omega \left(|u|^\gamma\right)^{n/n-1}\right]^{n-1/n} &\leq \frac{1}{\sqrt{n}} \int_\Omega D|u|^\gamma \leq \frac{\gamma}{\sqrt{n}} \int_\Omega |Du| \; u^{\gamma-1} \\ &\leq \frac{\gamma}{\sqrt{n}} \left(\int_\Omega |Du|^p\right)^{1/p} \left(\int_\Omega \left(u^{\gamma-1}\right)^{p/p-1}\right)^{\frac{p-1}{p}}.\end{aligned}$$

Now choose $\gamma = \dfrac{(n-1)p}{n-p}$, then

$$(\gamma-1)\frac{p}{p-1} = \left[\frac{(n-1)p}{n-p} - 1\right]\frac{p}{p-1} = \frac{np}{n-p}$$

and

$$\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}.$$

Therefore we have

$$\left(\int_\Omega |u|^{\frac{np}{n-p}}\right)^{\frac{n-p}{np}} \leq \frac{1}{\sqrt{n}} \frac{(n-1)p}{n-p} \left(\int_\Omega |Du|^p\right)^{1/p}.$$

RMK. Sobolev inequality is scaling invariant. Poincaré is scaling variant. For example given

$$\|u\|_{L^2(B_1)} \leq C_1 \|Du\|_{L^2(B_1)} \text{ for } u \in C_0^1(B_1).$$

For $w \in C_0^1(B_R)$, let $u(x) = w(Rx)$, then $Du(x) = RDw(Rx)$. And

$$\int_{B_1} u^2(x)\, dx = \int_{B_1} w^2(Rx)\, dx = \frac{1}{R^n} \int_{B_R} w^2(x)\, dx$$

$$\int_{B_1} |Du(x)|^2 dx = \int_{B_1} R^2 |Dw(Rx)|^2 dx = \frac{R^2}{R^n} \int_{B_R} |Dw(x)|^2 dx.$$

It follows

$$\|w\|_{L^2(B_R)} \leq RC_1 \|Dw\|_{L^2(B_R)}.$$