Lecture 9 $W^{1,2}$ or H^1 space, trace

- ◦ $W^{1,2}$ space different characterizations
- ◦ Trace
- ◦ Chain rule

Sobolev space $W^{1,2}$, different characterizations.

Approximation: $u \in W^{1,2}(\Omega)$ if there exist a sequence u_k s.t. $u_k \to u$ in $L^2(\Omega)$ and $\{Du_k\}$ is a Cauchy sequence in $L^2(\Omega)$.

IBP: $u \in W^{1,2}(\Omega)$ if $u \in L^2(\Omega)$ and $\int uD_e\varphi \le C\|\varphi\|_{L^2(\Omega)}$ for all $\varphi \in C_0^1(\Omega)$ and $e \in S^{n-1}$.

Finite difference: $u \in W^{1,2}(\Omega)$ if $u \in L^2(\Omega)$ and $\left\|\dfrac{u(x+he)-u(x)}{h}\right\|_{L^2(\Omega)} \le C$ for h-free C and all $e \in S^{n-1}$.

Proposition. The above approximation, IBP, and finite difference characterizations of Sobolev space $W^{1,2}(\Omega)$ are equivalent.

Proof. Approximation⇒IBP

We have

$$\int u_k D_e\varphi = -\int D_e u_k\ \varphi \le \|D_e u_k\|_{L^2}\|\varphi\|_{L^2} \le C\|\varphi\|_{L^2}.$$

Take limit of the left side, we have

$$\int uD_e\varphi \le C\|\varphi\|_{L^2}.$$

IBP⇒Finite difference

For test function to produce finite difference, one integrates the difference of two delta functions along the given direction. We construct $\Phi \in W^{1,\infty}(R^n)$ so that $\Phi_e = \phi$ where

$$\phi = \frac{1}{|B_\varepsilon|}\left[\chi_{B_\varepsilon(he)} - \chi_{B_\varepsilon(0)}\right].$$

figureS: graphs of ϕ and Φ

graph of Φ cylinder like $B_\varepsilon^{n-1} \times (0,h)$

Note that

$$\int |\Phi| \sim h.$$

Now for $\varphi \in C_0^1(\Omega)$

$$\int u(y)\, D_e(\varphi * \Phi)\, dy \le C_u\|\varphi * \Phi\|_{L^2} \le C_u\|\varphi\|_{L^2}\|\Phi\|_{L^1} \le C_u\ h\ \ \|\varphi\|_{L^2},$$

[0]January 7, 2014

and the left hand side equals

$$\int u(y)\left(\int \varphi(x)\,\phi(y-x)\,dx\right)dy = \int \varphi(x)\left(\int u(y)\,\phi(y-x)\,dy\right)dx$$

$$= \int \varphi(x)\left(\int u(y+x)\,\phi(y)\,dy\right)dx = \int \varphi(x)\left(\int u(y+x)\,\frac{1}{|B_\varepsilon|}\left[\chi_{B_\varepsilon(he)} - \chi_{B_\varepsilon(0)}\right]dy\right)dx.$$

Let $\varepsilon \to 0$, we have

$$\int \varphi(x)\left[u(he+x) - u(x)\right] \leq C_u\ h\ \ \|\varphi\|_{L^2}$$

that is

$$\int \varphi(x)\left[\frac{u(he+x) - u(x)}{h}\right] \leq C_u\ \ \|\varphi\|_{L^2}.$$

So by duality

$$\left\|\frac{u(x+he) - u(x)}{h}\right\|_{L^2(\Omega)} \leq C_u.$$

Finite difference$\Rightarrow$Approximation

As $L^2(\Omega)$ space is reflexive and separable, the sequence $\frac{u(x+he)-u(x)}{h}$ has a weak limit v in $L^2(\Omega)$, that is for all $\varphi \in C_0^1(\Omega)$

$$\int \frac{u(x+he) - u(x)}{h}\varphi(x) \to \int v\varphi.$$

The left hand side equals

$$\int u(x)\,\frac{\varphi(x-he) - \varphi(x)}{h} \to -\int u(x)\,D_e\varphi(x) \quad \text{as } h \to 0.$$

So

$$-\int u(x)\,D_e\varphi(x) = \int v\varphi.$$

Next we construct a Cauchy sequence by taking φ to be a usual mollifier $\rho \in C_0^\infty(R^n)$ with $\int \rho = 1$.

Claim: $D_e(u * \rho) = v * \rho$.

Indeed

$$\begin{aligned}
D_e(u*\rho) &= \int u(y)\,D_{e(x)}\rho(x-y)\,dy = \int u(y)\left[-D_{e(y)}\rho(x-y)\right]dy \\
&= \int v(y)\,\rho(x-y)\,dy \ \ \text{by the derived IBP identity above the claim.} \\
&= v * \rho.
\end{aligned}$$

It is also true that $D_e(u*\rho_\varepsilon) = v*\rho_\varepsilon$ with $\rho_\varepsilon(x) = \rho(x/\varepsilon)/\varepsilon^n$. Certainly $v*\rho_\varepsilon \to v$ in $L^2(\Omega)$. Therefore we have a Cauchy sequence with $u*\rho_\varepsilon \to u$ and $D_e(u*\rho_\varepsilon) \to v$ in $L^2(\Omega)$. Replace e with all axis directions we have the desired full gradient version.

Trace

figure: Q =flat piece Γ in x-R^{n-1} $\times$ δ height in y-R^1

We show that $u \in W^{1,2}$ has a well-defined restriction (trace) on the boundary. We first assume $u \in C^1$, then use approximation to reach the general conclusion. From

$$u(x,y) - u(x,0) = \int_0^y u_y(x,s)\,ds$$

it follows that

$$\begin{aligned}
\int_\Gamma |u(x,y) - u(x,0)|^2\,dx &\le \int_\Gamma \left[\int_0^y u_y(x,s)\,ds\right]^2 dx \\
&\le \int_\Gamma y \int_0^y u_y^2(x,s)\,dsdx \\
&\le y \int_Q |Du|^2
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\delta \int_\Gamma u^2(x,0)\,dxdy &= \int_0^\delta \int_\Gamma \left[-u(x,y) + \int_0^y u_y(x,s)\,ds\right]^2 dxdy \\
&\le 2\int_0^\delta \int_\Gamma u^2(x,y) + \left[\int_0^y u_y(x,s)\,ds\right]^2 dxdy \\
&\le 2\int_0^\delta \int_\Gamma \left[u^2(x,y) + \delta \int_0^\delta u_y^2(x,s)\,ds\right]\,dxdy \\
&\le 2\int_Q u^2 + 2\delta^2 \int_\Gamma \int_0^\delta u_y^2(x,s)\,dsdx \\
&\le 2\int_Q u^2 + 2\delta^2 \int_Q |Du|^2 .
\end{aligned}$$

The last inequality becomes

$$\int_\Gamma u^2(x,0)\,dx \le \frac{2}{\delta}\int_Q u^2 + 2\delta \int_Q |Du|^2 .$$

Therefore $u(\cdot,y)$ is a Cauchy sequence in $L^2(\Gamma)$ as $y \to 0$. The limit of the Cauchy sequence is defined as the trace of u on Γ.

RMK. Inspecting the above argument, we can also go along $W^{1,1}$ function to leave a trace, as long as notice that the integral of L^1 function over a small measure set is small:

$$\int_\Gamma |u(x,y) - u(x,0)|\,dx \le \int_\Gamma \int_0^y |Du|\,dsdx \to 0 \quad \text{as } y \to 0.$$

Chain Rule

Given $f \in C^1(R^1)$ and $u \in W^{1,2}(\Omega)$, then $f(u) \in W^{1,2}(\Omega)$ and $Df(u) = f'(u)Du$.

Proof. We use the approximation version of $W^{1,2}$ space. Let u_k be an approximating sequence for u in $W^{1,2}$ norm, that is $u_k \to u$ in $L^2(\Omega)$ and $Du_k \to Du$ or Du_k is a Cauchy sequence in $L^2(\Omega)$. We have

$$\int_\Omega |f(u_k) - f(u)|^2 dx \le \int_\Omega |Df(*)|^2 |u_k - u|^2 dx \le \sup_R |Df|^2 \int_\Omega |u_k - u|^2 dx \to 0$$

and

$$\begin{aligned}
&\int_\Omega |f'(u_k) Du_k - f'(u) Du|^2 dx \\
&\le 2\int_\Omega [f'(u_k)]^2 \ |Du_k - Du|^2 dx + 2\int_\Omega |f'(u_k) - f'(u)|^2 |Du|^2 dx \\
&\to 0 \quad \text{(1st limit by Cauchy } Du_k\text{, 2nd limit by Lebesgue dom. con.).}
\end{aligned}$$

Prop. For $u \in W^{1,2}$, we have $u \vee 0 \in W^{1,2}$ and $D(u \vee 0) = \begin{cases} Du & u > 0 \\ 0 & u \le 0 \end{cases}$ a.e. .

Proof. Smooth out the function $f(t) = \begin{cases} t & t > 0 \\ 0 & t \le 0 \end{cases}$ by convolution with an even (radial) mollifier ρ_ε in R^1. Then the mollified function $\tilde{f}_\varepsilon(t) = \begin{cases} t & t > \varepsilon \\ \sim \varepsilon & |t| \le \varepsilon \\ 0 & t < -\varepsilon \end{cases}$. Let

$$f_\varepsilon(t) = \tilde{f}_\varepsilon(t + \varepsilon).$$

By the Chain Rule, $Df_\varepsilon(u) = f'_\varepsilon(u) Du$. Hence for any $\varphi \in C_0^1(\Omega)$ (by Lebesgue D.C.T twice)

$$-\int_\Omega f(u) D\varphi = \lim_{\varepsilon \to 0} -\int_\Omega f_\varepsilon(u) D\varphi = \lim_{\varepsilon \to 0} \int_\Omega Df_\varepsilon(u)\varphi = \int_\Omega Du\chi_{\{u>0\}}\varphi.$$

Therefore (by IBP characterization of $W^{1,2}$: $f(u) \in W^{1,2}$ then $Df(u) = f'(u) Du$)

$$u \vee 0 = f(u) = \begin{cases} u & u > 0 \\ 0 & u \le 0 \end{cases} \in W^{1,2}.$$

RMK. One can just take $f_\varepsilon(t) = \begin{cases} \sqrt{t^2 + \varepsilon^2} - \varepsilon & t > 0 \\ 0 & t \le 0 \end{cases}$ as in [GT].

Cor. If u and v are in $W^{1,2}$, then so are the following: $u \wedge 0 = -(-u \vee 0)$, u^+ and u^- with $u = u^+ - u^-$, $|u| = u^+ + u^-$, and also $u \vee v = (u - v) \vee 0 + v = \begin{cases} u & u > v \\ v & v \ge u \end{cases}$, $u \wedge v = (u - v) \wedge 0 + v = \begin{cases} u & u < v \\ v & v \le u \end{cases}$.

eg. $\int |Du|^2 = \int |Du^+|^2 + \int |Du^-|^2 \ge \int |Du^+|^2$.