Lecture 10 Evans-Krylov-(Safonov)

∘ skip $C^{1,\alpha}$
∘ $C^{2,\alpha}$ estimate

Recall Krylov-Safonov for $C^0 \ni u \in S(\mu, 0)$. Now for μ-elliptic equation

$$F\left(D^2u\right) = 0$$

we have

i) $u \in C^\alpha$;
ii) $u \in C^{1,\alpha}$;
smooth version:

$$\sum F_{ij} D_{ij} u_e = 0.$$

C^0 version:

$$\frac{u(x+\varepsilon e) - u(x)}{\varepsilon} \in S(\mu, 0).$$

The strong argument already gives $u - v \in S(\mu, 0)$, then the uniqueness of viscosity solution,

$$\left\{ \begin{array}{c} F(D^2u) = F(D^2v) \quad \text{in } \Omega \\ u = v \quad \text{on } \partial\Omega \end{array} \right\} \Rightarrow u = v \text{ in } \Omega.$$

RMK. One can argue for the uniqueness "directly" when F is not uniformly elliptic, say only strictly elliptic or just elliptic. In such cases, there is no intermediate conclusion $u - v \in S(\mu, 0)$.

iii) $C^{1,1}/C^{2,\alpha}$ provided F is convex.
Smooth case: Analytic*

$$\sum F_{ij} D_{ij} u_{ee} + \overbrace{\sum F_{ij,kl} D_{ij} u_e D_{kl} u_e}^{\geq 0} = 0,$$

that is $\sum F_{ij} D_{ij} u_{ee} \leq 0$. Then

$$\left\{ \begin{array}{c} u_{ee} \in \bar{S}(\mu, 0) \\ F(D^2u) = 0 \end{array} \right\} \Rightarrow D^2u \in C^\alpha.$$

Continuous case:

$$\left\{ \begin{array}{c} \frac{u(x+\varepsilon e)+u(x-\varepsilon e)-2u(x)}{\varepsilon^2} \in \bar{S}(\mu, 0) \\ \\ F(D^2u) = 0 \end{array} \right\} \Rightarrow D^2u \in C^\alpha.$$

$$\left\{ \begin{array}{c} u^*_\rho = \frac{1}{\rho^2}\left[⨍_{\partial B_\rho(x)} u - u(x) \right] \in \bar{S}(\mu, 0) \\ \\ \text{local maximum principle} \end{array} \right\} \Rightarrow D^2u \in L^\infty.$$

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*Geometric

convex level figure

$$F(M)=0 \quad D^2u(x-\varepsilon e) \quad D^2u(x+\varepsilon e) \quad \nabla F\left(D^2u(x)\right)$$

$$\nabla F\cdot\frac{D^2u(x-\varepsilon e)\ +\ D^2u(x+\varepsilon e)-2D^2u(x)}{\varepsilon^2}\leq 0 \quad \text{or}$$
$$F_{ij}D_{ij}u_{ee}\leq 0.$$

Theorem 1 *Let $u\in C^2(C^4)$ be a solution to μ-elliptic equation $F(D^2u)=0$, F convex. Then $u\in C^{2,\alpha}$ and*

$$\left\|D^2u\right\|_{C^\alpha\left(B_{1/2}\right)}\leq C(n,\mu)\left\|D^2u\right\|_{L^\infty(B_1)},$$

where small $\alpha=\alpha(n,\mu)>0$.

Heuristic: Recall C^α estimate for solutions $0\leq u\leq 1$ to $\sum D_i(a_{ij}D_ju)=0$ or $\sum a_{ij}D_{ij}u=0$,

$$\operatorname*{osc}_{B_{1/2}}u\leq\theta\operatorname*{osc}_{B_1}u.$$

domain target pic

$$\mathcal{B}^+=\left[0,\frac{1}{2}\right] \quad \mathcal{B}^-=(\frac{1}{2},1].$$

Either i) $\left|u^{-1}(\mathcal{B}^+)\right|\geq\frac{1}{2}|B_1|$ or ii) $\left|u^{-1}(\mathcal{B}^-)\right|\geq\frac{1}{2}|B_1|$.

Case i) $u>0$ super solution satisfies

$$\inf_{B_{1/2}}u\geq C(n,\mu)\left(\int_{B_1}u^\varepsilon\right)^{1/\varepsilon}\geq C(n,\mu)\frac{1}{2}\left(\frac{1}{2}|B_1|\right)^\varepsilon\overset{\text{def}}{=}\eta(n,\mu).$$

Case ii) $1-u$ super solution satisfies

$$\inf_{B_{1/2}}(1-u)\geq C(n,\mu)\left(\int_{B_1}(1-u)^\varepsilon\right)^{1/\varepsilon}\geq C(n,\mu)\frac{1}{2}\left(\frac{1}{2}|B_1|\right)^\varepsilon\overset{\text{def}}{=}\eta(n,\mu).$$

Either way, we conclude

$$\operatorname*{osc}_{B_{1/2}}u\leq(1-\eta)\operatorname*{osc}_{B_1}u.$$

RMK. We really only used u along positive and negative directions are super solutions. One does similar things in the vector case:

∘ Fully nonlinear equations $F(D^2v)=0$, $u \dashrightarrow D^2v$, and v_{ee} directions are enough (note there is no negative direction now).

∘ Harmonic maps $\triangle U=Q(U,DU)$, $u\dashrightarrow U$ (no negative direction either).

Now heuristic for $C^2 \Rightarrow C^{2,\alpha}$ for $F(D^2u) = 0$.

$$
\begin{aligned}
&\quad\text{domain} \quad \text{target pic} \\
D^2u(B_1) &= \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3 \\
\text{Diam} D^2u(B_1) &= 1
\end{aligned}
$$

One of the preimages, say

$$\left| \left(D^2u\right)^{-1} \left(\mathcal{B}^2\right) \cap B_1 \right| \geq \frac{1}{3} |B_1|.$$

From $D^2u \in \{M: \ F(M) = 0\}$, we find $e \in \mathbb{R}^n$ such that

$$u_{ee} - \overbrace{\inf_{B_1} u_{ee}}^{m} \geq \bar{C}(n, \mu) > 0 \ \text{ in } \left(D^2u\right)^{-1} \left(\mathcal{B}^2\right).$$

Note

$$u_{ee} \in \bar{S}(\mu, 0) \quad (u \in C^4 \text{ straightforward for } u \in C^4, \ \text{ little involved for } u \in C^2).$$

From Krylov-Safonov, we obtain

$$
\begin{aligned}
\inf_{B_{1/2}} (u_{ee} - m) &\geq C(n, \mu) \left(\int_{B_1} (u_{ee} - m)^{\varepsilon} \right)^{1/\varepsilon} \\
&\geq C(n, \mu) \cdot \bar{C}(n, \mu) \left(\frac{1}{3} |B_1| \right)^{1/\varepsilon} \stackrel{\text{def}}{=} \eta(n, \mu) > 0.
\end{aligned}
$$

Then we can "drop" say $\mathcal{B}^3$ in the covering of $D^2u\left(B_{1/2}\right)$ or at least a fixed portion of $\mathcal{B}^3$. Iterate, we have D^2u-image shrinks as we shrink our domain, in a Hölder fashion, then Hölder for D^2u.

Lemma 2 *Assume F is μ-elliptic (no convexity assumption) and $F(M_1) = F(M_2)$. Then (in fact $\Longleftrightarrow$)*

$$\|M_1 - M_2\| \stackrel{C(\mu)}{\approx} \left\|(M_1 - M_2)^-\right\| \stackrel{C(\mu)}{\approx} \left\|(M_1 - M_2)^+\right\| \stackrel{C(n)}{\approx} \sup_{|e|=1} (M_1 - M_2) \cdot e^T e,$$

in particular

$$\|M_1 - M_2\| \geq \sup_{|e|=1} (M_1 - M_2) \cdot e^T e \geq \overbrace{C(n, \mu)}^{C_E} \|M_1 - M_2\|.$$

Here $\|M\|^2 = \sum M_{ij}^2$.

Proof. By μ-ellipticity, we have

$$\underline{F(M_1)} = F(M_1 - M_2 + M_2) \leq \underline{F(M_2)} + \mu^{-1} \left\|(M_1 - M_2)^+\right\| - \mu \left\|(M_1 - M_2)^-\right\|.$$

Then

$$\mu \left\|(M_1 - M_2)^-\right\| \leq \mu^{-1} \left\|(M_1 - M_2)^+\right\|.$$

By symmetry

$$\mu \left\|(M_1 - M_2)^+\right\| = \mu \left\|(M_2 - M_1)^-\right\| \leq \mu^{-1} \left\|(M_2 - M_1)^+\right\| = \mu^{-1} \left\|(M_1 - M_2)^-\right\|.$$

Next from

$$(M_1 - M_2)^+ = \begin{bmatrix} \lambda_1^+ & & & & & \\ & \cdots & & & & \\ & & \lambda_k^+ & & & \\ & & & 0 & & \\ & & & & \cdots & \\ & & & & & 0 \end{bmatrix}$$

we have

$$\begin{aligned} \left\|(M_1 - M_2)^+\right\| \leq \|M_1 - M_2\| &\leq \left\|(M_1 - M_2)^+\right\| + \left\|(M_1 - M_2)^-\right\| \\ &\leq \left(1 + \mu^{-2}\right) \left\|(M_1 - M_2)^+\right\|. \end{aligned}$$

Then

$$\sup_{|e|=1} (M_1 - M_2) \cdot e^T e \leq \left\|(M_1 - M_2)^+\right\| \leq \sqrt{n}\lambda_{\max}^+ = \sqrt{n} \sup_{|e|=1} e\,(M_1 - M_2)\, e^T$$

and

$$\sup_{|e|=1} e\,(M_1 - M_2)\, e^T \geq \overbrace{\frac{1}{\sqrt{n}\,(1 + \mu^{-2})}}^{c_E} \|M_1 - M_2\|.$$

Proof of the theorem (Caffarelli).

Step 0. Suppose $\operatorname{diam}(D^2 u\,(B_1)) = 1$. Otherwise let $v = u/\operatorname{diam}$, $G\,(M) = F\,(\operatorname{diam}\, M)\,/\operatorname{diam}$, then $G\,(D^2 v) = 0$ with G still being μ-elliptic and convex.

Step 1. There exists small $\varepsilon_0\,(n, \mu)$ (from weak Harnack) such that if $\{\mathcal{B}_{\varepsilon_0}\,(M_k)\}_{k=1}^{k=N}$ cover $D^2 u\,(B_1)$, then

either a) $D^2 u\left(B_{1/2}\right)$ has diameter less than $1/2$

or b) we can cover $D^2 u\left(B_{1/2}\right)$ with $N - 1$ balls.

Suppose a) does not happen, then $\operatorname{diam} D^2 u\,(B_1) \geq \operatorname{diam} D^2 u\left(B_{1/2}\right) \geq 1/2$. "Enlarge" the covering of $D^2 u\,(B_1)$ by N' (finitely many overlapping, "decoys") balls in $\mathbb{R}^{n\times n}$ $\{\mathcal{B}_h\,(H_l)\}_{l=1}^{l=N'}$ with $h = h\,(n, \mu) = \min\left\{\frac{1}{8}, \frac{1}{8}c_E\right\}$ (much larger than ε_0 such that $\left(h^{n\times n}\right)^{1/\varepsilon} > \varepsilon_0$) and c_E is from the above lemma.

domain target covering figure

We know $N'\,(n, \mu) \leq \left(\frac{1}{h}\right)^{n\times n}$, then there exists one ball, say $\mathcal{B}^1 = \mathcal{B}_h\,(H_1)$ and $H_1 = D^2 u\,(x_1)$ such that

$$\left|\left(D^2 u\right)^{-1} \left(\mathcal{B}^1\right)\right| \geq \frac{|B_1|}{N'} \text{ or } \frac{\left|B_{1/2}\right|}{N'}.$$

Also there exists H_* with $H_* = D^2 u(x_*)$ such that $\|H_1 - H_*\| \geq 1/4$. By the above lemma, there exists $e \in \mathbb{R}^n$ such that

$$u_{ee}(x_1) - u_{ee}(x_*) \geq c_E \left\| D^2 u(x_1) - D^2 u(x_*) \right\| \geq \frac{1}{4} c_E$$

and with $m = \inf_{B_1} u_{ee}(x) = u_{ee}(\underline{x})$

$$\begin{aligned} u_{ee}(x) - m \geq u_{ee}(x) - u_{ee}(x_*) &= u_{ee}(x) - u_{ee}(x_1) + u_{ee}(x_1) - u_{ee}(x_*) \\ &\geq - \left\| D^2 u(x) - D^2 u(x_1) \right\| + \frac{1}{4} c_E \\ &\geq -\frac{1}{8} c_E + \frac{1}{4} c_E = \frac{1}{8} c_E \end{aligned}$$

for all x satisfying $\|D^2 u(x) - D^2 u(x_1)\| \leq h \leq \frac{1}{8} c_E$.

Recall F is convex and $u \in C^4$ (C^0 is enough), then we have the important

$$u_{ee}(x) - m \in \bar{S}(\mu, 0).$$

By Krylov-Safonov, we derive

$$\begin{aligned} \inf_{B_{1/2}} (u_{ee}(x) - m) &\geq c(n, \mu) \left[\int_{B_1} (u_{ee} - m)^{\varepsilon} \right]^{1/\varepsilon} \\ &\geq c(n, \mu) \frac{1}{8} c_E \left(\frac{B_1}{N'} \right)^{1/\varepsilon} = \eta(n, \mu) > 0. \end{aligned}$$

Let, say $\mathcal{B}_{\varepsilon_0}(M_1)$ contain $D^2 u(\underline{x})$, then for $D^2 u(y) \in \mathcal{B}_{\varepsilon_0}(M_1)$

$$u_{ee}(y) - u_{ee}(\underline{x}) \leq \left\| D^2 u(y) - D^2 u(\underline{x}) \right\| \leq 2\varepsilon_0 < \eta$$

provided we (now) choose ε_0 such that $2\varepsilon_0(n, \mu) < \eta(n, \mu)$ (essentially $h^{n \times n/\varepsilon} > \varepsilon_0$).

$$D^2 u\left(B_{1/2}\right) \quad \text{and } \mathcal{B}_{\varepsilon_0}(M_1) \quad \text{figure}$$

Therefore, we can still cover $D^2 u\left(B_{1/2}\right)$ with $N - 1$ balls of $\{\mathcal{B}_{\varepsilon_0}(M_k)\}_{k=1}^{k=N}$, after throwing away one ball $\mathcal{B}_{\varepsilon_0}(M_1)$.

Step 2. Let

$$v(x) = 2^2 u(x/2) : B_1 \subset \mathbb{R}^n \to \mathbb{R}^1,$$

then

$$\begin{aligned} D^2 v(x) = D^2 u(x/2), \; D^2 v\left(B_{1/2}\right) = D^2 u\left(B_{1/4}\right) \\ F\left(D^2 v(x)\right) = F\left(D^2 u(x/2)\right) = 0. \end{aligned}$$

Repeat Step 1, $D^2 u\left(B_{1/4}\right) = D^2 v\left(B_{1/2}\right)$ is either a) or b). After $l < N \lesssim (1/\varepsilon_0)^{n \times n}$ many steps, we have

$$\operatorname{diam}\left(D^2 u\left(B_{1/2^l}\right)\right) \leq \frac{1}{2}.$$

Let $\gamma = \gamma(n, \mu) = 1/2^l$, then

$$\begin{aligned}
\operatorname{diam}\left(D^2u\left(B_\gamma\right)\right) &\leq \frac{1}{2}\operatorname{diam}\left(D^2u\left(B_1\right)\right)\\
\operatorname{diam}\left(D^2u\left(B_{\gamma^2}\right)\right) &\leq \frac{1}{2^2}\operatorname{diam}\left(D^2u\left(B_1\right)\right)\\
&\cdots\\
\operatorname{diam}\left(D^2u\left(B_{\gamma^k}\right)\right) &\leq \frac{1}{2^k}\operatorname{diam}\left(D^2u\left(B_1\right)\right).
\end{aligned}$$

Iterate, we obtain the desired Hölder estimate of D^2u. The proof of Evans-Krylov-(Safonov) is complete.

RMK. For complex Monge-Ampere equation $\det \partial\bar{\partial}u = 1$, one obtains real Hessian $\|D^2u\|_{C^\alpha}$ estimates in terms of complex Hessian $\left\|\partial\bar{\partial}u\right\|_{L^\infty}$ and $\|u\|_{L^\infty}$ as follows.

Curvature way (Yau): By Calabi $\left\|D\partial\bar{\partial}u\right\|_{L^\infty} \leq C\left(\left\|\partial\bar{\partial}u\right\|_{L^\infty}\right)$. By Schauder, $\|D^2u\|_{C^\alpha} \leq C\left(\left\|\partial\bar{\partial}u\right\|_{L^\infty}, \|u\|_{L^\infty}\right)$.

Bernstein way (X-J Wang): By Bernstein, $\|D^2u\|_{L^\infty} \leq C\left(\left\|\partial\bar{\partial}u\right\|_{L^\infty}, \|u\|_{L^\infty}\right)$. By Evans-Krylov-(Safonov), $[D^2u]_{C^\alpha} \leq C\left(\|D^2u\|_{L^\infty}\right)$.

Complex way: Replace real $e \bigotimes e$ by complex $\partial z \bigotimes \partial\bar{z}$, by Evans-Krylov-(Safonov), $\left[\partial\bar{\partial}u\right]_\alpha \leq C\left(\left\|\partial\bar{\partial}u\right\|_{L^\infty}\right)$. Then $trD^2u = \triangle u \in C^\alpha$. By Schauder, $\|D^2u\|_{C^\alpha} \leq C\left(\left\|\partial\bar{\partial}u\right\|_{L^\infty}\right) \cdot \left(\left\|\partial\bar{\partial}u\right\|_{L^\infty} + \|u\|_{L^\infty}\right)$.