

Lecture 1 Introduction

- o equations
- o source for equations
- o explicit solutions
- o reason for better behavior (than waves)
- o nonlinear theory
 - De Giorgi-Nash
 - Krylov-Safonov

The equations

u first derivatives Du and double derivatives $D^2u \sim \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Algebraically

Laplace $\Delta u = \sigma_1 = \lambda_1 + \cdots + \lambda_n = c$
 $\sigma_k = \lambda_1 \cdots \lambda_k + \cdots = c$
M-A $\det D^2u = \sigma_n = \lambda_1 \cdots \lambda_n = c$
 $\ln \lambda_1 + \cdots + \ln \lambda_n = C$
 $\arctan \lambda_1 + \cdots + \arctan \lambda_n = C$ (take tan, then algebraic)
Also $\ln \det(\partial \bar{\partial} u) = 0$.

figure

elliptic $\Leftrightarrow f(\lambda)$ monotonic

Geometrically

$\lambda \dashrightarrow \kappa = (\kappa_1, \dots, \kappa_n)$, principle curvatures of hypersurface $(x, u(x))$.

$\sigma_k(\kappa) = c$, in particular minimal surface $\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0$

Tangent way: Re-represent graph $(x, u(x))$ at $(x_0, u(x_0))$ over its tangent plane,
 $\bar{u}(\bar{x}) = 0 + 0 \cdot \bar{x} + \frac{1}{2}(\kappa_1 \bar{x}_1^2 + \cdots + \kappa_n \bar{x}_n^2)$.

Computational way: $II = \frac{D^2u}{\sqrt{1+|Du|^2}}$

$g = I + (Du)^T(Du)$ $g^{-1} = I - \frac{1}{1+|Du|^2}(Du)^T(Du)$

$g^{-1}II = B$ or $g^{-1/2}IIg^{-1/2} \sim \begin{bmatrix} \kappa_1 & & \\ & \ddots & \\ & & \kappa_n \end{bmatrix}$

$$\begin{aligned}
H &= \kappa_1 + \dots + \kappa_n = \frac{1}{\sqrt{1+|Du|^2}} \left(\Delta u - \frac{u_i u_j}{1+|Du|^2} u_{ij} \right) = \operatorname{div} \left(\frac{Du}{\sqrt{1+|Du|^2}} \right) \\
K &= \kappa_1 \cdots \kappa_n = \frac{1}{(1+|Du|^2)(1+|Du|^2)^{n/2}} \det D^2 u = \frac{1}{n} \partial_i \left[\frac{\partial \sigma_n(B)}{\partial B_{ij}} ? \frac{\partial_j u}{\sqrt{1+|Du|^2}} \right] \\
\sigma_k(\kappa) &= \frac{1}{k} \partial_i \left[\frac{\partial \sigma_k(B)}{\partial B_{ij}} ? \frac{\partial_j u}{\sqrt{1+|Du|^2}} \right]
\end{aligned}$$

Probability

$$\begin{aligned}
&\frac{2}{3} \lambda_{\max} + \frac{1}{3} \lambda_{\min} = 0 \\
&u = \varphi(x) \text{ on } \partial\Omega
\end{aligned}$$

figure

Random walk, when hits boundary, the pay off is $\varphi(x)$.

Let $u(x)$ be the maximal expectation of pay off, starting from interior point $x \in \Omega$, with directional probability $\frac{1}{3} \leq p \leq \frac{2}{3}$. Say we are in 2d case.

$$\begin{aligned}
u(x) &= p_h \left[\frac{u(x + \varepsilon e_h) + u(x - \varepsilon e_h)}{2} \right] + p_v \left[\frac{u(x + \varepsilon e_v) + u(x - \varepsilon e_v)}{2} \right] \\
&\Rightarrow 0 = p_h u_{hh} + p_v u_{vv}
\end{aligned}$$

Now $\frac{1}{3} \leq \begin{bmatrix} p_h \\ p_v \end{bmatrix} \leq \frac{2}{3}$, take max

$$\frac{2}{3} \lambda_{\max} + \frac{1}{3} \lambda_{\min} = \sup_{\substack{\frac{1}{3} \leq (a_{ij}) \leq \frac{2}{3} \\ \operatorname{tr} = 1}} a_{ij} D_{ij} u = 0$$

When compare with solution to

$$\begin{cases} \sum a_{ij} D_{ij} w = 0 & \text{in } \Omega \\ w = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

$$\begin{aligned}
\sum a_{ij} D_{ij} w &= 0 = \frac{2}{3} \lambda_{\max} + \frac{1}{3} \lambda_{\min} (u) \geq a_{ij} D_{ij} u \\
w &= u \quad \text{on } \partial\Omega
\end{aligned}$$

$\implies u \geq w$ by the maximum principle.

Other sources for equations.

- o Fluid mechanics

vector field \vec{V}

$$\text{incompressible } \operatorname{div}(\vec{V}) = 0$$

$$\text{irrotational } \operatorname{curl} \vec{V} = 0 \iff D\vec{V} = (D\vec{V})^T \implies \vec{V} = Du$$

$$\Rightarrow \Delta u = 0$$

- o Variational

$$E[u] = \int_{\Omega} F(Du) dx \\ \varphi \in C_0^\infty(\Omega)$$

$$\begin{aligned} \frac{d}{dt} \int F(Du + tD\varphi) dx|_{t=0} &= \int \sum F_{p_i}(Du) \frac{\partial \varphi}{\partial x_i} dx \\ &= \int - \sum \frac{\partial}{\partial x_i} [F_{p_i}(Du)] \varphi d \\ &\quad \sum \frac{\partial}{\partial x_i} [F_{p_i}(Du)] = 0. \end{aligned}$$

$$\text{eg1. } F(Du) = |Du|^2 \text{ Energy } F_{p_i} = 2Du \dashrightarrow \Delta u = 0.$$

$$\text{eg2. } F(Du) = \sqrt{1 + |Du|^2} \text{ Area } F_{p_i} = \frac{Du}{\sqrt{1+|Du|^2}} \dashrightarrow \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

$$\text{eg3. } E[u] = \int \sigma_{k-1}(\kappa) \sqrt{1 + |Du|^2} dx, \text{ E-L equation } \sigma_k(\kappa) = 0 \text{ (Reilly).}$$

RMK. One obvious thing

1d principle curvature of curve $(x, f(x))$

$$\kappa = \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} = \left(\frac{f_x}{\sqrt{1+f_x^2}}\right)_x$$

$$\text{also } \int \kappa ds = \int \frac{f_{xx}}{\left(\sqrt{1+f_x^2}\right)^3} \sqrt{1+f_x^2} dx = \int (\arctan f_x)_x dx = \arctan f_x|_{\partial}$$

Q. In nd similar thing should happen to the total Gauss curvature

$$\int \sigma_n(\kappa) \sqrt{1 + |Du|^2} dx?$$

More variationals

$$\text{Eg } \sigma_k : E[u] = -\frac{1}{k+1} \int u \sigma_k(D^2u) dx + \int u dx$$

E-L equation $\sigma_k(D^2u) = 1$. This can be derived using the following divergence structure.

$$\begin{aligned} k\sigma_k &= D_\lambda \sigma_k \cdot \lambda = \frac{\partial \sigma_k(D^2u)}{\partial m_{ij}} D_{ij} u \\ &= \frac{\partial}{\partial x_i} \left[\frac{\partial \sigma_k(D^2u)}{\partial m_{ij}} \partial_{x_j} u \right] + \underbrace{\frac{\partial}{\partial x_i} \left[\frac{\partial \sigma_k(D^2u)}{\partial m_{ij}} \right]}_0 \partial_{x_j} u. \end{aligned}$$

Eg Slag: $A[DU] = \int \sqrt{\det(I + (DU)^T DU)} dx$, $U : \Omega \rightarrow R^n$.

Insist **minimizer** irrotational, i.e. $U = Du$, then E-L

$$D \sum \arctan \lambda_i = 0 \Leftrightarrow \sum \arctan \lambda_i = c.$$

$$A[DU] = \int \sqrt{\det(I - (DU)^T DU)} dx, U : \Omega \rightarrow R^n.$$

Insist **maximizer** irrotational, i.e. $U = Du$, then E-L

$$D \sum \ln \frac{1 + \lambda_i}{1 - \lambda_i} = 0 \Leftrightarrow \sum \ln \frac{1 + \lambda_i}{1 - \lambda_i} = c \leftrightarrow \sum \ln \bar{\lambda}_i = c.$$

figure?

Explicit solutions

o $H = 0$

catenoid: $|(x, y)| = \cosh z$

helicoid: $z = \arctan \frac{y}{x}$

Scherk's surface: $z = \ln \frac{\cos y}{\cos x}$

o $H_k = \text{const.}$

unit sphere

o $\Delta u = 0$

complex analysis in even d: $u = \operatorname{Re} z^k, z^{-k}, e^z, z_1^3 e^{z_2}, \dots$

algebraic n-d $u = \sigma_k(x_1, \dots, x_n)$

radial

$$\Delta u = \partial_r^2 u + \frac{n-1}{r} \partial_r u + \frac{1}{r^2} \Delta_{S^{n-1}} u$$

$$u_{rr} + \frac{n-1}{r} u_r = 0$$

$$r^{n-1} u_{rr} + (n-1) r^{n-2} u_r = 0 \text{ or } (r^{n-1} u_r)_r = 0$$

$$u_r = \frac{c}{r^{n-1}}$$

$$u = \frac{c}{r^{n-2}}, \ln |(x_1, x_2)|, \text{ or } |x_1|$$

o $\arctan \lambda_1 + \dots + \arctan \lambda_3 = \pi/2$ or $\sigma_2 = 1$

$$u = |(x_1, x_2)| \cosh x_3$$

o $\arctan \lambda_1 + \dots + \arctan \lambda_3 = 0$ or $\Delta u = \det D^2 u$

$$u = x_3 \sinh^{-1} |(x_1, x_2)|$$

o $\arctan \lambda_1 + \arctan \lambda_2 = \pi/2$ or $\det D^2 u = 1$

$$u = \int^x du = \int^x u_1 dx_1 + u_2 dx_2$$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(t - Dh(t)) = \frac{1}{\sqrt{2}}(t_1 - 3t_1^2 + 3t_2^2, t_2 + 6t_1 t_2) \\ Du(x) = \frac{1}{\sqrt{2}}(t + Dh(t)) = \frac{1}{\sqrt{2}}(t_1 + 3t_1^2 - 3t_2^2, t_2 - 6t_1 t_2) \end{cases} .$$

$$v = \int^x dv = \int^x v_1 dx_1 + v_2 dx_2$$

$$\begin{cases} x = (t_1, -h_2) = (t_1, 6t_1 t_2) \\ Dv(x) = (h_1, t_2) = (3t_1^2 - 3t_2^2, t_2) \end{cases}$$

Exercise: Show that u and v are indeed solutions. Express u and v explicitly in terms of x . ($v = x_1^3 + \frac{x_2^2}{12x_1}$)

Reasons for better behavior (than wave equations)

R₀ harmonic

R₁ Energy minimizer

R₂ Comparison principle

two solutions cannot touch each other

figure

$$\Delta u = 0$$

$$D^2 u_1 > D^2 u_2 \Rightarrow 0 = \Delta u_1 - \Delta u_2 > 0 \rightarrow \leftarrow$$

two solutions can cross each other

figure

In contrast to $w_{tt} - w_{xx} = 0$, $w_1 = x^2 + t^2$, $w_2 = 0$.

R₃ Fourier transform

$$\Delta u = f \in C_0^\infty(R^n)$$

$$\widehat{\Delta u} = \widehat{f} \Rightarrow -|\xi|^2 \widehat{u} = \widehat{f}, \text{ or } \widehat{u} = -\frac{\widehat{f}}{|\xi|^2}$$

Exercise: Verify

$$u = c_n \int \frac{1}{|x-y|^{n-2}} f(y) dy.$$

$$\widehat{D_{ij}u} = -\xi_i \xi_j \widehat{u} = \underbrace{\frac{\xi_i \xi_j}{|\xi|^2} \widehat{f}}_{\text{bounded}}$$

$$f \in L^2 \Leftrightarrow \widehat{f} \in L^2 \Rightarrow \widehat{D_{ij}u} \in L^2 \Rightarrow D_{ij}u \in L^2$$

Also $f \in C^\alpha \Rightarrow D^2 u \in C^\alpha$.

Outlook:

Linear theory

$$\sum a_{ij}(x) D_{ij}u = f(x)$$

$a_{ij}(x) \in C^\alpha$ C^α invariant space Schauder

$a_{ij}(x) \in C^0/VMO$ L^p invariant space Calderon-Zygmund

Nonlinear theory

o Quasilinear equations

$$\sum F_{p_ip_j}(Du) D_{ij}u = 0$$

want $Du \in C^\alpha$, in order to apply Schauder.

Divergence equation

$$\begin{aligned} \partial_e : \sum \frac{\partial}{\partial x_i} [F_{p_i}(Du)] &= 0 \\ \sum \frac{\partial}{\partial x_i} \left[F_{p_ip_j}(Du) \frac{\partial}{\partial x_j} u_e \right] &= 0 \end{aligned}$$

De Giorgi-Nash $u_e \in C^\alpha$.

- Fully nonlinear equations

$$F(D^2u) = 0$$

want $D^2u \in C^\alpha$ in order to run Schauder, then continuity method.

$$\begin{aligned}\partial_e : \sum F_{u_{ij}} D_{ij} u_e &= 0 \\ \partial_e^2 : \sum F_{u_{ij}} D_{ij} u_{ee} + \sum F_{u_{ij} u_{kl}} D_{ij} u_e D_{kl} u_e &= 0\end{aligned}$$

Assume convexity to drop cubit derivatives. Thus need to study

Nondivergence equation

$$\sum a_{ij} (D^2u) D_{ij} w = 0$$

Krylov-Safonov $w \in C^\alpha$.