

Lecture 3 Schauder || Calderon-Zygmund

- o Statement
- o Examples and counterexamples

C^α Schauder 1930s

L^p Calderon-Zygmund 1950s

$$\underline{\text{Interior}} \quad \Delta u = f(x)$$

+ Positive

$$f \in C^\alpha(B_1) \Rightarrow D^2u \in C^\alpha(B_{1/2})$$

$$f \in L^p(B_1) \underset{1 < p < \infty}{\Rightarrow} D^2u \in L^p(B_{1/2})$$

$$f \in BMO \Rightarrow D^2u \in BMO$$

$$f \in H^1 \Rightarrow D^2u \in H^1$$

- Negative

$$f \in C^0 \not\Rightarrow D^2f \in C^0$$

$$f \in L^{1/\infty} \not\Rightarrow D^2f \in L^{1/\infty}$$

$$\text{Boundary} \quad \begin{cases} \Delta u = 0 & x_n > 0 \\ u(x, 0) = g(x) & x_n = 0 \end{cases}$$

+ Positive

$$g \in C^0, C^\alpha \Rightarrow u \in C^0, C^\alpha$$

$$g \in C^{1,\alpha} \Rightarrow u \in C^{1,\alpha}$$

$$g \in C^{2,\alpha} \Rightarrow u \in C^{2,\alpha}$$

$$g \in L^1, L^p, L^\infty \Rightarrow u \in L^1, L^p, L^\infty$$

$$g \in W^{1,p}_{1 < p < \infty} \Rightarrow u \in W^{1,p}$$

$$g \in W^{2,p}_{1 < p < \infty} \Rightarrow u \in W^{2,p}$$

Switch p to BMO or H^1 OK.

- Negative

$$g \in C^1 \not\Rightarrow u \in C^1$$

$$g \in C^2 \not\Rightarrow u \in C^2$$

$$g \in W^{1,1}, W^{1,\infty} \not\Rightarrow u \in W^{1,1}, W^{1,\infty}$$

$$g \in W^{2,1}, W^{2,\infty} \not\Rightarrow u \in W^{2,1}, W^{2,\infty}$$

Examples/Counterexamples

Interior

Eg 0.

$$\begin{aligned} u &= h(x)(\ln r)^{1/3} \quad \text{with } h(x) = \operatorname{Im} z^k \\ |D^k u(0)| &= \infty \\ \Delta u &= O\left(r^{k-2} \frac{1}{(\ln r)^{2/3}}\right) \in C^0, C^{k-2}. \end{aligned}$$

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More expensive ones.

$$u(x) = \Gamma * f(x) = \int_{\mathbb{R}^n} \frac{c_n}{|x-y|^{n-2}} f(y) dy \quad \text{for } f \in C_0^\alpha(\mathbb{R}^n).$$

$$\begin{aligned} & \ln|x-y| & n=2 \\ & |x-y| & n=1 \end{aligned}$$

Exercise: Try at least for $f \in C_0^\infty(\mathbb{R}^n)$, see [Gilbarg-Trudinger, p. 55 Lem4.2].

$$\begin{aligned} D_i u(x) &= \int_{\mathbb{R}^n} \Gamma_i(x-y) f(y) dy, \\ D_{ij} u(x) &= \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{B_R(x) \setminus B_\varepsilon(x)} \Gamma_{ij}(x-y) [f(y) - f(x)] dy \\ &= \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{B_R \setminus B_\varepsilon} \Gamma_{ij}(y) [f(x-y) - f(x)] dy = \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{B_R \setminus B_\varepsilon} \Gamma_{ij}(y) f(x-y) dy \\ &= \int_{\mathbb{R}^n} C_n \frac{y_i y_j}{|y|^{n+2}} [f(x-y) - f(x)] dy \neq \int_{\mathbb{R}^n} C_n \frac{y_i y_j}{|y|^{n+2}} f(x-y) dy, \\ D_{ii} u(x) &= \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{B_R(x) \setminus B_\varepsilon(x)} \Gamma_{ii}(x-y) [f(y) - f(x)] dy + \frac{1}{n} f(x) \\ &= \lim_{\substack{R \rightarrow \infty \\ \varepsilon \rightarrow 0}} \int_{B_R \setminus B_\varepsilon} \Gamma_{ii}(y) [f(x-y) - f(x)] dy + \frac{1}{n} f(x) \\ &= \int_{\mathbb{R}^n} C_n \frac{y_i^2 - \frac{1}{n}|y|^2}{|y|^{n+2}} [f(x-y) - f(x)] dy + \frac{1}{n} f(x) \neq \int_{\mathbb{R}^n} \Gamma_{ii}(x-y) f(y) dy. \end{aligned}$$

Eg 1. $f(x) = \chi_\Omega \in L^\infty(R^2)$, where $\Omega = \{x_1 x_2 \geq 0\} \cap B_1$.

figure

$$\begin{aligned} u_{12}(0) &= C_2 \int \int_{\Omega} \frac{y_1 y_2}{|y|^4} f(-y) dy \\ &\sim \int \int \frac{\frac{1}{2} \sin 2\theta}{r^2} r dr d\theta \sim \int \frac{1}{r} dr = \infty. \end{aligned}$$

RMK. One can smooth out f to get smooth counterexamples for $W^{2,\infty}$ estimates.

Eg 2. $f(x) = \frac{1}{|\ln|x||^{1/2}} \chi_{B_{1/2}} \in C^0$. figure?

$$\begin{aligned} u_{12}(0) &= C_2 \int \int_{B_{1/2}} \frac{y_1 y_2}{|y|^4} \frac{1}{|\ln|y||^{1/2}} dy \\ &\sim \int \frac{1}{r} \frac{1}{|\ln r|^{1/2}} dr = \infty. \end{aligned}$$

Eg 3. $f(x) = \frac{1}{\epsilon^n} \varphi\left(\frac{x}{\epsilon}\right) \in L^1$, $\varphi = \varphi(|x|) \in C_0^\infty$, $\varphi \geq 0$, and $\int \varphi = 1$. Say still on \mathbb{R}^2

$$\begin{aligned} u &= \Gamma * f = \Gamma(x) \text{ for } |x| \geq \epsilon \\ D_{12}u &= C_2 \frac{x_1 x_2}{|x|^4} \text{ for } |x| \geq \epsilon \\ C \|f\|_{L^1} &\not> \|D_{12}u\|_{L^1(B_1)} \geq \int_{B_1 \setminus B_\epsilon} \frac{x_1 x_2}{|x|^4} \sim \int_{B_1 \setminus B_\epsilon} \frac{1}{|x|^2} = |\ln \epsilon| \rightarrow \infty \end{aligned}$$

But $\|f\|_{L^1} = 1$.

Boundary situation.

Eg 0'. $u_x = g' \in L^\infty$, but $u_y \notin L^\infty$

$$u = \operatorname{Im} z \log z = x\theta + y \ln r$$

$$\begin{aligned} u(x, 0) &= x\theta = \begin{cases} 0 & x \geq 0 \\ \pi x & x < 0 \end{cases} \\ u_x(x, 0) &= \theta \in L^\infty \\ u_y &= \partial_y(x\theta + y \ln r) \text{ little tedious, instead} \\ u_y &= \partial_y \operatorname{Im} z \log z = \operatorname{Im} \partial_y(z \log z) \\ &= \operatorname{Im} \frac{1}{i} (\bar{\partial} - \partial)(z \log z) = \operatorname{Im} i\partial(z \log z) \quad \text{here } \begin{cases} \bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \\ \partial = \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \end{cases} \\ &= \operatorname{Re} \partial(z \log z) = \operatorname{Re}(\log z + 1) \\ &= \ln r + 1 \notin L^\infty. \end{aligned}$$

$u_x = g' \in C^0$, but $u_y \notin L^\infty$

$$\begin{aligned} u &= \operatorname{Im} z (\log z)^{1/3} = r (\ln^2 r + \theta^2)^{1/3} \sin\left(\frac{\theta}{3 \ln r} + \theta\right) \\ u(x, 0) &= \begin{cases} 0 & x \geq 0 \\ |x| (\ln^2 |x| + \pi^2)^{1/3} \sin\left(\frac{\pi}{3 \ln |x|} + \pi\right) & x < 0 \end{cases} \in C^0, \text{ of course near 0!} \\ u_y &= \operatorname{Re} \partial\left(z \log^{\frac{1}{3}} z\right) = \operatorname{Re} \left[\log^{\frac{1}{3}} z + \frac{1}{3} (\log z)^{-2/3} \right] \\ &= \underbrace{(\ln^2 r + \theta^2)^{1/3} \cos\left(\frac{\theta}{3 \ln r}\right)}_{\notin L^\infty} + \underbrace{\frac{1}{3} (\ln^2 r + \theta^2)^{-2/3} \cos\left(\frac{-2\theta}{3 \ln r}\right)}_{\in C^0}. \end{aligned}$$

Eg 0'' $u_{xx}(x, 0) = g'' \in L^\infty$, but $u_{xy} \notin L^\infty$.

$$\begin{aligned} u &= \operatorname{Im} z^2 (\log z) = 2xy \ln r + (x^2 - y^2)\theta \\ u(x, 0) &= \begin{cases} 0 & x \geq 0 \\ \pi x^2 & x < 0 \end{cases} \in C^{1,1} \end{aligned}$$

$$u_{xx}(x, 0) \in L^\infty$$

$$\begin{aligned} u_{xy} &= \partial_{xy} \operatorname{Im} z^2 (\log z) = \operatorname{Im} \frac{1}{i} (\bar{\partial}^2 - \partial^2) z^2 (\log z) \\ &= \operatorname{Re} \partial^2 [z^2 (\log z)] = \operatorname{Re} (\log z + 1) = \ln r + 1 \notin L^\infty. \end{aligned}$$

More expensive ones.

Poisson formula

$$u(x) = c_n \int_{R^{n-1}} \frac{x_n}{(|\xi|^2 + x_n^2)^{n/2}} g(x' - \xi) d\xi$$

$$\begin{aligned} D_{x'} u(x) &= u(x) = c_n \int_{R^{n-1}} \frac{x_n}{(|\xi|^2 + x_n^2)^{n/2}} D_{x'} g(x' - \xi) d\xi \\ D_{x'x'} u(x) &= u(x) = c_n \int_{R^{n-1}} \frac{x_n}{(|\xi|^2 + x_n^2)^{n/2}} D_{x'x'} g(x' - \xi) d\xi \end{aligned}$$

$D_{x_n} u(x)$ direct computation messy. Instead $\xi = x_n \bar{\xi}$

$$\begin{aligned} u(x) &= c_n \int_{R^{n-1}} \frac{1}{(|\bar{\xi}|^2 + 1)^{n/2}} g(x' - x_n \bar{\xi}) d\bar{\xi} \\ D_{x_n} u(x) &= c_n \int_{R^{n-1}} \frac{-\bar{\xi} \cdot D_{x'} g(x' - x_n \bar{\xi})}{(|\bar{\xi}|^2 + 1)^{n/2}} d\bar{\xi} \\ &= c_n \int_{R^{n-1}} \frac{-\bar{\xi}}{(|\xi|^2 + x_n^2)^{n/2}} \cdot D_{x'} g(x' - \xi) d\xi \end{aligned}$$

$$\begin{aligned} D_{x'x_n} u(x) &= c_n \int_{R^{n-1}} \frac{-\bar{\xi}}{(|\xi|^2 + x_n^2)^{n/2}} \cdot D_{x'x'} g(x' - \xi) d\xi \\ &= c_n \frac{-x'}{(|x'|^2 + x_n^2)^{n/2}} * D_{x'x'} g \end{aligned}$$

Now harmonic function in R^n

$$\frac{\xi}{(|\xi|^2 + x_n^2)^{n/2}} \stackrel{*}{=} c_n \int_{R^{n-1}} \frac{x_n}{(|\xi - x'|^2 + x_n^2)^{n/2}} \frac{x'}{|x'|^n} dx' = P_{x_n} * \frac{x'}{|x'|^n}$$

Exercise: Observe the integral make sense, because $P_{x_n} \in L^2(\mathbb{R}^{n-1}) \cap P_{x_n} \in L^{10,1}(\mathbb{R}^{n-1})$ matching the singular kernel $\frac{x'}{|x'|^n}$ on \mathbb{R}^{n-1} . Justify *.

Step 1.

$$\frac{\xi}{(|\xi|^2 + (x_n + \varepsilon)^2)^{n/2}} = c_n \int_{R^{n-1}} \frac{x_n}{(|\xi - x'|^2 + x_n^2)^{n/2}} \frac{x'}{(|x'| + \varepsilon^2)^n} dx' .$$

Step 2.

$$\lim_{\varepsilon \rightarrow 0+} c_n \int_{R^{n-1}} \frac{x_n}{(|\xi - x'|^2 + x_n^2)^{n/2}} \frac{x'}{(|x'| + \varepsilon^2)^n} dx' = \lim_{\varepsilon \rightarrow 0+} c_n \int_{R^{n-1} \setminus B_\varepsilon} \frac{x_n}{(|\xi - x'|^2 + x_n^2)^{n/2}} \frac{x'}{|x'|^n} dx'.$$

See [Stein-Weiss, p. 236 Th'm 4.17, p. 218 Th'm Lem1.2, p. 13 Th'm1.25].

$$u_{x_n} = P_{x_n} * \frac{-x'}{|x'|^n} * D_{x'} g$$

$$u_{x'x_n} = \begin{array}{c} P_{x_n} \\ \text{nice kernel} \end{array} * \begin{array}{c} -x' \\ \frac{-x'}{|x'|^n} \\ \text{singular kernel} \end{array} * D_{x'x'} g$$

$\frac{x'}{|x'|^n}$ on \mathbb{R}^{n-1} satisfies

- homogeneous degree $n - 1$
- $\int_{\partial B_1} \frac{x'}{|x'|^n} = 0$.

This implies $P_{x_n} * \frac{-x'}{|x'|^n} *$ preserves C^α $0 < \alpha < 1$ and L^p $1 < p < \infty$ norm.

It does not preserve C^0 , C^1 , L^1 , L^∞ norm.

- eg1' $g' \in L^\infty$, but $u_y \notin L^\infty$ - eg1'' $g' \in L^\infty$, but $u_{xy} \notin L^\infty$
- eg2' $g' \in C^0$, but $u_y \notin L^\infty$ - eg2'' $g' \in C^0$, but $u_{xy} \notin L^\infty$
- eg3' $g' \in L^1$, but $u_y \notin L^1$ - eg3'' $g' \in L^1$, but $u_{xy} \notin L^1$

$$\text{Eg1}, \begin{cases} \Delta u = 0 \text{ for } y > 0 \\ u(x, 0) = g(x) = |x| \in W^{1,\infty} \end{cases}$$

$$u(x, y) = C_2 \int_{R^1} \frac{y}{\xi^2 + y^2} g(x - \xi) d\xi$$

$$u_y(x, y) = C_2 \int_{R^1} \frac{-\xi}{\xi^2 + y^2} g'(x - \xi) d\xi$$

Now

$$u_y(0, 0) = C_2 \int_{R^1} \frac{-1}{\xi} g'(-\xi) d\xi = \infty.$$

In fact

$$u = \frac{-2}{\pi} \left(x\theta + y \ln r - \frac{\pi}{2}x \right), \text{ see eg 0'}$$

Eg 2' $g' = \frac{1}{\ln|x|} \in C^0$, yes near 0, or say C^0 $g' = \frac{1}{|\ln|x||^e}$ not from holomorphic functions, but $u_y(0, 0) = \infty$.

Eg 3' $g' = \frac{1}{\ln|x|} \in L^1$, yes near 0, but $u_y(x, 0) \notin L^1$.

Replace g' by g'' , and u_y by u_{xy} , we get eg1''2''3''.

RMK. Freezing coefficients, general version for

$$\sum a_{ij}(x) D_{ij} u + b_i(x) D_i u + c u = f(x)$$

C^α :

$$a_{ij} \in C^\alpha, b_i \in C^\alpha, c \in C^\alpha, f \in C^\alpha \implies D^2 u \in C^\alpha.$$

L^p :

$$a_{ij} \in C^0 \text{ (or VMO)}, b_i \in C^0 \text{ (or } L^\infty, L^q\text{)}, c \in C^0 \text{ (or } L^\infty, L^q\text{)}, f \in L^p \implies D^2 u \in L^p.$$