

Lecture 4 De Giorgi–Nash

- Statement
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- Liouville

Theorem 1 Let u be a weak solution to

$$\sum_{i,j=1}^n D_i (a_{ij}(x) D_j u) = 0 \quad \text{in } B_1 \subset \mathbb{R}^n$$

with

$$\begin{aligned} \mu I \leq (a_{ij}) &\leq \mu^{-1} I && \text{when } (a_{ij}) \stackrel{\text{def}}{=} A = A^T, \\ \mu I \leq (a_{ij}) &\quad \text{and } |a_{ij}| < \mu^{-1} && \text{when } A \neq A^T. \end{aligned} \tag{*}$$

Namely for all $v \in H_0^1(B_1)$

$$\int a_{ij}(x) D_i v D_j u = 0.$$

Then u is Hölder continuos in $B_{1/2}$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\mu, n) \|u\|_{L^2(B_1)} \quad \text{with } \alpha = \alpha(\mu, n) > 0.$$

RMK. The general equations

$$\sum_{i,j=1}^n D_i (a_{ij}(x) D_j u) + b_i D_i u + c u = 0$$

with $|b| \leq \mu^{-1}$ and $|c| \leq \mu^{-1}$ can be reduced to the above model case. To write the equation in full divergence form, let $\bar{c}_{x_1} = c$, then

$$\sum D_i (a_{ij}(x) D_j u) + \sum b_i D_i u - \bar{c} D_1 u + D_1(\bar{c} u) = 0.$$

Set

$$u^0(x^0, x) = x^0 u(x)$$

and

$$(a_{ij}^0) = \begin{bmatrix} a_{00} & \frac{x^0}{2}(b_1 - \bar{c}) & \frac{x^0}{2}b_2 & \cdots & \frac{x^0}{2}b_n \\ x^0 \bar{c} & & & & \\ 0 & & (a_{ij}) & & \\ \dots & & & & \\ 0 & & & & \end{bmatrix}.$$

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By choosing a_{00} large for, say $1 \leq x^0 \leq 3$, the nonsymmetric (a_{ij}^0) satisfies the uniform ellipticity condition (*).

Exercise: Verify $u^0(x^0, x)$ satisfies in both pointwise and integral senses the equation

$$\sum_{i,j=0}^n D_i (a_{ij}^0 (x) D_j u) = 0.$$

Motivation/Application 1. Minimal surface

$$\inf \int F(Du)$$

Euler-Lagrangian

$$\sum_{i,j=1}^n D_i (F_{p_i} (Du)) = 0$$

or

$$\sum_{i,j=1}^n F_{p_i p_j} (Du) D_{ij} u = 0.$$

In order to apply Schauder, or Calderon-Zygmund, need $F_{p_i p_j} (Du) \in C^\alpha$ or C^0 . Let $e \in \mathbb{R}^n$

$$D_e \sum_{i,j=1}^n D_i (F_{p_i} (Du)) = \sum_{i,j=1}^n D_i (F_{p_i p_j} (Du) D_j u_e) = 0.$$

Let us first assume

$$\mu I \leq (F_{p_i p_j} (Du)) \leq \mu^{-1} I.$$

De Giorgi–Nash then implies $u_e \in C^\alpha$.

M/A 2. Homogenization

$$\Delta_g u = \frac{1}{\sqrt{g}} D_i (\sqrt{g} g^{ij} D_j u) = 0$$

with the Riemannian metric g periodic.

periodic figure

Look at u from far away, what happens? We have solution u to

$$\sum_{i,j=1}^n D_i (a_{ij} (x) D_j u (x)) = 0.$$

What happens to $u_\varepsilon(x)$ as ε goes to 0?

$$\sum_{i,j=1}^n D_i \left(a_{ij} \left(\frac{x}{\varepsilon} \right) D_j u_\varepsilon(x) \right) = 0$$

De Giorgi–Nash says $u_\varepsilon(x) \rightarrow u \in C^a$ in $C^{\alpha-\delta}$ norm.

Eg. 1-d

$$D_x \left(\frac{1}{2 + \cos \frac{x}{\varepsilon}} D_x \left(2x + \varepsilon \sin \frac{x}{\varepsilon} \right) \right) = 0$$

$u_\varepsilon(x) = 2x + \varepsilon \sin \frac{x}{\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 2x$, which satisfies

$$D_x \left(\frac{1}{2} D_x(2x) \right) = 0.$$

Equations for limit solutions are very interesting to know.

Proof.

Step 1. Caccioppoli, ($\|Du\|_2 \leq \|u\|_2$)

Step 2. L^∞ bound

Step 3. Oscillation

Step 1. Let $v = \eta^2 u$, (technical, trial-error picking)

$$\int \underbrace{D(\eta^2 u)}_{\eta D(\eta u) + \eta u D\eta} A D u = 0.$$

Let us move one η to the other u , in this the non-commutative situation

$$0 = \int [D(\eta u) + u D\eta] A [D(\eta u) - u D\eta].$$

Then

$$\begin{aligned} \mu \int |D(\eta u)|^2 &\leq \int D(\eta u) A D(\eta u) \stackrel{\#}{=} \int u^2 D\eta A D\eta + u D(\eta u) A D\eta - u D\eta A D(\eta u) \\ &= \int u^2 D\eta A D\eta + u D(\eta u) \left(\frac{A - A^T}{2} \right) D\eta - u D\eta \left(\frac{A - A^T}{2} \right) D(\eta u) \\ &\quad \left\{ \begin{array}{l} \mu^{-1} \int u^2 |D\eta|^2 \quad \text{when } A \text{ is symmetric} \\ C(n, \mu) \int u^2 |D\eta|^2 + \bar{C}(n, \mu) \underbrace{\int u |D\eta| |D(\eta u)|}_{\leq \frac{[\bar{C}(n, \mu)]^2}{2\mu} \int u^2 |D\eta|^2 + \frac{\mu}{2} \int |D(\eta u)|^2} \quad \text{otherwise} \end{array} \right. . \end{aligned}$$

Thus

$$\int |D(\eta u)|^2 \leq C(n, \mu) \int u^2 |D\eta|^2.$$

RMK 1. In the “intrinsic” case

$$\begin{aligned}
0 &= \int \eta^2 u \Delta_g u dv_g = - \int \sum D_i (\eta^2 u) g^{ij} D_j u dv_g \\
&= - \int \langle \nabla_g (\eta^2 u), \nabla_g u \rangle_g dv_g = - \int \langle \nabla_g (\eta u) + \nabla_g \eta, \nabla_g (\eta u) - \nabla_g \eta \rangle_g dv_g \\
&= - \int \left(|\nabla_g (\eta u)|_g^2 - u^2 |\nabla_g \eta|_g^2 \right) dv_g,
\end{aligned}$$

that is, we only have #

$$\int |\nabla_g (\eta u)|_g^2 dv_g = \int u^2 |\nabla_g \eta|_g^2 dv_g.$$

RMK 2.

$$\begin{aligned}
\int_{B_{1/2}} |Du|^2 &\leq C(n, \mu) \frac{1}{(1/2)^2} \int_{B_1 \setminus B_{1/2}} u^2 \\
\int_{B_{1/2}} |Du|^2 &\leq C(n, \mu) \frac{1}{\varepsilon^2} \int_{B_{\frac{1}{2}+\varepsilon} \setminus B_{\frac{1}{2}}} u^2
\end{aligned}$$

figure

RMK 3. Let $v = \eta^2 (u - a)^+$ (be a positive test function), we have for (sub) solution u

$$0 \geq \int D [\eta^2 (u - a)^+] A Du = \int D [\eta^2 (u - a)^+] A D (u - a)^+.$$

Hence by repeating the above argument, we get

$$\int |D [\eta (u - a)^+]|^2 \leq C(n, \mu) \int [(u - a)^+]^2 |D \eta|^2.$$

Recall Sobolev

$$\begin{aligned}
\left[\int |\eta u|^{\frac{2n}{n-2}} \right]^{\frac{n-2}{2n}} &\leq C(n) \left[\int |D(\eta u)|^2 \right]^{\frac{1}{2}} \quad \text{when } n > 2 \\
\left[\int_{B_1} |\eta u|^p \right]^{\frac{1}{p}} &\leq \underset{\text{scaling variant}}{C(p, B_1)} \left[\int |D(\eta u)|^2 \right]^{\frac{1}{2}} \quad \text{for all } p < \infty \text{ when } n = 2.
\end{aligned}$$

Step 2. Claim: There exists $\varepsilon_0(n, \mu) > 0$ small such that (for sub solution u)

$$\int_{B_1} (u^+)^2 \leq \varepsilon_0 \Rightarrow \sup_{B_{\frac{1}{2}}} u \leq 1.$$

With the claim in hand, $w = \varepsilon_0^{1/2} u / \|u\|_{L^2}$ still a solution satisfies

$$\begin{aligned} \int_{B_1} w^2 &\leq \varepsilon_0 \\ \sup_{B_{\frac{1}{2}}} w &\leq 1 \quad \text{and} \quad \sup_{B_{\frac{1}{2}}} -w \leq 1. \end{aligned}$$

So

$$\|u\|_{L^\infty(B_{1/2})} \leq \frac{1}{\sqrt{\varepsilon_0}} \|u\|_{L^2(B_1)} = C(n, \mu) \|u\|_{L^2(B_1)}.$$

Now to the claim.

Domain goes from 1 to $\frac{1}{2}$, range goes from $\frac{1}{2}$ to 1.

domain range figure

Set

$$\begin{aligned} \eta_k(x) &= \begin{cases} 1 & \text{for } |x| < \frac{1}{2} + \frac{1}{2^{k+1}} \\ \text{linear interpolation in between} \\ 0 & \text{for } |x| > \frac{1}{2} + \frac{1}{2^k} \end{cases} \\ u_k &= \left[u - \left(1 - \frac{1}{2^k} \right) \right]^+. \end{aligned}$$

Observe $u_k \leq u_{k-1}$, and $u_{k-1} > \frac{1}{2^k}$ when $u_k > 0$. Set

$$A_k \stackrel{\text{def}}{=} \int_{B_1} (\eta_k u_k)^2 \geq \int_{B_{1/2}} [(u - 1)^+]^2.$$

We prove

$$|\{u > 1\} \cap B_{1/2}| = 0$$

via iteration

$$A_k \leq [b(n, \mu)]^{S(k)} A_1^{S(k)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Case $n > 2$.

$$\begin{aligned}
A_k &\leq \left[\int_{B_1} (\eta_k u_k)^{2\frac{n}{n-2}} \right]^{\frac{n-2}{n}} \left(\int_{\eta_k u_k > 0} 1 \right)^{\frac{2}{n}} \\
&\stackrel{\text{Sobolev}}{\leq} C(n) \int_{B_1} |D(\eta_k u_k)|^2 \left(\int_{\eta_k u_k > 0} 1 \right)^{\frac{2}{n}} \\
&\stackrel{\text{Step } 1}{\leq} C(n, u) \int_{B_1} |D\eta_k|^2 u_k^2 \left(\int_{\eta_k u_k > 0} 1 \right)^{\frac{2}{n}} \\
&\leq C(n, \mu) (2^{k+1})^2 \int_{B_1} \eta_{k-1}^2 u_k^2 \left(\int_{\eta_{k-1} u_{k-1} > \frac{1}{2^k}} 1 \right)^{\frac{2}{n}} \\
&\leq C(n, \mu) (2^{k+1})^2 \int_{B_1} \eta_{k-1}^2 u_{k-1}^2 \left(\int_{B_1} \left(\frac{\eta_{k-1} u_{k-1}}{\frac{1}{2^k}} \right)^2 \right)^{\frac{2}{n}} \\
&\leq C(n, \mu) 4 \cdot 16^k \left(\int_{B_1} \eta_{k-1}^2 u_{k-1}^2 \right)^{1+\frac{2}{n}}.
\end{aligned}$$

Case $n = 2$. Say $p = 4$ in the Sobolev, then

$$\begin{aligned}
C(n, \mu) &\dashrightarrow C(n, \mu, \overset{=4}{p}) \\
\frac{2}{n} &\dashrightarrow \frac{1}{2}.
\end{aligned}$$

Any way both $1 + \frac{2}{n} > 1$ and $1 + \frac{1}{2} > 1$. At this point we have

$$\begin{aligned}
A_k &\leq b^k A_{k-1}^\beta \quad \text{with} \\
b &= C(n, \mu) \quad \text{and } \beta = 1 + \frac{2}{n} \text{ or } 1 + \frac{1}{2}.
\end{aligned}$$

To skip “tedious” iteration, let us go with a short cut. We need

$$\begin{aligned}
d^k A_k &\leq (d^{k-1} A_{k-1})^\beta \quad \text{and really} \\
b^k &\leq \frac{(d^{k-1})^\beta}{d^k} = d^{\beta(k-1)-k}
\end{aligned}$$

or

$$\frac{k \ln b}{(\beta - 1) k - \beta} \leq \ln d.$$

This can be achieved for $k \geq k_0(n, \mu)$ with $\ln d > 0$, $d > 1$. Thus we have

$$d^k A_k \leq (d^{k-1} A_{k-1})^\beta \leq (d^{k-2} A_{k-2})^{\beta^2} \leq \dots \leq (d^{k_0} A_{k_0})^{\beta^{k-k_0}}.$$

It follows that

$$\begin{aligned} A_k &\leq \frac{1}{d^k} (d^{k_0} A_{k_0})^{\beta^{k-k_0}} \\ &\leq \frac{1}{d^k} \left[d^{k_0} \int_{B_{-1}} (u^+)^2 \right]^{\beta^{k-k_0}} \\ &\rightarrow 0 \quad \text{as } k \rightarrow \infty \end{aligned}$$

provided

$$\int_{B_{-1}} (u^+)^2 \leq \varepsilon_0(n, \mu) \quad \text{small enough.}$$

Therefore

$$\int_{B_{1/2}} [(u - 1)^+]^2 = 0$$

and

$$\sup_{B_{1/2}} u^+ \leq 1.$$

Step 3. Claim: Assume (sub) solution u in B_2 satisfying

$$\begin{aligned} u &\leq 1 \quad \text{in } B_2 \\ \frac{|\{u \leq 0\} \cap B_1|}{|B_1|} &\geq \delta_0 > 0. \end{aligned}$$

Then

$$u \leq 1 - \varepsilon(\delta_0, n, \mu) \quad \text{in } B_1.$$

Consequence: “Full” solution u

$$\operatorname{osc}_{B_1} u \leq 2 \Rightarrow \operatorname{osc}_{B_{1/2}} u \leq 2\theta(n, \mu) < 2.$$

In fact, by linearity of the equation, suppose $-1 \leq u \leq 1$.

$$\text{Case } \frac{|\{u \leq 0\} \cap B_1|}{|B_1|} \geq 1/2 \stackrel{u \text{ sub sol}}{\Rightarrow} u \leq 1 - \varepsilon\left(\frac{1}{2}, n, \mu\right) \text{ in } B_{1/2} \Rightarrow$$

$$\operatorname{osc}_{B_{1/2}} u \leq 2 - \varepsilon\left(\frac{1}{2}, n, \mu\right) = 2 \cdot \frac{2 - \varepsilon\left(\frac{1}{2}, n, \mu\right)}{2} = 2\theta;$$

$$\text{Case } \frac{|\{u \leq 0\} \cap B_1|}{|B_1|} < 1/2, \text{ then}$$

$$\frac{|\{-u \leq 0\} \cap B_1|}{|B_1|} \geq 1/2 \stackrel{u \text{ sub sol}}{\Rightarrow} -u \leq 1 - \varepsilon\left(\frac{1}{2}, n, \mu\right) \text{ in } B_{1/2} \Rightarrow$$

$$\begin{aligned} \operatorname{osc}_{B_{1/2}} -u &\leq 2 - \varepsilon\left(\frac{1}{2}, n, \mu\right) = 2 \cdot \frac{2 - \varepsilon\left(\frac{1}{2}, n, \mu\right)}{2} = 2\theta. \\ &= \operatorname{osc}_{B_{1/2}} u \end{aligned}$$

Now we prove the claim.

Case almost: $\frac{|\{u \leq 0\} \cap B_1|}{|B_1|} \geq 1 - h_0$ with small enough $h_0 = \frac{\varepsilon_0}{4|B_1|}$. We have

$$\int_{B_{-1}} (u^+)^2 \leq 1^2 h_0 |B_1| \leq \frac{1}{4} \varepsilon_0.$$

Apply Step 2 to $2u$, we get $2u < 1$ in $B_{1/2}$ or $u < 1/2$ in $B_{1/2}$.

Case $\delta_0 : \frac{|\{u \leq 0\} \cap B_1|}{|B_1|} \geq \delta_0$ with $\delta_0 < 1 - h_0$. The proof is through next

Claim 2: $v \in H^1(B_1)$ $0 \leq v \leq 1$, $\Sigma_0 = \{v = 0\}$ $\Sigma_1 = \{v = 1\}$. Then

$$|\Sigma_0| |\Sigma_1| \leq C(n) \|Dv\|_{L^2(B_1)} |\{0 < v < 1\}|^{1/2}.$$

Assuming Claim 2, let us proceed.

$$u \text{ from } 1 - \frac{1}{2^{k-1}} \text{ to } 1 - \frac{1}{2^k} \text{ figure}$$

Apply Step 1 Caccioppoli to test function $\eta^2 u_k$ with

$$u_k = \min \left\{ 2^k \left[u - \left(1 - \frac{1}{2^{k-1}} \right) \right]^+, -1 \right\} \in W^{1,2},$$

we get

$$\int_{B_1} |Du_k|^2 \leq C(n, \mu) \int_{B_2 \setminus B_1} u_k^2 \leq C(n, \mu) \int_{B_2 \setminus B_1} 1.$$

By Claim 2 applying to $v = u_k$,

$$\begin{aligned} & \underbrace{\left| \left\{ u \leq 1 - \frac{1}{2^{k-1}} \right\} \cap B_1 \right|}_{\geq \delta_0 |B_1|} \left| \left\{ 1 \geq u \geq 1 - \frac{1}{2^k} \right\} \cap B_1 \right| \\ & \leq C(n) \|Du_k\|_{L^2} \left| \left\{ 1 - \frac{1}{2^{k-1}} < u < 1 - \frac{1}{2^k} \right\} \right|^{1/2}. \end{aligned}$$

If

$$\left| \left\{ 1 \geq u \geq 1 - \frac{1}{2^k} \right\} \cap B_1 \right| \geq h_0 |B_1| \quad \text{for all } k = 1, 2, 3, \dots,$$

then

$$\delta_0 |B_1| h_0 |B_1| C(n) \leq \left| \left\{ 1 - \frac{1}{2^{k-1}} < u < 1 - \frac{1}{2^k} \right\} \cap B_1 \right|^{1/2}.$$

It implies

$$|\{0 < u \leq 1\} \cap B_1| > \sum_{k=1}^{\infty} \left| \left\{ 1 - \frac{1}{2^{k-1}} < u < 1 - \frac{1}{2^k} \right\} \cap B_1 \right|^{1/2} = \infty.$$

This contradiction shows there exists (large) $k_0 = k_0(n, \mu)$ such that

$$\frac{|\{1 \geq u \geq 1 - \frac{1}{2^{k_0}}\} \cap B_1|}{|B_1|} < h_0.$$

Now (sub) solution $w = [u - (1 - \frac{1}{2^{k_0}})] 2^{k_0}$ satisfies

$$\int_{B_1} (w^+)^2 \leq 1 |B_1| h_0 \leq \frac{\varepsilon_0}{4}.$$

Applying Step 2 to w we get

$$\sup_{B_{1/2}} 2w \leq 1$$

or

$$\left[u - \left(1 - \frac{1}{2^{k_0}} \right) \right] \leq \frac{1}{2} \frac{1}{2^{k_0}} \quad \text{in } B_{1/2}.$$

That is

$$u \leq 1 - \frac{1}{2} \frac{1}{2^{k_0}} \quad \text{in } B_{1/2} = 1 - \varepsilon(n, \mu).$$

Finally we prove Claim 2.

“solid angle” from x to reach all Σ_1

$S_x(\Sigma_1)$ “solid” angle from $x \in \Sigma_0$ to reach all $y \in \Sigma_1$. First

$$1 = v(y) - v(x) = \int_0^{|y-x|} D_\rho v(x + \rho w) d\rho \leq \int_0^{|y-x|} |Dv(x + \rho w)| d\rho.$$

Next fix x and integrate over $\int_{S_x(\Sigma_1)} d\omega$

$$\begin{aligned} |S_x(\Sigma_1)| &= \int_{S_x(\Sigma_1)} 1 d\omega \leq \int_{S_x(\Sigma_1)} \int_0^{|y-x|} \frac{\left| Dv \left(\overbrace{x + \rho w}^z \right) \right|}{\rho^{n-1}} \rho^{n-1} d\rho d\omega \\ &\leq \int_{B_1} \frac{|Dv(z)|}{|z - x|^{n-1}} dz. \end{aligned}$$

Then integrate over $\int_{\Sigma_0} dx$

$$\begin{aligned} |\Sigma_0| \min_{x \in \Sigma_0} |S_x(\Sigma_1)| &= \int_{\Sigma_0} |S_x(\Sigma_1)| dx \leq \int_{B_1} \int_{B_1} \frac{|Dv(y)|}{|z-x|^{n-1}} dz dx \\ &\leq C(n) \int_{B_1} |Dv(z)| dz \\ &\leq C(n) \left(\int_{B_1} |Dv(z)|^2 dz \right)^{1/2} \left(\int_{|Dv(z)| \neq 0} 1 dz \right)^{1/2} \\ &= C(n) \|Dv\|_{L^2(B_1)} |\{0 < v < 1\}|^{1/2}. \end{aligned}$$

Lastly we solve the puzzle $\min_{x \in \Sigma_0} |S_x(\Sigma_1)| \geq c(n) |\Sigma_1|$. This is because

$$\begin{aligned} |\Sigma_1| &= \int_{\Sigma_1} dy \leq \int_{\rho_1(y)}^{\rho_2(y)} \int_{S_x(\Sigma_1)} \rho^{n-1} d\omega d\rho \\ &\leq \int_0^2 \int_{S_x(\Sigma_1)} \rho^{n-1} d\omega d\rho = \frac{2^n}{n} |S_x(\Sigma_1)|. \end{aligned}$$

Thus $\min_{x \in \Sigma_0} |S_x(\Sigma_1)| \geq \frac{n}{2^n} |\Sigma_1|$.

RMK. For $u \in C_0^\infty(\Omega)$ we have

$$u(y) = - \int_0^\infty D_\rho u(y + \rho\omega) d\rho$$

then

$$\begin{aligned} u(y) |\partial B_1| &= - \int_{\partial B_1} \int_0^\infty D_\rho u(y + \rho\omega) d\rho d\omega = - \int_{\Omega} \frac{\langle z - y, Du(z) \rangle}{|z - y|^n} dz \\ &= \frac{1}{(n-2)} \int_{\Omega} \left\langle D_z \frac{1}{|z-y|^{n-2}}, Du(z) \right\rangle dz = \frac{-1}{n-2} \int_{\Omega} \frac{1}{|z-y|^{n-2}} \Delta u(z) dz, \end{aligned}$$

that is

$$u(y) = \int_{\Omega} \frac{-1}{(n-2) |\partial B_1| |z-y|^{n-2}} \Delta u(z) dz.$$

Also we have

$$\begin{aligned} u(y) &= \frac{z}{-\|\partial B_1\| |z|^n} * Du \quad \text{and} \\ \|u\|_{L^r(\Omega)} &\leq \left\| \frac{z}{-\|\partial B_1\| |z|^n} \right\|_{L^p(\Omega)} \|Du\|_{L^q(\Omega)} \quad \text{with say} \\ r &= p < \frac{n}{n-1} \quad \text{and } q = 1 \quad \text{in the condition } 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \end{aligned}$$

By the way last Young's inequality is proved as follows: decompose

$$fg = f^{\frac{p}{r}} g^{\frac{q}{r}} f^{p(\frac{1}{p}-\frac{1}{r})} g^{q(\frac{1}{q}-\frac{1}{r})} \quad \text{with } 1 = \frac{1}{r} + \left(\frac{1}{p} - \frac{1}{r} \right) + \left(\frac{1}{q} - \frac{1}{r} \right);$$

apply Hölder

$$\int f(x-y)g(y)dy \leq \left[\int f^p(x-y)g^q(y)dy \right]^{\frac{1}{r}} \left[\int f^p(x-y)dy \right]^{\left(\frac{1}{p}-\frac{1}{r}\right)} \left[\int g(y)dy \right]^{\left(\frac{1}{q}-\frac{1}{r}\right)};$$

integrate

$$\begin{aligned} \int \left[\int f(x-y)g(y)dy \right]^r dx &\leq \int \int f^p(x-y)g^q(y)dydx \left[\int f^p \right]^{r\left(\frac{1}{p}-\frac{1}{r}\right)} \left[\int g \right]^{r\left(\frac{1}{q}-\frac{1}{r}\right)} \\ &= \left[\int f^p \right]^{1+r\left(\frac{1}{p}-\frac{1}{r}\right)} \left[\int g \right]^{1+r\left(\frac{1}{q}-\frac{1}{r}\right)}; \end{aligned}$$

simplify

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Immediate application: Assume

$$\begin{aligned} u \in W_{loc}^{1,2}(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n), \text{ say } |Du| \leq \text{a google} \\ \sum_i D_{x_i}(F_{p_i}(Du)) = 0, \text{ say } F(p) = \sqrt{1 + |p|^2}. \end{aligned}$$

Then u is a linear function.

In fact

$$\begin{aligned} \frac{1}{\varepsilon} \sum_{\text{osc } i} D_{x_i} [F_{p_i}(Du(x+\varepsilon e)) - F_{p_i}(Du(x))] &= 0 \\ \sum_i D_{x_i} \left[F_{p_i p_j} (*) D_{x_j} \left(\frac{Du(x+\varepsilon e) - Du(x)}{\varepsilon} \right) \right] &= 0. \end{aligned}$$

Now De Giorgi-Nash implies

$$\begin{aligned} \text{osc}_{B_1} \frac{Du(x+\varepsilon e) - Du(x)}{\varepsilon} &\leq \theta^k \text{osc}_{B_{2^k}} \frac{Du(x+\varepsilon e) - Du(x)}{\varepsilon} \\ &\leq \theta^k \|Du\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

where positive $\theta = \theta(n, \|Du\|_{L^\infty(\mathbb{R}^n)}) < 1$. Thus

$$\text{osc}_{B_1} [Du(x+\varepsilon e) - Du(x)] = 0$$

for all $\varepsilon, e, B_1(x_0)$.

Exercise: Relying on this zero oscillation, show that u is a linear function. (Try not to use derivative.)

RMK. We use the Euclidean structure (only) in deriving the measure control of Claim 2 in Step 3. This part is messy on minimal surfaces. In fact it is not true in general. Otherwise (since Step 1 and Step 2 (Sobolev) generalize to minimal

surfaces), one would have Hölder estimate for harmonic functions. One consequence is that Hölder growth of non constant harmonic functions

$$\operatorname{osc}_{B_{2^{k+1}}} h \geq \frac{1}{\theta^k} \operatorname{osc}_{B_1} h = (2^k)^{\overbrace{\log_2 \theta^{-1}}^{\alpha}} \operatorname{osc}_{B_1} h.$$

But the height of catenoid $z = ch^{-1} |x|$ satisfies

$$\begin{aligned}\Delta_g z &= 0 \\ z &\sim \ln |x| \ll \rho^\alpha.\end{aligned}$$

This contradiction indicates that Claim 2 is not true on Catenoid.