Lecture 5 Moser

- Statement
- strong maximal principle

Theorem 1 (Moser) *Let u be a weak solution to*

$$\sum_{i,j=1}^{n} D_i \left(a_{ij}\left(x\right) D_j u\right) = 0 \quad in \ B_1 \subset \mathbb{R}^n$$

with

$$\begin{aligned} &\mu I \leq (a_{ij}) \leq \mu^{-1} I && when \ \ (a_{ij}) \overset{def}{=} A \ = A^T, \\ &\mu I \leq (a_{ij}) \quad and \ \ |a_{ij}| < \mu^{-1} && when \ \ A \ \neq A^T. \end{aligned} \tag{*}$$

Suppose u satisfies

$$\begin{aligned} u \geq 0 \quad &in \ B_1\left(0\right), \\ u\left(0\right) \leq 1. & \end{aligned}$$

Then

$$\sup_{B_{1/2}} u \leq C\left(n, \mu\right).$$

Recall the examples r^{2-n} and $x_1 r^{1-n}$.
Proof.
Step 1. Distribution estimate of solution
Step 2. Divergent sequence

Step 1. Claim: Suppose (**super**) solution $v \geq 0$ in Q_2 cube and $v\left(0\right) \leq 1$. Then

$$\frac{\left|\left\{v > t\right\} \cap Q_1\right|}{\left|Q_1\right|} \leq \frac{2}{t^{\gamma}},$$

where $\gamma = \gamma\left(n, \mu\right) > 0$, could be small.

RMK. Norm $\|v\|_{L^2(Q_1)}$ is not available. One cannot normalize so that $\|v\|_{L^2(Q_1)} = 1$ and $v\left(0\right) \leq 1$ simultaneously. Otherwise the claim the trivial with $\gamma = 2$.

RMK. The assumption $v\left(0\right) \leq 1$ is a conflicting condition for positive solution v, hence the reverse control of the large distribution of the positive solution v.

Let $\Sigma_k = \left\{v \geq N^k\right\} \cap Q_1$ with $N = N\left(n, \mu\right)$ to be chosen in the inductive step.

Recall Step 3 in the proof of De Giorgi: Suppose

$$\begin{gathered} v \ \ \text{sub solution in } B_2 \\ v \leq 1 \ \ \text{in } B_2 \\ \frac{\left|\left\{v \leq 0\right\} \cap B_1\right|}{\left|B_1\right|} \geq \delta_0 \end{gathered}$$

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Then $v < 1 - \varepsilon(\delta_0, n, \mu)$ in $B_{1/2}$, where $\delta_0 = 2^{-2n-1}$ is chosen in the inductive step.

Now a "variant" claim: Suppose

$$v \text{ sub solution in } Q_2$$
$$v \le 1 \text{ in } Q_2$$
$$\frac{|\{v \le 0\} \cap Q_1|}{|Q_1|} \ge \delta_0 \quad \text{or} \quad \frac{|\{v \le 0\} \cap Q_2|}{|Q_2|} \ge \frac{\delta_0}{2^n}.$$

Then $v < 1 - \varepsilon(\delta_0 2^{-n}, n, \mu)$.

Initial Step. $|\Sigma_1| < \frac{1}{2}$ with $N = \frac{2}{\varepsilon}$.

Otherwise if $|\Sigma_1| \ge \frac{1}{2}$, we seek a contradiction. Now the (sub) solution

$$w = 1 - \frac{v}{N} = 1 - \frac{\varepsilon}{2} v$$

satisfies

$$w \le 1 \text{ in } Q_2$$
$$\frac{|\{w \le 0\} \cap Q_1|}{|Q_1|} \ge \frac{1}{2}.$$

By the "variant" claim, $w \le 1 - \varepsilon$ in Q_1 or $v > 2$ in Q_1, which contradicts $v(0) \le 1$.

Inductive step. $|\Sigma_{k+1}| \le \frac{1}{2}|\Sigma_k|$.

RMK. The strategy is to prove $|\Sigma_{k+1}| \le \frac{1}{2}|\Sigma_k|$ at every small scale, namely $|\Sigma_{k+1} \cap Q| \le \frac{1}{2}|\Sigma_k \cap Q|$ for all Qs. Only density points of Σ_{k+1} make contributions toward its measure. We (Calderon-Zygmund) decompose Q_1 forever.

cube Q_1

Case splitting: $\frac{|\Sigma_{k+1} \cap Q|}{|Q|} < \frac{1}{2}$, continue splitting.

Case keeping: $\frac{|\Sigma_{k+1} \cap Q|}{|Q|} \ge \frac{1}{2}$, keep Q. And in this case the predecessor Q^* of Q satisfies $\frac{|\Sigma_{k+1} \cap Q^*|}{|Q^*|} < \frac{1}{2}$ and

$$Q^* \subset \Sigma_k.$$

Indeed consider sub solution

$$w = 1 - \frac{v/N^k}{N} < 1 \text{ in } 2Q^*$$
$$\frac{|\{w \le 0\} \cap 2Q^*|}{|2Q^*|} \ge \frac{\frac{1}{2}|Q|}{|2Q^*|} = 2^{-2n-1}.$$

By the "variant" claim, $w \le 1 - \varepsilon = 1 - \frac{2}{N}$ in Q^* or $v > 2N^k$ in Q^*, which implies $Q^* \subset \Sigma_k$.

Now let the (disjoint) collection of Q be $\{Q^j\}$, we have

$$\left|\Sigma_{k+1}\right| \overset{\text{Lebesgue}}{\leq} \sum_j \left|Q^j \cap \Sigma_{k+1}\right| \leq \sum_l \underset{\text{not all predecessor}}{\left|\left(Q^l\right)^* \cap \Sigma_{k+1}\right|}$$
$$\overset{\text{case splitting}}{<} \frac{1}{2} \sum_l \left|\left(Q^l\right)^*\right| \overset{Q^* \subset \Sigma_k}{\leq} \frac{1}{2} \left|\Sigma_k\right|.$$

So we have the claim

$$\left|\{v \geq t\}\right| \leq \left|\left\{v \geq N^k\right\}\right| \leq \frac{1}{2^k} = \frac{2}{2^{k+1}} = \frac{2}{\left(N^{\log_N 2}\right)^{k+1}} = \frac{2}{\left(N^{k+1}\right)^{\log_N 2}} < \frac{2}{t^{\log_N 2}} = \frac{2}{t^\gamma},$$

where

$$N^k \leq t < N^{k+1}$$
$$\gamma = \log_{N(n,\mu)} 2 > 0.$$

Step 2. Claim: The positive solution u in the theorem satisfies

$$\sup_{Q_{1/2}} u \leq M(n,\mu), \quad \text{large enough to be chosen in the end.}$$

Otherwise, there exist $\{x_k\} \subset Q_1$ such that

$$u(x_k) \geq l^{k-1} M \to \infty \ \text{ with } 1 < l = l(n,\mu) \text{ to be chosen shortly.}$$

blow up sequence figure

This contradiction proves the claim. Now let us find a blow-up sequence.

Step x_1. There exists $x_1 \in Q_{1/2}$ such that $u(x_1) \geq M$.

Step x_2. From Step 1.

$$\left|\left\{u > \frac{M}{2}\right\} \cap Q_1\right| \leq \frac{2}{\left(\frac{M}{2}\right)^\gamma} = \frac{1}{2}\left(\frac{h_1}{2}\right)^n \quad \text{with } h_1 = 2\left[\frac{4}{\left(\frac{M}{2}\right)^\gamma}\right]^{1/n}.$$

Then

$$\frac{\left|\left\{u \leq \frac{M}{2}\right\} \cap Q_{h_1/2}(x_1)\right|}{\left|Q_{h_1/2}(x_1)\right|} > \frac{1}{2}. \tag{*2}$$

From this we show that there exists $x_2 \in Q_{h_1}(x_1)$ such that $u(x_2) \geq lM$. Suppose otherwise, then $u(x) < lM$ in $Q_{h_1}(x_1)$.

(The heuristic idea of the following argument is, to look down u from lM with $l = 1 + \frac{1}{100000000}$, then **relatively** $u(x_1) \geq M$ is near lM, but $M/2$ is far away from

lM. By Step 1, the $M/2$ far away distribution of the "flipped" solution is small. The competition of distributions from two ends then leads to a collision.)

flip figure

We have (sub) solution

$$w=\frac{lM-u}{lM-M}\geq 0 \text{ in } Q_{h_1}(x_1)$$
$$w(x_1)\leq 1.$$

By "scaled" Step 1,

$$\frac{\left|\left\{w\geq\frac{lM-\frac{M}{2}}{lM-M}\right\}\cap Q_{h_1/2}(x_1)\right|}{\left|Q_{h_1/2}(x_1)\right|}\leq\frac{2}{\left(\frac{l-\frac{1}{2}}{l-1}\right)^{\gamma}}<\frac{1}{2}$$

if $l=l(n,\mu)>1$ and close to 1. In terms of u

$$\frac{\left|\left\{u\leq\frac{M}{2}\right\}\cap Q_{h_1/2}(x_1)\right|}{\left|Q_{h_1/2}(x_1)\right|}<\frac{1}{2}.$$

This contradicts (*2).

Step x$_3$. Again from Step 1

$$\left|\left\{u>\frac{lM}{2}\right\}\cap Q_1\right|\leq\frac{2}{\left(\frac{lM}{2}\right)^{\gamma}}=\frac{1}{2}\left(\frac{h_2}{2}\right)^n \quad\text{with } h_2=2\left[\frac{4}{\left(\frac{M}{2}\right)^{\gamma}}\right]^{1/n}\frac{1}{\left(l^{1/n}\right)^{\gamma}}.$$

Then

$$\frac{\left|\left\{u\leq\frac{lM}{2}\right\}\cap Q_{h_2/2}(x_2)\right|}{\left|Q_{h_2/2}(x_2)\right|}>\frac{1}{2}. \tag{*3}$$

From this we show that there exists $x_3\in Q_{h_2}(x_2)$ such that $u(x_3)\geq l^2M$. Suppose otherwise, then $u(x)<l^2M$ in $Q_{h_2}(x_2)$. We have (sub) solution

$$w=\frac{l^2M-u}{l^2M-lM}\geq 0 \text{ in } Q_{h_2}(x_2)$$
$$w(x_2)\leq 1.$$

By "scaled" Step 1, we have

$$\frac{\left|\left\{w\geq\frac{l^2M-\frac{lM}{2}}{l^2M-lM}\right\}\cap Q_{h_2/2}(x_2)\right|}{\left|Q_{h_2/2}(x_2)\right|}\leq\frac{2}{\left(\frac{l-\frac{1}{2}}{l-1}\right)^{\gamma}}<\frac{1}{2}.$$

In terms of u, it is

$$\frac{\left|\left\{u\leq\frac{lM}{2}\right\}\cap Q_{h_2/2}(x_2)\right|}{\left|Q_{h_2/2}(x_2)\right|}<\frac{1}{2}.$$

It contradicts (*3).

$\cdots$

In particular

$$
\begin{aligned}
& h_1 + h_2 + h_3 + \cdots \\
& = h_1 \left(1 + \frac{1}{\left(l^{\frac{\gamma}{n}} \right)} + \frac{1}{\left(l^{\frac{\gamma}{n}} \right)^2} + \cdots \right) \\
& = 2 \left[\frac{4}{\left(\frac{M}{2} \right)^{\gamma}} \right]^{1/n} \frac{1}{1 - \frac{1}{\left(l^{\frac{\gamma}{n}} \right)}} < 1
\end{aligned}
$$

provided we choose $M = M(\gamma, n, \mu)$ large enough. The proof of Moser is complete.

Strong Maximum Principle. Suppose $W^{1,2}$

$$
\begin{gathered}
u \ \text{ is a weak solution to } \sum D_i \left(a_{ij} D_j u \right) = 0 \\
u \geq 0 \ \text{ in } B_1 \\
u(0) = 0.
\end{gathered}
$$

Then $u \equiv 0$.

Proof. For arbitrarily large K, $Ku \geq 0$ in B_1, $Ku(0) = 0$. By Moser

$$
\begin{gathered}
Ku \leq C(n, \mu) \quad \text{in } B_{1/2} \ \text{ or} \\
0 \leq \sup_{B_{1/2}} u \leq \frac{C(n, \mu)}{K} \to 0 \ \text{ as } K \to \infty.
\end{gathered}
$$

Similarly $u \equiv 0$ in $B_{\frac{1}{2} + \frac{1}{4}}, B_{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}}, \cdots, B_1$.