Lecture 6 Quick applications of Harnack

- ∘ minimal graph cone
  codimension 1
  3-d and high codimension
- ∘ estimates for Green's function

<u>Application 1</u>. Minimal graph cones of codimension 1 must be planes.

cone figure

Analytically

Theorem 1 *Any homogeneous order one solution* $u(x) = |x|\, u(x/|x|)$ *to*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \quad \textit{or}$$

$$\sum D_{x_i}\left(F_{p_i}(Du)\right) = 0 \quad \textit{with } F = \sqrt{1+|p|^2}, \ \textit{say,}$$

must be linear.

Proof. First Du is bounded, since $Du(x) = Du(x/|x|)$. Then

$$\mu I \le \left(F_{p_i p_j}\right) \le \mu^{-1} I.$$

F figure

For any $e \in \mathbb{R}^n$, we have

$$\sum D_{x_i}\left(F_{p_i p_j}(Du)\, D_{x_j} u_e\right) = 0 \quad \text{or}$$

$$\sum D_{x_i}\left(F_{p_i p_j}(Du)\, D_{x_j}(u_e - m)\right) = 0,$$

with $m = \min u_e$.

min figure and homog figure

By Harnack, we have the strong maximum principle. Apply it to u_e, we get

$$\sup(u_e - m) \le C(n,\mu) \inf(u_e - m) = 0.$$

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Thus $u_e \equiv const.$ As e is arbitrary, we see $Du = const.$ and u is linear. ■

RMK. Direct strong maximum principle way. D_e the above equation,

$$\sum F_{p_i p_j}(Du) D_{ij} u_e + \underbrace{F_{p_i p_j p_k}(Du) D_k u_i}_{b_j} D_j u_e = 0.$$

The usual Hopf strong maximum principle applies to u_e, and one gets the same linearity conclusion.

Application 2. Three dimensional minimal graph cones of any codimension must be planes.

cone figure

Analytically, one is dealing with

$$\sum_{i,j=1}^{\mathbf{3}} g^{ij}(DU) D_{ij} U = 0 \text{ with } U(x) = |x| U(x/|x|).$$

The argument is via strong maximum principle for derivative of solution u_e if $\sum_{i,j=1}^{\mathbf{2}} a_{ij} D_{ij} u = 0$. Usually we only have strong maximum principle for w with $\sum_{i,j=1}^{\mathbf{n}} a_{ij} D_{ij} w = 0$ or $\sum_{i,j=1}^{\mathbf{n}} D_i (D_j a_{ij} u) = 0$.

Proposition 2 *Let u be a $W^{2,2}$ strong solution for*

$$\sum_{i,j=1}^{\mathbf{2}} a_{ij} D_{ij} u = 0 \text{ with } \mu I \le (a_{ij}) \le \mu^{-1} I.$$

Then $u \in C^{1,\alpha}$ and Du satisfies the strong maximum principle ***componentwise****.*

RMK. The condition $|x| U(x/|x|) = U(x) \in W^{1,2}$ makes

$$\int \sqrt{\det g} = \int \sqrt{\det\left(I + (DU)^T DU\right)}$$

integrable.

RMK. In $R^2 = C^1$, $\triangle u = 0$, then $H = u_x - i u_y$ is holomorphic. Then $\ln|H| = \operatorname{Re}\ln H$ satisfies the strong maximum principle, or $|H| = |Du|$ satisfies the strong maximum principle.

Proof. The equation is

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} = 0 \quad \text{or}$$
$$u_{xx} + \frac{2a_{12}}{a_{11}} u_{xy} + \frac{a_{22}}{a_{11}} u_{yy} = 0\ .$$

D_y the last equation

$$D_x(1 D_x u_y) + D_y\left(\frac{2a_{12}}{a_{11}} D_x u_y\right) + D_y\left(\frac{a_{22}}{a_{11}} D_y u_y\right) = 0.$$

The nonsymmetric coefficients satisfy

$$\mu^2 I \leq \begin{pmatrix} 1 & 0 \\ \frac{2a_{12}}{a_{11}} & \frac{a_{22}}{a_{11}} \end{pmatrix} \text{ and } \left|\frac{2a_{12}}{a_{11}}\right| \leq 2\mu^{-2} - 2, \quad \left|\frac{a_{22}}{a_{11}}\right| \leq \mu^{-2}.$$

Apply De Giorgi-Nash to $W^{1,2}$ weak solution u_y, we obtain $u_y \in C^\alpha$. Apply Moser, we see u_y satisfies the strong maximum principle. Similarly $u_x \in C^\alpha$ and u_x satisfies the strong maximum principle. ■

RMK. The above proposition fails in 3d and above. The 4d counterexample is "easy". Consider the Hopf map

$$H(z_1, z_2) = \frac{\left(|z_1|^2 - |z_2|^2, 2\bar{z}_1 z_2\right)}{|z|} : \mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{R}^3.$$

One can cook up coefficients for the saddle surface $\frac{|z_1|^2 - |z_2|^2}{|z|}$ for the following equations. Or Lawson-Osserman noticed that $\left(z, \frac{\sqrt{5}}{2} H(z)\right)$ is a minimal graph cone in $\mathbb{R}^7$, (later Harvey-Lawson discovered that this minimal cone is volume minimizing, in fact a calibrated submanifold in $\operatorname{Im}\mathbb{O} = \mathbb{R}^7$). Thus with

$$g = I + \frac{5}{4}(DH)^T DH$$

we have the minimal surface system in both nondivergence and divergence forms

$$\sum_{i,j=1}^{4} g^{ij}(x) D_{ij} H = 0 \text{ or } \sum_{i,j=1}^{4} D_i \left(g^{ij}(x) D_j H\right) = 0.$$

RMK. Recall Euclid triple for right angle triangles: $(m^2 - n^2, 2mn, m^2 + n^2)$.

Theorem 3 *Let u be a smooth ($W^{2,2}_{loc}$ strong) solution to*

$$\sum_{i,j=1}^{\mathbf{3}} a_{ij} D_{ij} u = 0 \quad \textit{and}$$

$$u(x) = |x| \, u(x/|x|).$$

Then u is a linear function.

RMK. Heuristically, $\Sigma = Du(S^2)$ is a saddle and closed surface in $\mathbb{R}^3$, then can only be a point.

touching figure

Plane with normal e touches Σ at $Du(e)$ or $Du(-e)$, even Σ is singular.

Claim: Σ is saddle.

Rodrigue formula $K_d dX = d\gamma$, where γ is the unit normal to a hypersurface X, e_d is a principle direction.

hypersurface figure

Now $X = Du(x)$, in this case $\gamma = x/|x|$.

Convex case.

support function figure

The support function u is defined as

$$u(w) = \sup_{y\in\Sigma} w\cdot y, \quad \text{or } u(rw) = \sup_{y\in\Sigma} rw\cdot y \text{ that is } u(x) = \sup_{y\in\Sigma} x\cdot y,$$

then we have

$$Du(x) = y(x) + \underbrace{x}_{\text{normal}} \cdot \overbrace{D_x y(x)}^{\text{tangent}} = y(x).$$

General case. One abandons the support function approach. Assume along x direction there exists a tangent plane (locally uniquely) to Σ at y. Define

$$u(x) = x\cdot y(x).$$

This way we also have $Du(x) = y(x)$.

Next $D^2u(x) = \frac{1}{|x|}D^2u(x/|x|)$ has one zero eigenvalue with eigendirection ∂_r. This is because for all e, $\partial_r u_e = 0$, this implies $\langle D^2u\partial_r, e\rangle = 0$. Then $D^2u\partial_r = 0\partial_r$.

Lastly Rodrigue becomes

$$K_d d^2u = K_d du = d\gamma = dx,$$

from which we obtain

$$(\kappa_1, \cdots \kappa_{n-1}) = \left(\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_{n-1}}\right), \quad \lambda_n = 0!$$

Now the proof of the theorem (is via 2-d equation). Suppose $Du \neq const$. Let

$$h(x_1, x_2) = u(x_1, x_2, 1).$$

For $x_3 > 0$

$$u(x_1, x_2, x_3) = x_3 h\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),$$

$$u_1 = h_1\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right), \quad u_2 = h_2\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),$$
$$u_3 = h\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) - \frac{x_1}{x_3}h_1\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right) - \frac{x_2}{x_3}h_2\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}\right),$$

$$D^2u(x_1, x_2, 1) = \begin{bmatrix} 1 & & \\ & 1 & \\ -x_1 & -x_2 & 1 \end{bmatrix} \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & & -x_1 \\ & 1 & -x_2 \\ & & 1 \end{bmatrix}.$$

From the equation $Tr\left[(a_{ij})\, D^2u\right]=0$, we have

$$Tr\left\{\begin{bmatrix}1 & & -x_1\\ & 1 & -x_2\\ & & 1\end{bmatrix}(a_{ij})\begin{bmatrix}1 & & \\ & 1 & \\ -x_1 & -x_2 & 1\end{bmatrix}\begin{bmatrix}h_{11} & h_{12} & 0\\ h_{21} & h_{22} & 0\\ 0 & 0 & 0\end{bmatrix}\right\}=0,$$

that is

$$\sum_{i,j=1}^{2}A_{ij}\left(x\right)D_{ij}h=0 \ \text{ with}$$
$$\mu\left(x\right)I\leq\left(A_{ij}\left(x\right)\right)\leq\mu^{-1}\left(x\right)I.$$

By the above 2d proposition: maximum principle for h_1, $\sup h_1$ only occurs at ∞. Recall

$$u_1\left(x\right)=u_1\left(\frac{x_1}{x_3},\frac{x_2}{x_3},1\right)=h_1\left(\frac{x_1}{x_3},\frac{x_2}{x_3}\right),$$

then

$$\sup_{\mathbb{R}^3}u_1\left(x\right)=u_1\left(x_1^*,x_2^*,0\right).$$

To x_1, direction x_2 and x_3 are symmetric, thus similar arguments with $u=x_2u\left(\frac{x_1}{x_2},1,\frac{x_3}{x_2}\right)$ give

$$\sup_{\mathbb{R}^3}u_1\left(x\right)=u_1\left(x_1^*,0,x_3^*\right).$$

Thus

$$\sup_{\mathbb{R}^3}u_1\left(x\right)=u_1\left(x_1^*,0,0\right)=u_1\left(1,0,0\right)\ \text{ or } u_1\left(-1,0,0\right).$$

The same argument leads to

$$\sup_{\mathbb{R}^3}u_3\left(x\right)=u_3\left(0,0,1\right)\ \text{ or } u_1\left(0,0,-1\right),\ \text{ say } u_3\left(0,0,1\right).$$

(In fact, we only need this info for u_3.)

Next

$$\begin{aligned}u_3\left(x_1,x_3,1\right)&=h\left(x_1,x_2\right)-x_1h_1\left(x_1,x_2\right)-x_3h_2\left(x_1,x_2\right)\\&\leq u_3\left(0,0,1\right)=h\left(0,0\right).\end{aligned}$$

The Taylor expansions for h, Dh at $x=0$ are

$$h\left(x\right)=h\left(0\right)+h_1\left(0\right)x_1+h_2\left(0\right)x_2+\frac{1}{2}h_{11}\left(0\right)x_1^2+h_{12}\left(0\right)x_1x_2+\frac{1}{2}h_2\left(0\right)x_2^2+o\left(|x|^2\right),$$
$$h_1\left(x\right)=h_1\left(0\right)+h_{11}\left(0\right)x_1+h_{12}\left(0\right)x_2+o\left(|x|\right),$$
$$h_2\left(x\right)=h_2\left(0\right)+h_{21}\left(0\right)x_1+h_{22}\left(0\right)x_2+o\left(|x|\right).$$

It follows that

$$\begin{aligned}h\left(0\right)-\left[\frac{1}{2}h_{11}\left(0\right)x_1^2+h_{12}\left(0\right)x_1x_2+\frac{1}{2}h_2\left(0\right)x_2^2\right]+o\left(|x|^2\right)&=h\left(x\right)-x_1h_1\left(x\right)-x_2h_2\left(x\right)\\&\underset{*}{\leq}h\left(0\right).\end{aligned}$$

But $Tr\left[AD^2h\right]=0$, then $-\left[\frac{1}{2}h_{11}(0)x_1^2+h_{12}(0)x_1x_2+\frac{1}{2}h_2(0)x_2^2\right]$ is a saddle surface, in fact a hyperbola, in turn, cannot stay below 0. Thus * is a contradiction. Note that we can choose a point on S^2 such that $D^2u\neq 0$, because we assume $Du\neq const$. For convenience, say the point is $(0,0,1)$. Then the saddle surface (hyperbola) is not degenerate. The proof is complete.

RMK. In $W^{2,2}(R^2)$ or $W^{2,n/2}(R^n)$ case, the Taylor expansion is true (by another result of Calderon-Zygmund).

Application 3. Estimates for Green's function.

Let g be Green's function for μ-elliptic divergence equation

$$\begin{cases} -\sum D_i\left(a_{ij}(x)D_jg\right)=\delta(0) \quad \text{in } B_1 \\ g(x)=0 \quad \text{on } \partial B_1 \\ \lim_{x\to 0}g(x)=\infty. \end{cases}$$

Then for $n\geq 3$ and $|x|\leq 1/2$

$$\frac{c(n,\mu)}{|x|^{n-2}}\leq g(x)\leq \frac{c^{-1}(n,\mu)}{|x|^{n-2}} \quad .$$

Proof. The argument is through comparing to the model case $-\triangle h=\delta(0)$. We assume $a_{ij}(x)\in C^\infty$.

Step 1. Define

$$\begin{aligned} Cap_A(\Omega) &= \inf_{\substack{u\in H_0^1(B_1)\\ u\geq 1 \quad \text{on } \Omega}}\int Du\ ADu=\int_{B_1\backslash\Omega}DV\ ADV \\ &= \int_{\partial(B_1\backslash\Omega)}V\ V_{A\nu}=\int_{\partial\Omega}-V_{A\gamma}, \end{aligned}$$

where γ is the outward unit normal of Ω and V is the unique minimizer (the existence of V is straightforward for the convex quadratic energy functional) satisfying

$$\begin{cases} -\operatorname{div}(ADV)=0 \ \text{ in } B_1\backslash\Omega \\ V=0 \quad \text{on } \partial B_1 \\ V=1 \ \text{ on } \partial\Omega. \end{cases}$$

Ω inside B_1 figure

Observation 1. $\Omega_1\subset\Omega_2$ then $Cap_A(\Omega_1)\leq Cap_A(\Omega_2)$.

Observation 2. $\mu\ Cap_I(\Omega)\leq Cap_A(\Omega)\leq\mu^{-1}\ Cap_A(\Omega)$.

RMK. $Cap_A(\{0\})=0$.

Step 2. Let $m=\min_{|x|=r}g(x)$ and $M=\max_{|x|=r}g(x)$.

$\Omega_M\subset\Omega_m\subset B_1$ figure

By the maximum principle applied to g, we get

$$\Omega_M \overset{\text{def}}{=} \{x: \ g(x) \geq M\} \subset B_r \subset \{x: \ g(x) \geq m\} \overset{\text{def}}{=} \Omega_m.$$

We calculate

$$\begin{aligned} Cap_A(\Omega_m) &= \frac{1}{m^2}\int_{B_1\backslash\Omega_m} \overbrace{DgADg}^{div(gADg)} \overset{\text{need Sard}}{=} \frac{1}{m^2}\int_{\partial\Omega_m} g\langle ADg, -\gamma\rangle = \frac{1}{m}\int_{\partial\Omega_m} -g_{A\gamma} \\ &= \frac{1}{m}\int_{\Omega_m} -div(ADg) = \frac{1}{m}\int_{\Omega_m}\delta(0) = \frac{1}{m}. \end{aligned}$$

Now

$$Cap_A(\Omega_m) \geq Cap_A(B_r) \geq \mu Cap_I(B_r) = \mu\ \frac{1}{h(r)} = \frac{\mu(n-2)|\partial B_1|}{r^{2-n}-1},$$

where $h = \frac{1}{(n-2)|\partial B_1|}\left(|x|^{2-n}-1\right)$ satisfies

$$\begin{cases} -\triangle h = \delta(0) & \text{in } B_1 \\ h = 0 & \text{on } \partial B_1 \end{cases}.$$

So

$$m \leq \frac{r^{2-n}-1}{\mu(n-2)|\partial B_1|}.$$

Similarly we get

$$\begin{aligned} &Cap_A(\Omega_M) = \frac{1}{M}, \\ &Cap_A(\Omega_M) \leq Cap_A(B_r) \leq \mu^{-1}Cap_I(B_r) = \mu^{-1}\ \frac{1}{h(r)}, \text{ and} \\ &M \geq \mu\ \frac{1}{(n-2)|\partial B_1|}\left(r^{2-n}-1\right). \end{aligned}$$

Step 3. Apply Moser's Harnack along the ring ∂B_r to positive solution g

$$-div(ADg) = 0 \ \text{ in } B_{2r}\backslash\{0\},$$

we get

$$M \leq C(n,\mu)\, m$$

and then

$$\begin{gathered} c(n,\mu)\frac{1}{|x|^{n-2}} \leq \frac{M}{C(n,\mu)} \leq m \\ \leq g(x) \leq \\ M \leq C(n,\mu)\, m \leq c^{-1}(n,\mu)\ \frac{1}{|x|^{n-2}} \end{gathered}$$

for $|x| \leq 1/2$.

The proof of the estimates for Green's function is complete. ∎