Lecture 7 Minimal Surface equations

- ∘ non-solvability
- ∘ strongly convex functional
- ∘ further regularity

Consider minimal surface equation

$$\begin{cases} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \ \text{ in } \Omega \\ u = \varphi \ \text{ on } \partial\Omega \end{cases}.$$

The solution is a critical point or the minimizer of

$$\inf_{u|_{\partial\Omega}=\varphi} \int_\Omega \sqrt{1+|Du|^2}.$$

But the integrand $F(p) = \sqrt{1+|p|^2}$ is not strongly convex, that is $D^2F \ngeq \delta I$, only $D^2F > 0$. The loss of strong convexityor convexity causes non-solvability, or non minimizer for general domains, unlike $\triangle u = 0$ with $\int_\Omega |Du|^2$ case.

Eg1. Let the boundary data be $u = t$ on ∂B_2 and $u = 0$ on ∂B_1 with $\Omega = B_2 \backslash B_1$.

catenoid on annulus figure

The minimizer should be radial (by symmetry), or just we consider radial solutions. Necessarily we have a constraint

$$0 = \int_{B_r \backslash B_1} div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \int_{\partial B_r} \frac{Du}{\sqrt{1+|Du|^2}} \cdot \partial_r \, dA - \int_{\partial B_1} \frac{Du}{\sqrt{1+|Du|^2}} \cdot \partial_r \, dA.$$

We infer

$$\frac{u_r}{\sqrt{1+u_r^2}} r^{n-1} = \frac{1}{C} \neq 0$$

$$u_r = \frac{1}{\sqrt{(Cr^{n-1})^2 - 1}} \quad \text{with } \ C \geq 1.$$

Then

$$u(r) = \int_1^r \frac{1}{\sqrt{(C\rho^{n-1})^2 - 1}} d\rho \leq \int_1^2 \frac{1}{\sqrt{(\rho^{n-1})^2 - 1}} d\rho < \infty$$

$$\overset{\text{eg } n=2}{=} ch^{-1}(2).$$

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Now $u = t$, say 100000000 on ∂B_2 contradicts the above inequality.

RMK. Mean curvature of $\partial\Omega$ is necessary and sufficient (use distance to the boundary as barrier) in solving minimal surface equation with arbitrary boundary condition. In our lecture, we only consider strongly convex Ω and $\varphi \in C^{1,1}(\partial\Omega)$.

Eg2. Consider the non-convex functional

$$\inf_{\substack{u(0)=0\\u(1)=1}} \int_0^1 F(u_x)\,dx \quad \text{with } F(p) = \begin{cases} p^2 (p-2)^2 \ \text{ for } \ |p| \le 10 \\ \text{quadratic extension for } \ |p| > 10 \end{cases},$$

double well F figure

The Euler-Lagrangian equation is $D_x (F_p(u_x)) = F''(u_x)\, u_{xx} = 0$.

- $u = x$ is a critical point, not minimizer, $\int_0^1 F(x')\,dx = 1$.
- $v = \cdots$ with $v' = 0$ or 2, minimizers, not smooth, not unique, $\int_0^1 F(v')\,dx = 0$.

various critical pts figure

Next we solve the minimal surface equation via strongly convex functionals.

Step δ. Let $F^\delta(p) = \sqrt{1+|p|^2} + \delta |p|^2$, then $2\delta I \le (D^2 F^\delta) \le (1+2\delta) I$. ($\left(\sqrt{1+x^2}\right)_x = \frac{x}{\sqrt{1+x^2}} = \sin\theta$, $\left(\sqrt{1+x^2}\right)_{xx} = \cos\theta \cdot \theta_x = \frac{1}{\sqrt{1+x^2}} \frac{1}{1+x^2}$) Parallel to the minimizing process to $\int_\Omega |Du|^2$, we minimize

$$J[u] = \int_\Omega F^\delta(Du).$$

Let the minimizing sequence $u^k \in H^1$ with $u^k = \varphi$ on $\partial\Omega$

$$J\left[u^k\right] \to \inf J[u] = m.$$

We claim: $\left\{u^k\right\}$ is a Cauchy sequence in H^1.

convexity figure

For any (small) positive ε, we have for all large k and l

$$\begin{aligned} m &\le J\left[u^k\right] \le m + \varepsilon \\ m &\le J\left[\frac{1}{2}\left(u^k + u^l\right)\right] = \int F^\delta\left(\frac{1}{2}\left(Du^k + Du^l\right)\right) \\ &\le \int \frac{1}{2}\left[F^\delta\left(Du^k\right) + F^\delta\left(Du^l\right)\right] - \frac{1}{2}\text{``}\min D^2 F''\text{''} \left|\frac{Du^k - Du^l}{2}\right|^2 \\ &\le m + \varepsilon - \frac{\delta}{4} \int_\Omega \left|Du^k - Du^l\right|^2. \end{aligned}$$

Hence

$$\frac{\delta}{4}\int_\Omega \left|Du^k - Du^l\right|^2 \le \varepsilon.$$

So we know

· $u^k \to u^\delta$ in H^1

· the minimizer is unique (by setting $\varepsilon = 0$) and satisfies for all $\phi \in H_0^1(\Omega)$

$$\int_\Omega \sum D_{x_i}\phi \; F^\delta_{p_i}\left(Du^\delta\right) = 0.$$

Step 0. Now

$$\inf_{\substack{u\in W^{1,1}\\ u=\varphi \text{ on } \partial\Omega}} \int \sqrt{1+|Du|^2} = \inf .$$

To make sense the functional, $W^{1,1}$ is the right space. Note $W^{1,1}$ functions also have L^1 trace on $\partial\Omega$. (Going with the $W^{1,2}$ minimizing sequence woud not lead to a $W^{1,2}$ Cauchy sequence. Even if one has a $W^{1,2}$ minimizer, the minimizer is only in $W^{1,2}$ space, not in $W^{1,1}$ space.)

For any $\varepsilon > 0$, there exists $v \in W^{1,1}$ with $v = \varphi$ on $\partial\Omega$ such that

$$\int \sqrt{1+|Dv|^2} \le \inf + \varepsilon.$$

To move from $W^{1,1}$ v to $W^{1,2}$, let $V_\eta \in C^\infty$ be the approximation for v, then

$$\int \sqrt{1+|DV_\eta|^2} \le \inf + \varepsilon + \varepsilon.$$

Also there is $\delta = \delta\left(\varepsilon, \|DV_\eta\|_{L^2}\right)$ such that

$$\int_\Omega \sqrt{1+|DV_\eta|^2} + \delta\,|DV_\eta|^2 \le \inf + \varepsilon + \varepsilon + \varepsilon.$$

So the minimizer $u^\delta \in H^1$ with $u^\delta = \varphi$ on $\partial\Omega$ for $\int \sqrt{1+|Du|^2} + \delta\,|Du|^2$ satisfies

$$\int_\Omega \sqrt{1+|Du^\delta|^2} + \delta\left|Du^\delta\right|^2 \le \inf + 3\varepsilon.$$

Thus

$$\int_\Omega \sqrt{1+|Du^\delta|^2} \overset{\delta\to 0}{\to} \inf$$

provided

$$\left\|Du^\delta\right\|_{L^2(\Omega)} \le C \;\text{ independent of } \delta \;\text{ (to be justified).} \tag{*}$$

Observe a δ dependent bound of $\left\|Du^\delta\right\|_{L^2(\Omega)}$ is already there. Further there is $u \in H^1$ such that $u^\delta \rightharpoonup u$ in H^1 weakly by weak compactness of H^1 space. Let us show that

$$\int_\Omega \sqrt{1+|Du|^2} \le \liminf_{\delta\to 0} \int_\Omega \sqrt{1+|Du^\delta|^2} = \inf .$$

This is because the functional $\int \sqrt{1+|Du|^2}$ is convex and a convex combination of $u^\delta \to u$ in H^1 strongly. Then

$$\begin{aligned}\int_\Omega \sqrt{1+|Du|^2} &\leftarrow \int_\Omega \sqrt{1+|D \text{ convex combination of } u^\delta|^2} \\ &\overset{\text{Jessen}}{\leq} \text{convex combination of } \int_\Omega \sqrt{1+|Du^\delta|^2} \\ &\leq \inf + \delta C.\end{aligned}$$

At this point, we have already obtained a minimizer u and its H^1 regularity then its uniqueness ($\varepsilon = 0$ argument pushed further), if we furnish (*). Indeed we show a stronger claim by taking advantage of the boundary data ($C^{1,1}$ at this point, further relaxed C^0 condition depending on Bombieri-De Giorgi-Miranda's a priori gradient estimate.) and $C^{1,1}$ boundary:

$$\left\|Du^\delta\right\|_{L^\infty(\Omega)} \leq C\left(\|\varphi\|_{C^{1,1}}, \partial\Omega\right), \quad \text{independent of } \delta.$$

RMK. First note from De Giorgi-Nash, we already know $u^\delta \in C^{1,\alpha}$ inside Ω, though we do not know yet a uniform $C^{1,\alpha}$ norm. Further by a similar, but simpler argument than De Giorgi-Nash, one can get C^ε regularity of u^δ up to the Lipschitz boundary with C^β boundary data φ. In the following we are just drawing uniform estimates independent of parameter δ.

Boundary. For any linear function L, $\sum D_i\left(F^\delta_{p_i}(DL)\right) = 0$. We compare L to u^δ satisfying $\sum D_i\left(F^\delta_{p_i}\left(Du^\delta\right)\right) = 0$. We have

$$\begin{gathered}\sum D_i\left(F^\delta_{p_ip_j}(*)\, D_j\left(L-u^\delta\right)\right) = 0 \\ 2\delta I \leq \left(F^\delta_{p_ip_j}\right) \leq (1+2\delta)\, I.\end{gathered}$$

The (strong) maximum principle implies that the inf and sup of $L - u^\delta$ achieves on the boundary.

Recall/Exercise: For $C^{1,1}$ boundary $\partial\Omega$ strongly κ_0-convex, that is the principle curvatures $(\kappa_1, \cdots, \kappa_{n-1}) \geq \kappa_0$ componentwise, and $C^{1,1}$ boundary data φ, we have

$$x_n = |x'|^2 \text{ boundary figure}$$

$$\begin{gathered}\overbrace{\varphi(0) + D_{x'}\varphi(0)\cdot x' - Mx_n}^{L^-} \leq \varphi(x) \leq \overbrace{\varphi(0) + D_{x'}\varphi(0)\cdot x' + Mx_n}^{L^+} \\ L^- \leq u^\delta \leq L^+ \quad \text{on } \partial\Omega.\end{gathered}$$

Hint: $x_n \geq \kappa_0 |x'|^2$.

Apply the maximum principle (either after Moser, or a simpler argument to be found in the end of this lecture), we get

$$L^- \leq u^\delta \leq L^+ \quad \text{in } \Omega.$$

It implies

$$\left|D_{x_n} u^{\delta}(0)\right| \leq M.$$

Thus

$$\left|Du^{\delta}\right| = \left|\left(D' u^{\delta}, D_n u\right)\right| \leq M\left(\|\varphi\|_{C^{1,1}}, \kappa_0\right) \quad \text{on } \partial\Omega, \quad \delta\text{-free.} \tag{Bdry Lip}$$

Interior to Boundary. For any $e \in R^n$, for any $x \in \partial\Omega$, really boundary of $\Omega \cap \{\Omega - \varepsilon e\}$ with ε small. By the boundary Lip (Bdry Lip), we have for any fixed boundary point $x = x_0$ and for all $\varepsilon \leq \varepsilon_0(x_0)$

$$u^{\delta}(x + \varepsilon e) \leq u^{\delta}(x) + 2M\varepsilon.$$

By the compactness of $\partial\Omega$, we have the above inequality at all boundary points of $\Omega \cap \{\Omega - \varepsilon e\}$ for all $\varepsilon \leq \varepsilon_{\partial\Omega}$. Observe that both $u^{\delta}(x + \varepsilon e)$ and $u^{\delta}(x)$ are $W^{1,2}$ weak solutions to

$$\sum D_{x_i}\left(F^{\delta}_{p_i}(Dv)\right) = 0 \quad \text{in } \Omega \cap \{\Omega - \varepsilon e\}.$$

By the (strong) maximum principle

$$u^{\delta}(x + \varepsilon e) \leq u^{\delta}(x) + 2M\varepsilon \quad \text{in } \Omega \cap \{\Omega - \varepsilon e\},$$

from which we infer for all $x \in \Omega \cap \{\Omega - \varepsilon e\}$

$$\frac{u^{\delta}(x + \varepsilon e) - u^{\delta}(x)}{\varepsilon} \leq 2M.$$

Similarly we obtain

$$-2M \leq \frac{u^{\delta}(x + \varepsilon e) - u^{\delta}(x)}{\varepsilon}.$$

By letting $\varepsilon \to 0$, we get

$$\left\|Du^{\delta}\right\|_{L^{\infty}(\Omega)} \leq 2M, \quad \delta\text{-free.}$$

Let δ go to zero, (Summary:) we have got the minimizer for $\int_{\Omega} \sqrt{1 + |Dv|^2}$ with $u = \varphi \in C^{1,1}(\partial\Omega)$ on the strongly convex boundary, such that

$$\|Du\|_{L^{\infty}(\bar{\Omega})} \leq C\left(\|\varphi\|_{C^{1,1}}, \kappa_0(\partial\Omega)\right).$$

Step $C^{2,\alpha}$. Regularity for the critical point u.

First we have

$$\sum D_{x_i}\left(F_{p_i p_j}(*)\, D_{x_j}\left(\frac{u(x + \varepsilon e) - u(x)}{\varepsilon}\right)\right) = 0.$$

De Giorgi-Nash implies

$$\begin{aligned}\left\|\frac{u(x + \varepsilon e) - u(x)}{\varepsilon}\right\|_{C^{a}\left(B_{1/2}\right)} &\leq C \left\|\frac{u(x + \varepsilon e) - u(x)}{\varepsilon}\right\|_{L^2(B_1)} \\ &\leq C \ \|Du\|_{L^2(B_1)} \leq C\left(\|\varphi\|_{C^{1,1}}, \kappa_0(\partial\Omega)\right).\end{aligned}$$

Thus $u \in C^{1,\alpha}$ and u is a weak solution to $\sum D_i\left(F_{p_i}(Du)\right) = 0$.

Next we show that $u \in C^{2,\alpha}$, then $\sum F_{p_i p_j}(Du)\, D_{ij} u = 0$. The proof is through the $C^{2,\alpha}$ solution to a Dirichlet problem by Schauder theory. Let $a_{ij}(x) = F_{p_i p_j}(Du(x))$ (like the regularity for viscosity/Perron solution to $\triangle u = 0$), we know how to solve

$$\begin{cases} \sum a_{ij}(x)\, D_{ij} w = 0 & \text{in } B_\eta \\ w = u & \text{on } \partial B_\eta \end{cases} \tag{Schauder}$$

by weighted norm method. Then we have $C^{2,\alpha}(B_\eta)$ solution w.

Proposition 1 *Let $u \in C^{1,\alpha}$ be a weak solution to $\sum D_i\left(F_{p_i}(Du)\right) = 0$ in B_η and $w \in C^{2,\alpha}$ solution to (Schauder). Then $u \equiv w$ in B_η.*

The idea of the proof is to show $\sum D_i\left(F_{p_i}(D\mathbf{w})\right) = 0$ by a "viscosity" way. The technical execution is to modify w to $v \in C^{2,\alpha}$ such that

$$\begin{aligned} \sum D_i\left(F_{p_i}(Dv)\right) = \sum F_{p_i p_j}(Dv)\, D_{ij} v &\geq 0 \\ v \geq u \ \text{ and } v(x_1) &> u(x_1) \\ v = u \ \text{ on boundary.} \end{aligned}$$

Then contradicts $\sum D_i\left(F_{p_i}(Du)\right) = 0$.

Proof. Suppose $u \neq w$ in B_η, say $\max_{B_\eta}(w-u) = (w-u)(x_0) = t > 0$ and $x_0 \in \mathring{B}_\eta$.

w over u at x_0 figure

Frist step toward a sub soluution v : Let $w_t = w + \frac{t}{2}\left(|x|^2 - \eta^2\right)$, then $w_t = u$ on ∂B_η and

$$w_t(x_0) = w(x_0) + \frac{t}{2}\left(|x_0|^2 - \eta^2\right) = u(x_0) + t + \frac{t}{2}\left(|x_0|^2 - \eta^2\right) > u(x_0),$$

where we assumed that we started with $\eta \leq 1$. Then there exists C_t such that

$$\begin{aligned} w_t - C_t &\leq u \text{ in } B_\eta \\ w_t - C_t &= u \ \text{ at } x_1 \in \mathring{B}_\eta \quad (x_1 \text{ may not be } x_0) \\ Dw_t(x_1) &= Du(x_1). \end{aligned}$$

v over u over w_t figure

Second step toward a sub solution v : Let $v = w_t - C_t - \frac{t}{4}|x - x_1|^2 + \gamma$, then $v \geq u$ in a neighborhood N_γ of x_1. And N_γ shrinks to the point x_1 as γ goes to zero.

Since $w \in C^{2,\alpha}$, $a_{ij}(x) \in C^{\alpha}$, and $Dv(x_1) = Du(x_1)$, we can choose γ small so that N_γ small, then Dv is close to Du in N_γ and eventually so that

$$\begin{aligned}
\sum D_i\left(F_{p_i}(Dv)\right) &= \sum F_{p_ip_j}(Dv)\, D_{ij}v \\
&= \sum \underbrace{\left(F_{p_ip_j}(Dv) - F_{p_ip_j}(Du)\right)}_{o(1)} \underbrace{D_{ij}v}_{\text{bounded}} + \sum a_{ij}(x)\, D_{ij}v \\
&= o(1) + \overrightarrow{\sum a_{ij}(x)\, D_{ij}w^0}^{0} + \sum \underbrace{a_{ij}(x)}_{\geq \mu} \frac{t}{4}\delta_{ij} \\
&\geq \mu \frac{t}{8} \quad \text{for small } \gamma.
\end{aligned}$$

Now

$$\left\{\begin{array}{ll} \sum D_i\left(F_{p_i}(Du)\right) = 0 & \\ \sum D_i\left(F_{p_i}(Dv)\right) \geq 0 & \text{in } N_\gamma \\ u = v \quad \text{on } \partial N_\gamma & \end{array}\right. ,$$

or

$$\sum D_i\left(F_{p_ip_j}(*)\, D_j(v-u)\right) \geq 0.$$

Take a test function $(v-u)^+ \in H_0^1(N_\gamma)$, we get

$$\begin{aligned}
0 &\leq \int_{N_\gamma} (v-u)^+ D_i\left(F_{p_ip_j}(*)\, D_j(v-u)\right) \\
&\overset{\text{Sard}}{=} -\int_{N_\gamma} \sum D_i(v-u)^+ F_{p_ip_j}(*)\, D_j(v-u) \\
&\leq -\mu \int_{N_\gamma} \left|D_i(v-u)^+\right|^2.
\end{aligned}$$

It follows that $\int_{N_\gamma} \left|D_i(v-u)^+\right|^2 = 0$, then $(v-u)^+ \equiv 0$ or $v \leq u$ in N_γ. But $v - u = \gamma > 0$ at x_1 in N_γ.

This contradiction shows that $u \equiv w \in C^{2,\alpha}$ in B_η.

Exercise: Let u be a $C^{2,\alpha}$ solution to $\sum F_{p_ip_j}(Du)\, D_{ij}u = 0$ and $\mu I \leq \left(F_{p_ip_j}\right) \leq \mu^{-1}I$. Show that $u \in C^{3,\alpha}$.