○ $C^{1,1}/W^{2,n}$ version
○ viscosity version statement

Alexandrov-Bakelman-Pucci maximum principle:
Let $u \in C^{1,1}$ be a solution to

$$\sum a_{ij}(x) D_{ij} u = f$$
$$u \geq 0 \ \text{ on } B_1,$$

where

$$\mu I \leq (a_{ij}) \leq \mu^{-1} I.$$

Then

$$\sup_{B_1} u^- \leq C(n, \mu) \left[\int_{B_1} (f^+)^n \right]^{1/n}.$$

Proof.

cone figure

Consider the cone with base ∂B_1 and vertex $(x_0, u(x_0))$, where $u(x_0) = \inf_{B_1} u$. Then

$$\sup_{B_1} u^- \leq \overbrace{2}^{\text{diameter}} \cdot \overbrace{\inf |D\text{cone}|}^{m}.$$

We next estimate m from above. Consider the $\widetilde{\text{cone}}$ with vertex $(x_0, u(x_0))$ and a uniform slope m. For each and every tangent plane to the $\widetilde{\text{cone}}$ (along its generator), we move down parallelly the plane until it touched u (and leaves u after further down). At the touching points, which we denote by Σ,

$$Du = D\text{plane} = D\text{cone}$$
$$u < 0$$
$$D^2 u \geq 0.$$

It follows that

$$B_m \subset Du(\Sigma).$$

RMK. To better describe the contact set Σ, we let Γ be the convex envelope of $\min(u, 0)$ in B_1 (mainly for viscosity version). We have

$$\Sigma = \{u = \Gamma\}$$
$$L(x) \leq \Gamma \leq u$$
$$L(x^*) = u(x^*) \quad x^* \ \text{ touching point.}$$

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We infer

$$|B_1|\ m^n \leq |Du\,(\Sigma)| = \int_{Du(\Sigma)} dy$$

$$\overset{u \in C^{1,1}}{\underset{D^2 u \geq 0}{=}} \int_\Sigma \det D^2 u = \int_\Sigma \frac{\det (AD^2 u)}{\det A}$$

$$\leq \int_\Sigma \frac{1}{\det A} \left[\frac{tr\,(AD^2 u)}{n} \right]^n$$

$$= \int_\Sigma \frac{1}{\det A} \left(f^+ \right)^n \leq C\,(n,\mu) \int_\Sigma \left(f^+ \right)^n .$$

Thus we get

$$\sup_{B_1} u^- \leq C\,(n,\mu) \left[\int_\Sigma \left(f^+ \right)^n \right]^{1/n} = C\,(n,\mu) \left[\int_{\Gamma = u} \left(f^+ \right)^n \right]^{1/n} .$$

In order to have a short notation, mainly to avoid linearization of fully nonlinear equations, we introduce the viscosity sub and super classes: for $\mu > 0$ and $f \in C^0$

$$\underline{S}\,(\mu, f) = \left\{ u :\ M^+ \left(D^2 u \right) \geq f \text{ in viscosity sense} \right\}$$

$$\bar{S}\,(\mu, f) = \left\{ u :\ M^- \left(D^2 u \right) \leq f \text{ in viscosity sense} \right\}$$

$$S = \underline{S} \cap \bar{S}.$$

Here the sub viscosity sense means whenever a quadratic Q (touching u from below, "$\triangle Q$" $\geq f\,(x^0)$) satisfying

$$\left\{ \begin{matrix} u \geq Q \ \text{ near } x^0 \\ = \ \text{ at } x_0 \end{matrix} \right\}$$

then

$$M^+ \left(D^2 Q \right) = \sum_{\lambda_i \geq 0} \mu^{-1} \lambda_i + \sum_{\lambda_i < 0} \mu \lambda_i \geq f\left(x^0 \right) ;$$

and the super viscosity sense means whenever a quadratic Q (touching u from above, "$\triangle Q$" $\leq f\,(x^0)$) satisfying

$$\left\{ \begin{matrix} u \leq Q \ \text{ near } x^0 \\ = \ \text{ at } x_0 \end{matrix} \right\}$$

then

$$M^- \left(D^2 Q \right) = \sum_{\lambda_i \geq 0} \mu \lambda_i + \sum_{\lambda_i < 0} \mu^{-1} \lambda_i \leq f\left(x^0 \right) .$$

Theorem 1 (Caffarelli) *Let $u \in C^0\,(B_1)$ with $u \geq 0$ on ∂B_1 and $u \in \bar{S}\,(\mu, f)$. Then*

$$\sup_{B_1} u^- \leq C\,(n,\mu)\,C\,(n,\mu) \left[\int_\Sigma \left(f^+ \right)^n \right]^{1/n} = C\,(n,\mu) \left[\int_{\Gamma = u} \left(f^+ \right)^n \right]^{1/n} .$$

The proof is via justifying Γ is $C^{1,1}$ $\cdots$ to be added.