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BEHAVIOR NEAR THE BOUNDARY OF POSITIVE SOLUTIONS OF SECOND ORDER PARABOLIC EQUATIONS. II

E. B. FABES, M. V. SAFONOV, AND YU YUAN

Abstract. A *boundary backward* Harnack inequality is proved for positive solutions of second order parabolic equations in *non-divergence* form in a bounded cylinder $Q = \Omega \times (0,T)$ which vanish on $\partial_x Q = \partial\Omega \times (0,T)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. This inequality is applied to the proof of the Hölder continuity of the quotient of two positive solutions vanishing on a portion of $\partial_x Q$.

1. Introduction

In this paper we are concerned with the boundary behavior of positive solutions u and v of the parabolic *non-divergence* equation

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t)\, D_{ij} u(x,t) - D_t u(x,t) = 0 \tag{1}$$

near an open portion of the *Lipschitz* lateral boundary where u and v are assumed to vanish. We prove that $\frac{v}{u}$ is locally Hölder continuous up to that portion of the lateral boundary (Theorem 4.6 and related results in Sec. 4.3).

In our proof of the Hölder continuity of the quotient we first derive so called *boundary backward* Harnack inequality (Theorem 3.7) which is of considerable interest in itself. It states that any non-negative solution of (1) in a bounded cylinder $Q = \Omega \times (0,T)$, which vanishes on the entire lateral boundary $\partial\Omega \times (0,T)$, satisfies

$$u(x,s) \le N u(x,t) \tag{2}$$

uniformly for all $(x,t) \in Q$, such that $s \ge t \ge s - d^2 \ge \delta^2 = \text{const} > 0$, with N independent of u, where $d = \text{dist}(x, \partial\Omega)$. Note that (2) is different from the usual Harnack inequality (5), which is formulated in Theorem 2.2 below; here we have $u(Y) \le N u(X)$ with $X = (x,t)$, $Y = (x,s)$.

The above two results in parabolic *divergence* case were proved in a recent paper [FS]. The *boundary backward* Harnack inequality (2) was first obtained in [G] and [FGS] for the *non-divergence* and *divergence* cases when the coefficients are time-independent. Notice that the constant N in (2) does not depend on $d = \text{dist}(x, \partial\Omega)$. A weaker estimate with N depending also on d is called the *interior backward* Harnack inequality. In combination with the usual Harnack inequality (5), this

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inequality (2) implies the *elliptic-type* Harnack inequality (Theorem 3.6), which was proved in [G] for the *non-divergence* case and in [FGS] for the *divergence* case. The Hölder continuity of the quotient for two positive solution of heat equation $\triangle u - u_t = 0$ was proved in [ACS]. The corresponding results for the (*non-divergence* and *divergence*) elliptic equation is contained in [B], [AC] and [FGMS]. When the domain Ω is smooth, the estimate (2) for *non-divergence* can be extended for all $s \geq t \geq \delta^2$.

The results and methods of this paper are independent of [FS], though the structure of these two papers is similar. Moreover, our approach provides an alternative proof of the Hölder continuity of quotients in [ACS], [FS] for the parabolic *divergence* case and also [JK], [B], [AC], [FGMS] for the elliptic case, where it was derived via the estimates for the Green's functions and the *doubling property* for the corresponding L-caloric (L-harmonic) measures. The examples in [FK] and [S] show that the appropriate estimates for Green's functions in the *non-divergence* case fail. Regarding the *doubling property* in the parabolic *divergence* case, it follows automatically from the *backward* Harnack inequality (see [FGS]); for the *non-divergence* case it is proved in a forthcoming paper [SY].

Some intermediate results here (Statements 3.1–3.6) are basically contained in [G]. We give their simplified proofs for completeness of presentation.

In this paper, we assume the coefficients $a_{ij}(x,t)$ are measurable and for all $X = (x,t) \in \Omega \times (0,\infty)$, $\xi \in R^n$,

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(X)\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \tag{3}$$

with a constant $\nu \in (0,1]$. However, by means of appropriate approximation procedures, all our estimates for solutions can be reduced to the similar estimates with smooth a_{ij} and u. It is important only that these estimates do not depend on the smoothness of a_{ij} and u. So we may assume that all the functions a_{ij} and u in (1) are smooth.

The preliminary draft of the paper was ready before Professor E. B. Fabes passed away unexpectedly. We hope the present paper meets his high standards, though we are responsible for this final version.

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2. Assumptions and Known Results

For an arbitrary domain $V \subset \mathbb{R}^{n+1}$, we define its *parabolic boundary* $\partial' V$ as the set of all points $X \in \partial V$ such that there is a continuous curve lying in $V \cup \{X\}$ with initial point X, along which t is non-decreasing. In particular, for $Q = \Omega \times (0,T)$ we have

$$\partial' Q = \partial_x Q \cup \partial_t Q,$$

where the lateral boundary $\partial_x Q = \partial\Omega \times (0,T)$, and $\partial_t Q = \overline{\Omega} \times \{0\}$.

The following *comparison principle* is well-known.

Theorem 2.1. *Let V be a bounded domain in $\mathbb{R}^{n+1}$, $u, v \in C^2(V) \cap C(\overline{V})$, $Lu \leq Lv$ in V, and $u \geq v$ on $\partial' V$. Then $u \geq v$ on $\overline{V}$.*

For $X = (x,t) \in \mathbb{R}^{n+1}$ and $r > 0$, a "standard" cylinder

$$C_r(X) = C_r(x,t) = B_r(x) \times (t - r^2, t),$$

where $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$. For $\delta = const > 0$, $\Omega \subset \mathbb{R}^n$, $Q = \Omega \times (0,T)$ we set

$$\begin{aligned} &\Omega^\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\} = \{x \in \Omega : \overline{B}_\delta(x) \subset \Omega\}, \\ &Q^\delta = \Omega^\delta \times (\delta^2, T) = \{X \in Q : \overline{C}_\delta(X) \subset Q\}. \end{aligned} \tag{4}$$

Theorem 2.2 (Harnack Principle). *Let u be a nonnegative solution of $Lu = 0$ in a bounded $Q = \Omega \times (0,T)$, $\delta = const > 0$ such that Ω^δ is a connected set, $X = (x,t), Y = (y,s) \in Q^\delta$, and $s - t \ge \delta^2$. Then*

$$u(X) \le N u(Y), \tag{5}$$

where the constant $N = N(n, \nu, \operatorname{diam}\Omega, T, \delta)$. For cylinders $Q = C_r$, $r > 0$, the constant $N = N(n, \nu, \frac{\delta}{r})$.

This theorem was proved in [KS], see also [K, Chap. 4], for the divergence case it was proved in [M1], [M2], see also [FSt].

As in [FGS] and [FS], we assume that a bounded domain $\Omega \subset \mathbb{R}^n$ satisfies the following *Lipschitz* condition with some positive constants r_0 and m.

Assumptions: For each $y \in \partial\Omega$, there is an orthonormal coordinate system centered at y such that

$$\Omega \cap B_{r_0}(y) = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > \varphi(x'), |x| < r_0\},$$

where $\|\nabla\varphi\|_{\mathfrak{L}^\infty} \le m$.

In such local coordinates, $y \in \partial\Omega$ is represented as $(0,0)$ and $(0,r) \in \Omega$ for all $r \in (0, r_0]$. For $Q = \Omega \times (0,T)$, $Y = (y,s) = (0,0,s) \in \partial_x Q = \partial\Omega \times (0,T)$, and $r > 0$, we set

$$\overline{Y}_r = (0, r, s + r^2), \quad \underline{Y}_r = (0, r, s - 2r^2).$$

Throughout this paper, N denotes various positive constants depending only on the original quantities.

3. Backward Harnack Inequalities

3.1. **Estimates of Solutions Near the Boundary.**

Lemma 3.1. *Let u be a nonnegative solution of $Lu = 0$ in $Q = \Omega \times (0,T)$. Then for any $Y = (y,s) \in \partial_x Q$ and $0 < r \le \frac{1}{2}\min(r_0, \sqrt{T - s})$, we have*

$$M = \sup_{Q_{2r}} d^\gamma u \le N r^\gamma u(\overline{Y}_r), \tag{6}$$

where $Q_{2r} = Q \cap C_{2r}(Y)$, $\quad d = d(X) = \sup\{\rho > 0 : C_\rho(X) \subset Q_{2r}\}$, and γ, N are positive constants depending only on n, ν, m.

Proof. We fix a point $X \in Q_{2r}$. By simple geometrical considerations one can see that there exists a finite sequence $\{X^{(i)} = (x^{(i)}, t^{(i)}) : i = 0, 1, \cdots, k\} \subset Q$ such that

$$X^{(0)} = X, \quad X^{(k)} = \overline{Y}_r, \quad d_i \ge \delta_0 q^i d_0, \tag{7}$$

$$C_{\delta d_i}(X^{(i-1)}) \subset C_{d_i}(X^{(i)}) \subset Q, \quad t^{(i)} - t^{(i-1)} \ge \delta^2 d_i^2, \tag{8}$$

for all $i = 1, 2, \cdots, k$, where $d_i = \sup\{\rho > 0 : C_\rho(X^{(i)}) \subset Q\}$, $i \geq 0$, and constants $\delta_0, \delta \in (0,1)$, $q > 1$ depend only on m. From (7) it follows

$$d = d(X) \leq d_0 \leq \delta_0^{-1} q^{-k} d_k \leq \delta_0^{-1} q^{-k} r. \tag{9}$$

Further, by Theorem 2.2 we have $u(X^{(i-1)}) \leq N u(X^{(i)})$, $i = 1, \cdots, k$, where $N = N(n, \nu, m) > 1$. We represent this constant N in the form $N = q^\gamma$. Then

$$u(X) = u(X^{(0)}) \leq q^{k\gamma} u(X^{(k)}) = q^{k\gamma} u(\overline{Y}_r).$$

Together with (9) this yields $d^\gamma u(X) \leq N r^\gamma u(\overline{Y}_r)$. Since X is an arbitrary point in Q_{2r}, we arrive at the desired estimate (6). □

Lemma 3.2. *Let $Q = \Omega \times (0, T)$, $Y = (y, s) \in \partial_x Q$, and $0 < r \leq \frac{1}{2}\min(r_0, \sqrt{s})$ be fixed, and let u be a nonnegative solution of $Lu = 0$ in $Q_{2r} = Q \cap C_{2r}(Y)$. Then*

$$u(\underline{Y}_r) \leq N r^\gamma \inf_{Q_r} d^{-\gamma} u, \tag{10}$$

where $d = d(x) = \operatorname{dist}(x, \partial\Omega)$ for $X = (x, t) \in Q_r = Q \cap C_r(Y)$, and γ, N are positive constants depending only on n, ν, m.

Proof. It follows the lines of the proof of the previous lemma, only we replace $\overline{Y}_r$ by $\underline{Y}_r$ in (7), and instead of (8) we now take

$$C_{\delta d_i}(X^{(i+1)}) \subset C_{d_i}(X^{(i)}) \subset Q, \quad t^{(i)} - t^{(i+1)} \geq \delta^2 d_i^2.$$

Then we have (9) and

$$u(\underline{Y}_r) = u(X^{(k)}) \leq q^{k\gamma} u(X^{(0)}) = q^{k\gamma} u(X) \leq N r^\gamma d^{-\gamma} u(X),$$

which proves (10). □

The next theorem is a *boundary Harnack inequality*. Such kind of estimate is also referred to as Carleson type inequality. The estimate (12) was first proved by S. Salsa in [Sl] (Theorem 3.1) for *divergence* case, and by N. Garofalo in [G] (Theorem 2.3) for *non-divergence* case.

Theorem 3.3. *Let $Y = (y, s) \in \partial_x Q$ and $0 < r \leq \frac{1}{2}\min(r_0, \sqrt{T - s}, \sqrt{s})$ be fixed. Then for any nonnegative solution of $Lu = 0$ in Q, which continuously vanishes on $\Gamma = \partial_x Q \cap C_{2r}(Y)$, we have*

$$M_0 = \sup_{Q_{2r}} d_0^\gamma u \leq N r^\gamma u(\overline{Y}_r), \tag{11}$$

where

$$d_0 = d_0(X) = \sup\{\rho > 0 : C_\rho(X) \subset C_{2r}(Y)\},$$

and γ, N are positive constants depending only on n, ν, m. In particular,

$$\sup_{Q_r} u \leq N u(\overline{Y}_r). \tag{12}$$

First we prove the following elementary estimate; such kinds of estimates usually serve as intermediate steps in the proof of boundary Hölder estimates (in the divergence case, see [LSU, Chap. II], Sec. 8, and [T, Sec. 4]).

Lemma 3.4. *Let a domain* $U \subset C_{2r} = C_{2r}(Y)$, *where* $r > 0$ *and* $Y = (y, s) \in \mathbb{R}^{n+1}$. *Let* $Z = (z, \tau)$ *and* $0 < \varepsilon \leq 1$ *be such that*

$$B_{\varepsilon r}(z) \times \{\tau\} \subset C_{2r} \backslash U, \quad s - 4r^2 < \tau \leq s - 2r^2. \tag{13}$$

Then for any u *satisfying* $Lu \geq 0$ *in* U, $u \leq 0$ *on* $(\partial' U) \backslash (\partial' C_{2r})$, *and* $\sup_U u > 0$, *we have*

$$\sup_{U \cap C_r} u \leq \theta \sup_U u \tag{14}$$

with a constant $\theta = \theta(n, \nu, \varepsilon) \in (0, 1)$.

Proof. We fix $X_0 = (x_0, t_0) \in U \cap C_r$. Without loss of generality we may assume $\sup_U u = 1$, $r = 1$, and $Z = (z, \tau) = (0, 0)$. Then from (13) it follows that $|x_0| \leq 3$, $1 \leq t_0 \leq 4$. Consider the function $v = e^{-\lambda t} w^2$, where $w = \varepsilon^2 - |x - tl|^2$, $l = x_0/t_0$, and $\lambda =$ const, on the slant cylinder

$$V = \{(x, t) : |x - tl| < \varepsilon, \ \ 0 < t < t_0\}.$$

We have

$$Lv = e^{-\lambda t}\left(\lambda w^2 + 2wLw + F\right), \quad \text{where} \quad F = 2\sum_{i,j} a_{ij} D_i w D_j w \geq 2\nu |Dw|^2$$

and $|Lw| \leq N = N(n, \nu)$ in V. Since $F \geq \nu \varepsilon^2$ and w is small near

$$\partial' V = \{(x, t) : |x - tl| = \varepsilon, \ 0 < t < t_0\},$$

there exists $\varepsilon_1 = \varepsilon_1(n, \nu, \varepsilon) \in (0, \varepsilon)$ such that $Lv \geq 0$ for $\varepsilon_1 \leq |x - tl| \leq \varepsilon$, $0 \leq t \leq t_0$, and arbitrary $\lambda \geq 0$. On the remaining part of U, we also have $Lv \geq 0$, provided $\lambda = \lambda(n, \nu, \epsilon) > 0$ is large enough.

Further, the parabolic boundary $\partial'(U \cap V) = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1 \subset \partial_x V \cap \overline{U}$, $\Gamma_2 = (\partial' U \backslash \partial' C_2) \cap V$. Since $u \leq \sup_U u \leq 1$, $v = 0$ on Γ_1 and $u \leq 0$, $v \leq \varepsilon^4 \leq 1$ on Γ_2, we have $u + v \leq 1$ on $\partial'(U \cap V)$. Moreover, $L(u + v) \geq 0$ in $U \cap C_{2r}$. By the comparison principle, $u + v \leq 1$ in $\overline{U \cap V}$. Hence

$$u(X_0) \leq 1 - e^{-4\lambda} \varepsilon^4 = \theta = \theta(n, \nu, \varepsilon) \in (0, 1).$$

Since $X_0 \in U \cap C_{2r}$ is arbitrary, we get the estimate (14). □

By iterating the estimate (14), we get the following Hölder estimate (cf. [G, Lemma 2.1]).

Corollary 3.5. *Under the assumption of* Theorem 3.3, *for* $\rho \leq r$, *we have*

$$\sup_{Q_\rho} u \leq 2^\alpha \left(\frac{\rho}{r}\right)^\alpha \sup_{Q_r} u$$

where $\alpha = \alpha(n, \nu, m) = -\log_2 \theta > 0$.

Proof of Theorem 3.3. Comparing (11) with (6), we see that it suffices to prove the estimate

$$M_0 \leq N_0 M, \quad \text{with} \ \ N_0 = N_0(n, \nu, m). \tag{15}$$

We choose $\varepsilon_0 = \varepsilon_0(n, \nu, m) \in (0, \frac{1}{3})$ small enough, so that

$$\theta_0 = (1 - 3\varepsilon_0)^{-\gamma} \theta < 1, \tag{16}$$

where $\theta < 1$ is the constant in Lemma 3.4. For arbitrary $X = (x, t) \in Q_{2r}$, we consider separately two possible cases (a) and (b).

(a) $d = d(X) \le \varepsilon_0 d_0(X)$. In this case, $d = \operatorname{dist}(x, \partial\Omega) = |x - x_0|$ for some $x_0 \in \partial\Omega$. By Lemma 3.4 applied to u in $Q_{2d}(X_0) = Q \cap C_{2d}(X_0)$, $X_0 = (x_0, t)$, we have

$$u(X) \le \sup_{Q_d(X_0)} u \le \theta \sup_{Q_{2d}(X_0)} u. \tag{17}$$

Further, $d_0(X) \le d_0(Z) + |X - Z|$, where

$$|X - Z| = \max(|x - z|, |t - \tau|^{\frac{1}{2}}) \le 3d \le 3\varepsilon_0 d_0(X)$$

for arbitrary $Z = (z, \tau) \in Q_{2d}(X_0)$. Therefore, $(1 - 3\varepsilon_0) d_0(X) \le d_0(Z)$ for such Z, and together with (17), (16), (11), this gives us

$$d_0^\gamma u(X) \le (1 - 3\varepsilon_0)^{-\gamma} \theta \sup_{Q_{2d}(X_0)} d_0^\gamma u \le \theta_0 M_0. \tag{18}$$

(b) $d = d(X) > \varepsilon_0 d_0(X)$. Obviously, in this case,

$$d_0^\gamma u(X) \le \varepsilon_0^{-\gamma} d^\gamma u(X) \le N_0 M \quad \text{with } N_0 = \varepsilon_0^{-\gamma}. \tag{19}$$

Combining (18) and (19), we now have

$$M_0 = \sup_{Q_{2r}} d_0^\gamma u \le \max(\theta_0 M_0,\, N_0 M) = N_0 M,$$

because $\theta_0 < 1$. So the estimate (15) is proved. □

3.2. **Backward Harnack Inequalities.** The following *elliptic-type* Harnack inequality is similar to Theorem 2.6 in [G] (see also [FGS], Theorem 1.3, for the *divergence* case).

Theorem 3.6. *Let u be a nonnegative solution of $Lu = 0$ in $Q = \Omega \times (0, T)$ which continuously vanishes on $\partial_x Q$, and let $0 < \delta \le \frac{1}{2}\min(r_0, \sqrt{T})$. Then there exists a positive constant $N = N(n, \nu, m, r_0, \operatorname{diam}\Omega, T, \delta)$, such that*

$$\sup_{Q^\delta} u \le N \inf_{Q^\delta} u, \tag{20}$$

where Q^δ is defined in (4).

Proof. Applying the maximum principle, the boundary Harnack inequality (Theorem 3.3) and the Harnack principle (Theorem 2.2), we have

$$\sup_{Q^\delta} u \le \sup_{x \in \Omega} u\left(x, \frac{\delta^2}{4}\right) \le N_1 \sup_{x \in \Omega^{\mu\delta}} u\left(x, \frac{\delta^2}{2}\right) \le N \inf_{Q^\delta} u$$

where $\mu = \mu(m) > 0$, $N_1 = N_1(n, \nu, m)$, and $N = N(n, \nu, m, r_0, \operatorname{diam}\Omega, T, \delta)$. □

The *boundary backward* Harnack inequality is formulated as follows.

Theorem 3.7. *Let u be a nonnegative solution of $Lu = 0$ in $Q = \Omega \times (0, T)$ which continuously vanishes on $\partial_x Q$, and let $\delta = const > 0$. Then there exists a positive constant $N = N(n, \nu, m, r_0, \operatorname{diam}\Omega, T, \delta)$, such that*

$$u(x, s) \le N u(x, t) \tag{21}$$

where $T > s \ge t \ge s - d^2 \ge \delta^2 = \text{const} > 0$, $d = \operatorname{dist}(x, \partial\Omega)$.

We first prove an auxiliary result. For given $Y=(y,s)$, $r>0$ and $k>0$, we set

$$V_1=\Omega_{kr}\times(s-r^2,s),\quad V_2=\Omega_{2kr}\times(s-4r^2,s), \tag{22}$$

where $\Omega_{kr}=\Omega\cap B_{kr}(y)$, $\Omega_{2kr}=\Omega\cap B_{2kr}(y)$. The parabolic boundary of V_2 is represented in the form $\partial' V_2=\Gamma_0\cup\Gamma_1\cup\Gamma_2$, where

$$\begin{aligned}&\Gamma_0\subset Q_0\backslash\partial' Q_0,\quad \Gamma_1\subset\partial_x Q_0,\quad \Gamma_2\subset\partial_t Q_0,\\ &Q_0=B_{2kr}(y)\times\left(s-4r^2,s\right).\end{aligned} \tag{23}$$

Lemma 3.8. *Let $Y=(y,s)\in\overline{Q}$ and let positive constants r and γ be given. There exists a constant $k=k(n,\nu,\gamma)\geq 8$ such that, for any nonnegative solution of $Lu=0$ in V_2 which continuously vanishes on Γ_0, from the inequality*

$$M=\sup_{V_1}u>(2k)^{-\gamma}\sup_{\Gamma_1}u \tag{24}$$

it follows

$$\sup_{\Gamma_2}u>\frac{1}{2}M. \tag{25}$$

Proof. Using the transformations:

$$x\longrightarrow(2r)^{-1}(x-y),\ t\longrightarrow(2r)^{-2}(t-s)+1,\ \text{and } u\longrightarrow const\cdot u,$$

we reduce the proof to the case

$$y=0,\ \ s=1,\ \ r=\frac{1}{2},\ \ \sup_{\Gamma_1}u=1,$$

so that $V_1=\Omega_{\frac{k}{2}}\times(\frac{3}{4},1)$, $V_2=\Omega_k\times(0,1)$. Next, we show that for the proof of the lemma it suffices to construct a function $v(x,t)$ satisfying the inequalities

$$v\geq 0,\ \ Lv\leq 0\ \text{ in } V_2,\ \ v\leq Ne^{-\frac{k^2}{N}}\ \text{ in } V_1,\ \text{ and } v\geq 1\ \text{ on } \Gamma_1 \tag{26}$$

with constants $N=N(n,\nu)>0$. Indeed, then we have

$L(u-v)=-Lv\geq 0$ in V_2, and $u-v\leq 0$ on $\Gamma_0\cup\Gamma_1=\partial' V_2\setminus\Gamma_2$. Therefore,

$$\sup_{\Gamma_2}u\geq\sup_{\Gamma_2}(u-v)=\sup_{\partial' V_2}(u-v)$$

$$=\sup_{V_2}(u-v)\geq\sup_{V_1}(u-v)\geq M-Ne^{-\frac{k^2}{N}},$$

where $N=N(n,\nu)>0$. We can choose $k=k(n,\nu,\gamma)\geq 8$ such that

$$Ne^{-\frac{k^2}{N}}\leq\frac{1}{2}(2k)^{-\gamma}=\frac{1}{2}(2k)^{-\gamma}\sup_{\Gamma_1}u.$$

This gives us

$$\sup_{\Gamma_2}u\geq M-\frac{1}{2}(2k)^{-\gamma}\sup_{\Gamma_1}u.$$

This inequality together with (24) yields (25). So it remains to construct $v(x,t)$ satisfying (26).

Consider the function

$$v=k_0s^{-\alpha}e^{\frac{\beta|x|^2}{s}},\quad \text{where } s=2-t,\ \ k_0=2^{\alpha}e^{-\frac{\beta k^2}{2}},$$

and α, β are constants. We have $v \geq 0$,

$$\begin{aligned} Lv &= \sum_{i,j} a_{ij} D_{ij} v + v_s \\ &= \frac{v}{s}\left(2\beta \sum_i a_{ii} - \alpha\right) + \frac{\beta v}{s^2}\left(4\beta \sum_{i,j} a_{ij} x_i x_j - |x|^2\right) \leq 0 \end{aligned}$$

in V_2 for some α, $\beta > 0$ depending only on n, ν. Further,

$$|x| < \frac{k}{2}, \quad v \leq k_0 s^{-\alpha} e^{\frac{\beta k^2}{4s}} \leq 2^\alpha e^{-\frac{\beta k^2}{4}} \leq N e^{-\frac{k^2}{N}} \quad \text{in } V_1.$$

Finally, $|x| = k$ and $1 < s < 2$ imply $v \geq 1$ on Γ_1. Hence v satisfies (26), and the proof of the lemma is completed. □

Proof of Theorem 3.7. Denote $\rho_0 = \frac{1}{2}\min(r_0, \delta_0) > 0$. If $d > \rho_0$, then (21) follows directly from (20) with $\delta = \rho_0$, so we may assume $d \leq \rho_0$.

We choose $y \in \partial\Omega$ such that $|x - y| = d$, set $Y = (y, s)$, and consider the function

$$f(\rho) = \rho^{-\gamma} \sup_{Q_\rho} u, \quad \text{where } Q_\rho = Q \cap C_\rho(Y),$$

and $\gamma = \gamma(n, \nu, m) > 0$ is the constant in Lemma 3.2. Now we define

$$r = \max\{\rho : d \leq \rho \leq \rho_0,\ f(\rho) \geq f(d)\}.$$

The inequality $f(d) \leq f(r)$ implies

$$u(x, s) \leq \sup_{Q_d} u \leq \left(\frac{d}{r}\right)^\gamma \sup_{Q_r} u.$$

By Lemma 3.2 we also have

$$u(\underline{Y}_r) \leq N \left(\frac{r}{d}\right)^\gamma u(x, t).$$

These two estimates reduce the proof of (21) to the inequality

$$M_1 = \sup_{Q_r} u \leq N u(\underline{Y}_r). \tag{27}$$

In the proof of this inequality, we consider separately two cases (a) and (b).

(a) $\frac{\rho_0}{2k} \leq r \leq \rho_0$, where k is a constant in Lemma 3.8 corresponding to our γ. Since r is comparable with ρ_0, from Theorem 3.6 it follows $u(\overline{Y}_r) \leq N u(\underline{Y}_r)$. This estimate and (12) yield (27).

(b) $d \leq r < \frac{\rho_0}{2k}$. By definition of r, $f(r) > f(2kr)$; hence

$$M_1 = \sup_{Q_r} u > (2k)^{-\gamma} \sup_{Q_{2kr}} u. \tag{28}$$

It is easy to see that $Q_r \subset V_1$, $\overline{Q_{2kr}} \supset \Gamma_1$, where V_1, Γ_1 are defined in (22), (23). Therefore, (28) implies (24) and (25). The last estimate means

$$M_1 < 2u(Z_0) \quad \text{for some } Z_0 = (z_0,\, s - 4r^2),\ z_0 \in \overline{\Omega_{2kr}}. \tag{29}$$

If $\operatorname{dist}(z_0, \partial\Omega) < r$, then $|z_0 - z| < r$ for some $z \in \partial\Omega$, and by (12)

$$u(Z_0) \leq \sup_{Q_r(Z)} u \leq N u(\overline{Z}_r).$$

The Harnack inequality (5) guarantees $u(\overline{Z}_r) \le Nu(\underline{Y}_r)$, where by "scaling" arguments N depends only on n, ν, m. Therefore,

$$u(Z_0) \le Nu(\underline{Y}_r). \tag{30}$$

If $\operatorname{dist}(z_0, \partial\Omega) \ge r$, then we can use the Harnack inequality directly to get (30). Finally, (29) and (30) provide the estimate (27). □

4. Hölder Continuity of Quotients

We now begin the discussion leading to the proof of the second main result: the quotient of two solutions of a parabolic *non-divergence* equation, both solutions vanishing on a portion of the lateral boundary, is locally Hölder continuous up to that portion of the boundary. In this section we need some further *Notation:* for $Y = (y, s) \in \mathbb{R}^{n+1}$ with $y \in \partial\Omega$, $0 < r \le R$, we set

$$\begin{aligned}
\Omega_R(y) &= B_R(y) \cap \Omega, \\
\Omega_{R,r}(y) &= \Omega_R(y) \cap \{x \in \Omega : d(x) < r\}, \\
Q_R(Y) &= Q_R(y,s) = \Omega_R(y) \times (s - R^2, s), \\
Q_{R,r}(Y) &= Q_{R,r}(y,s) = \Omega_{R,r}(y) \times (s - R^2, s), \\
S_{R,r}(Y) &= \{(x,t) \in \partial' Q_{R,r}(Y) : d(x) = r\}, \\
\Gamma_{R,r}(Y) &= \{(x,t) \in \partial' Q_{R,r}(Y) : 0 < d(x) < r\},
\end{aligned}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$.

4.1. **Some Lemmas.** We first present two auxiliary lemmas, which are interesting for their own sake.

Lemma 4.1. *Let $Y = (y,s) \in \mathbb{R}^{n+1}$ with $y \in \partial\Omega$, $r > 0$, $K \ge 6$, $Kr \le r_0$, and let u be a solution of $Lu \ge 0$ in $Q_{Kr,r}(Y)$ satisfying the following conditions:* 1. $u \le 1$ *in $Q_{Kr,r}(Y)$,* 2. $u \le 0$ *on $(\partial' Q_{Kr,r}) \setminus \Gamma_{Kr,r}$. Then we have*

$$\sup_{Q_r(Y)} u \le e^{-NK}, \tag{31}$$

where $N = N(n, \nu, m) > 0$.

Proof. We prove this decay estimate by iteration. By scaling $x \to r^{-1}x$, $t \to r^{-2}t$, we may assume $r = 1$. Let $j \ge 1$, $2j + 1 \le K$, $\sup_{Q_{2j-1,1}} u = u(X_j)$ for some $X_j \in \partial' Q_{2j-1,1}$. Since $\partial\Omega$ satisfies the Lipschitz condition, we may apply Lemma 3.4 with a constant $\varepsilon = \varepsilon(n, \nu, m) \in (0,1)$ to $U = C_2(X_j) \cap Q_{K,1} \subset Q_{2j+1,1}$, then we have

$$\sup_{Q_{2j-1,1}} u = u(X_j) \le \theta \sup_U u \le \theta \sup_{Q_{2j+1,1}} u,$$

where $\theta = \theta(n, \nu, m) \in (0,1)$. Notice that $Q_{1,1} = Q_1$. Iterating the above estimate, we obtain

$$\sup_{Q_1} u \le \theta^k \sup_{Q_{2k+1,1}} u \le \theta^k$$

where $2k + 1 \le K \le 2k + 3$. Since $k > \frac{K-3}{2} \ge \frac{K}{4}$, we get the desired estimate (31) with $N = N(n, \nu, m) = -\frac{1}{4} \ln \theta > 0$. □

Lemma 4.2. *Let $Y = (y,s) \in \partial\Omega \times \mathbb{R}^1$ be fixed, and let u, v satisfy the conditions*

$$\begin{array}{ll} 1.\ Lu \le 0, \quad u \ge 0 \quad \text{in} \quad Q_{Kr,r}, \quad u \ge 1 \quad \text{on} \quad S_{Kr,r}, \\ 2.\ Lv \ge 0, \quad v \le 1 \quad \text{in} \quad Q_{Kr,r}, \quad v \le 0 \quad \text{on} \quad (\partial' Q_{Kr,r}) \setminus \Gamma_{Kr,r}, \end{array} \tag{32}$$

where $K \ge 6$, $0 < Kr \le r_0$. Then we have

$$v \le u \qquad in \quad Q_r = Q_r(Y), \tag{33}$$

provided $K = K(n,\nu,m)$ is large enough.

Proof. As before, we may assume $r = 1$. First we prove the estimate

$$u(X) = u(x,t) \ge 2\delta d^\gamma(x) \ \text{ in } \ Q_1 = Q_1(Y) \tag{34}$$

with positive constants δ and γ depending only on n, ν, m.

We choose $R = R(m) \ge 6$ and $\tilde{y} \in \Omega$, such that $|\tilde{y} - y| = R$ and $B_2(\tilde{y}) \subset \Omega_{R+2} = \Omega \cap B_{R+2}(y)$, and assume $K \ge R$. Next, we define $\tilde{u}$ in $Q_R = Q \cap C_R(Y)$ as the (unique) solution of the equation $L\tilde{u} = 0$ in Q_R with the boundary values

$$\tilde{u} = \min(u,1) \quad \text{on} \quad (\partial' Q_R) \cap (\partial' Q_{R,1}), \qquad \tilde{u} = 1 \quad \text{on} \quad (\partial' Q_R) \setminus (\partial' Q_{R,1}).$$

By the comparison principle, $0 \le \tilde{u} \le 1$ in Q_R. Moreover, since

$$u \ge 1 \quad \text{on} \quad S_{K,1} \supseteq S_{R,1} \supseteq (\partial' Q_{R,1}) \setminus (\partial' Q_R),$$

we have $u \ge \tilde{u}$ on $\partial' Q_{R,1}$, and hence $u \ge \tilde{u}$ in $Q_{R,1} \supseteq Q_1$. Further, we set

$$z = \tilde{y} + R^{-1}(\tilde{y} - y), \quad \tilde{z} = \tilde{y} - R^{-1}(\tilde{y} - y), \quad \tilde{Y} = (\tilde{y}, s-4), \quad \tilde{Z} = (\tilde{z}, s-4).$$

It is easy to see that

$$B_1(z) \subset B_2(\tilde{y}) \setminus \Omega_R, \qquad B_1(\tilde{z}) \subset B_2(\tilde{y}) \cap \Omega_R.$$

We can apply Lemma 3.4 to the function $1 - \tilde{u}$ in $U = Q_R(Y) \cap C_2(\tilde{Y})$, which vanishes on $(\partial' Q_R) \setminus (\partial' Q_{R,1}) \supseteq (\partial' U) \setminus (\partial' C_2(\tilde{Y}))$. This gives us

$$1 - \tilde{u}(\tilde{Z}) \le \sup_{U \cap C_1(\tilde{Y})} (1 - \tilde{u}) \le \theta \sup_U (1 - \tilde{u}) \le \theta = \theta(n,\nu) < 1,$$

and $\tilde{u}(\tilde{Z}) \ge 1 - \theta > 0$. By the Harnack principle, Theorem 2.2,

$$\tilde{u}(\underline{Y}_1) \ge N^{-1} \tilde{u}(\tilde{Z}) \ge \delta_0 = \delta_0(n,\nu,m) > 0.$$

Now applying Lemma 3.2 to $\tilde{u}$ in $Q_R \supset Q_2 \supset Q_1$, we have

$$\tilde{u}(X) = \tilde{u}(x,t) \ge N^{-1} d^\gamma(x) \tilde{u}(\underline{Y}_1) \ge 2\delta d^\gamma(x) \quad \text{in } Q_1,$$

where $\gamma = \gamma(n,\nu,m) > 0$, $\delta = \delta(n,\nu,m) > 0$. Since $u \ge \tilde{u}$ in Q_1, the estimate (34) is proved for $K \ge R$. In particular, for such K we have

$$u \ge 2\delta K^{-\gamma} \quad \text{in} \quad \overline{Q_1 \setminus Q_{1,K^{-1}}} \supset S_{1,K^{-1}}.$$

It follows from Lemma 4.1 that

$$v \le e^{-NK} \le \delta K^{-\gamma} \quad \text{in} \quad Q_1,$$

provided $K = K(n,\nu,m)$ is chosen large enough. Then

$$\begin{aligned} u_1 &= \frac{K^\gamma}{2\delta} u \ge 0 \quad \text{in} \quad Q_{1,K^{-1}}, \quad u_1 \ge 1 \quad \text{in} \quad \overline{Q_1 \setminus Q_{1,K^{-1}}} \supset S_{1,K^{-1}}, \\ v_1 &= \frac{K^\gamma}{2\delta}(2v - u) \le \frac{K^\gamma}{\delta} v \le 1 \quad \text{in} \quad Q_1 \supset Q_{1,K^{-1}}, \quad v_1 \le 0 \quad \text{on} \quad S_{1,K^{-1}}, \end{aligned}$$

and hence

$$u_1 - v_1 = \frac{K^\gamma}{\delta}(u-v) \geq 0 \quad \text{in} \quad \overline{Q_1 \backslash Q_{1,K^{-1}}}.$$

In particular, u_1, v_1 satisfy the same assumption as (32) with $r = K^{-1}$. By iteration, we can construct u_j, v_j such that

$$u_j - v_j = \left(\frac{K^\gamma}{\delta}\right)^j (u-v) \geq 0 \quad \text{in} \quad \overline{Q_{K^{1-j}} \backslash Q_{K^{1-j},K^{-j}}}$$

for all $j = 1, 2, 3, \cdots$. As a consequence,

$$u - v \geq 0 \quad \text{on} \quad I(Y) = \bigcup_{j=1}^{\infty} \overline{Q_{K^{1-j}}(Y) \backslash Q_{K^{1-j},K^{-j}}(Y)}.$$

For arbitrary $X_0 = (x_0, t_0) \in Q_1 = Q_1(Y)$, we can take $Y_0 = (y_0, t_0)$ with $y_0 \in \partial\Omega$ satisfying $d(x_0) = \operatorname{dist}(x_0, \partial\Omega) = |x_0 - y_0|$. Then $X_0 \in I(Y_0)$. Moreover, $|y_0 - y| \leq |x_0 - y| + |x_0 - y_0| < 2$; therefore,

$$Q_{K,1}(Y_0) \subset Q_{K+2,1}(Y), \qquad S_{K,1}(Y_0) \subset S_{K+2,1}(Y).$$

Replacing K with $K+2$, we conclude $u - v \geq 0$ on $I(Y_0) \ni X_0$. Since X_0 is an arbitrary point in Q_1, we arrive at (33). □

Remark. In terms of the L-caloric measure ω^X (see [FGS, p. 540]) corresponding to L and $Q_{Kr,r}(Y)$, (33) says

$$\omega^X(S_{Kr,r}) \geq \omega^X(\Gamma_{Kr,r}) \quad \text{for} \quad X \in Q_r(Y).$$

4.2. **Boundedness of Quotients.**

Theorem 4.3. *Fix* $Y = (y,s) \in \partial\Omega \times (0,\infty)$ *with* $0 < Kr < \frac{1}{2}\min(r_0, \sqrt{s})$, *where* K *is the constant in* Lemma 4.2. *Assume* u *and* v *are two nonnegative solutions of* $Lu = 0$ *in* $\Omega \times (0,\infty)$, *and* $v = 0$ *on* $C_{2Kr}(Y) \cap (\partial\Omega \times (0,\infty))$*; then*

$$\sup_{Q_r(Y)} \frac{v}{u} \leq N(n,\nu,m) \frac{v(\overline{Y}_{Kr})}{u(\underline{Y}_{Kr})}. \tag{35}$$

Proof. By scaling, we may assume $r = 1$, $u(\underline{Y}_K) = v(\overline{Y}_K) = 1$. By the boundary Harnack inequality, Theorem 3.3,

$$v \leq N_0(n,\nu,m) \qquad \text{in} \qquad Q_K \supset Q_{K,1}.$$

By Lemma 3.2 (or Theorem 2.2),

$$u \geq \frac{1}{N_0(n,\nu,m)} \qquad \text{on} \qquad S_{K,1}.$$

Applying Lemma 4.2 to the functions $u_0 = N_0 u$ and $v_0 = N_0^{-1} v - u_0$, we get

$$\sup_{Q_1} \frac{v}{u} = N_0^2 \sup_{Q_1}\left(\frac{v_0}{u_0} + 1\right) \leq 2N_0^2 = N(n,\nu,m),$$

the desired estimate (35). □

Remark. The above estimate (35) was first proved in [G] (Theorem 3.1) for the *non-divergence* case and C^2-domains and in [FGS] (Theorem 1.6) for the *divergence* case and Lipschitz domains. For elliptic equations (*divergence* and *non-divergence*), it was proved in [CFMS] (Theorem 1.4), [B] (Theorem 2.1), and [FGMS] (Theorem I.3.7).

4.3. **Oscillation Decay.** In the following two theorems, we use the *notation*

$$\omega(X,r) = \operatorname*{osc}_{Q_r(X)} \frac{v}{u}, \text{ where } Q_r(X) = Q \cap C_r(X).$$

Theorem 4.4. *Assume u and v are two strictly positive solutions of $Lu = 0$ in $Q = \Omega \times (0,\infty)$ and also $u = 0$ on $\partial_x Q = \partial\Omega \times (0,\infty)$.*

(a) Let $X = (x,t) \in Q$, $t > \delta^2 = \text{const} > 0$, and $0 < r \le \frac{1}{2} d(X)$, where $d(X) = \min(d(x), \sqrt{t})$. Then

$$\omega\left(X, \frac{r}{2}\right) \le \theta_0 \omega(X,r), \tag{36}$$

where $\theta_0 = \theta_0(n,\nu,m,r_0, \text{diam}\,\Omega, \delta) \in (0,1)$.

(b) Let $Y = (y,s) \in \partial_x Q$, $s \ge \delta^2 = \text{const} > 0$ and $0 < Kr \le \frac{1}{2}\min(r_0, \sqrt{s})$, where K is the constant in Lemma 4.2. *Let $v = 0$ on $C_{Kr}(Y) \cap \partial_x Q$. Then*

$$\omega\left(Y, \frac{r}{2K}\right) \le \theta_1 \omega(Y,r), \tag{37}$$

where $\theta_1 = \theta_1(n,\nu,m,r_0, \text{diam}\,\Omega, \delta) \in (0,1)$.

Proof. (a) Denote $X^{\pm} = (x, t \pm r^2/2)$. We may assume

$$0 \le \frac{v}{u} \le 1 = \omega(X,r) = \operatorname*{osc}_{C_r(X)} \frac{v}{u} \text{ in } C_r(X), \text{ and } \frac{v}{u}(X^-) \ge \frac{1}{2};$$

otherwise in place of v we take $c_1 u + c_2 v$ with some constants c_1, c_2. By the Harnack principle (Theorem 2.2),

$$v(X^-) \le Nv, \quad u \le Nu(X^+) \text{ in } C_{\frac{r}{2}}(X).$$

Moreover, by the boundary backward Harnack inequality (Theorem 3.7),

$$u(X^+) \le N_1 u(X^-).$$

Thus

$$\frac{1}{2} \le \frac{v}{u}(X^-) \le N_2 \frac{v}{u} \le N_2 \text{ in } C_{\frac{r}{2}}(X),$$

which implies (36) with $\theta_0 = 1 - \frac{1}{2N_2}$.

(b) According to Theorem 4.3, $\omega(Y,r) < \infty$. As before, we may assume

$$0 \le \frac{v}{u} \le 1 = \omega(Y,r) \quad \text{in} \quad Q_r(Y), \text{ and } \frac{v}{u}\left(\underline{Y}_{\frac{r}{2}}\right) \ge \frac{1}{2}.$$

Applying Theorem 4.3 again and then Theorem 3.7, we get

$$\sup_{Q_{\frac{r}{2K}}} \frac{u}{v} \le N \frac{u\left(\overline{Y}_{\frac{r}{2}}\right)}{v\left(\underline{Y}_{\frac{r}{2}}\right)} \le N_3 \frac{u}{v}\left(\underline{Y}_{\frac{r}{2}}\right) \le 2N_3 = 2N_3(n,\nu,m,r_0,\text{diam}\,\Omega,\delta).$$

Thus

$$\frac{1}{2N_3} \le \frac{v}{u} \le 1 \text{ in } Q_{\frac{r}{2K}}(Y),$$

which implies (37) with $\theta_1 = 1 - \frac{1}{2N_1}$. □

Theorem 4.5. *Let u and v be two strictly positive solutions of $Lu = 0$ in $Q = \Omega \times (0,\infty)$ such that $u = 0$ on $\partial_x Q = \partial\Omega \times (0,\infty)$, and $v = 0$ on $C_{Kr_1}(Y) \cap \partial_x Q$, where $Y = (y,s) \in \partial_x Q$, $s \ge \delta^2 = \text{const} > 0$, K is the constant in Lemma 4.2, and $r_1 = \frac{1}{6K}\min(r_0, \sqrt{s})$. Then the quotient $\frac{v}{u}$ is Hölder continuous in $\overline{Q}_{r_1} = \overline{Q_{r_1}}(Y)$.*

Proof. By Theorem 4.3, we have

$$\omega(Y, 3r_1) \le N \frac{v\left(\overline{Y}_{3Kr_1}\right)}{u\left(\underline{Y}_{3Kr_1}\right)} = N_0 < \infty. \tag{38}$$

For arbitrary $X = (x,t) \in \overline{Q}_{r_1}$ and $0 < r \le r_1$, we deal with three cases:

(a) $0 < r \le d \le r_1$, (b) $0 \le d < r \le \frac{1}{2}r_1$, and (c) $d < r$, $\frac{1}{2}r_1 < r \le r_1$, where $d = d(x) = \mathrm{dist}(x, \partial\Omega)$.

Case (a). Iterating (36), we get

$$\omega(X,r) \le 2^{\alpha_0}\left(\frac{2r}{d}\right)^{\alpha_0}\omega\left(X, \frac{d}{2}\right) \quad \text{for } 0 < r \le \frac{d}{2},$$

where $\alpha_0 = -\log_2\theta_0 > 0$. This implies the estimate

$$\omega(X,r) \le \left(\frac{4r}{d}\right)^{\alpha_0}\omega(X,d), \tag{39}$$

which is also true for $\frac{d}{2} < r \le d$.

Case (b). In this case, $Q_r(X) \subset Q_{2r}(X_0)$ for some $X_0 = (x_0, t) \in \partial_x Q$ with $|x - x_0| = d = d(x)$. By iterating estimate (37), we have

$$\omega(X,r) \le \omega(X_0, 2r) \le \left(\frac{4Kr}{r_1}\right)^{\alpha_1}\omega(X_0, r_1) \quad \text{for} \quad d \le r \le \frac{1}{2}r_1, \tag{40}$$

where $\alpha_1 = -\log_{2K}\theta_1 > 0$.

Combining (39) and (40), we get

$$\omega(X,r) \le \left(\frac{4r}{d}\right)^{\alpha_0}\omega(X,d) \le \left(\frac{4r}{d}\right)^{\alpha_0}\left(\frac{4Kd}{r_1}\right)^{\alpha_1}\omega(X_0,r_1) \quad \text{for } 0 < r \le d \le \frac{1}{2}r_1.$$

Notice that $Q_{r_1}(X_0) \subset Q_{2r_1}(X) \subset Q_{3r_1}(Y)$; hence by virtue of (38), $\omega(X_0, r_1) \le \omega(Y, 3r_1) \le N_0$. We set $\alpha = \min(\alpha_0, \alpha_2)$. We may assume that the constants θ_0, θ_1 are close to 1, so that $\alpha_0, \alpha_1, \alpha \in (0,1)$. Then the above estimate gives us

$$\omega(X,r) \le 16K\left(\frac{r}{r_1}\right)^{\alpha} N_0 \tag{41}$$

for arbitrary $X \in \overline{Q}_{r_1}$ and $0 < r \le d \le \frac{1}{2}r_1$. If $0 < r \le \frac{1}{2}r_1 < d \le r_1$, it is a consequence of (39). If $d < r \le \frac{1}{2}r_1$, this estimate follows immediately from (40). In case (c) and the remaining of case (a), we have $r > \frac{1}{2}r_1$, which also implies (41). Thus the estimate (41) holds for all $X \in \overline{Q}_{r_1}$, $0 < r \le r_1$, and this provides the Hölder continuity of the quotient $\frac{v}{u}$. □

Finally, we are ready to prove the Hölder continuity of the quotient of $\frac{v}{u}$ with u and v vanishing on an open portion of the lateral boundary.

Theorem 4.6. *Let u and v be strictly positive solutions of $Lu = 0$ in $Q = \Omega \times (0,\infty)$, vanishing on $C_{2r}(Y_0) \cap \partial_x Q$, where $Y_0 = (y_0, s_0) \in \partial_x Q = \partial\Omega \times (0,\infty)$ and $s_0 \ge 4r^2 > 0$. Then $\frac{v}{u}$ is Hölder continuous in $\overline{Q_r}(Y_0)$.*

Proof. First we assume $u \equiv 0$ on $\partial_x Q$. The Hölder continuity of u and v in any subdomain $Q' \subset \overline{Q'} \subset Q$, which is known from [KS], implies the same property for $\frac{v}{u}$ (it can also be obtained by iteration of (39)). Moreover, by Theorem 4.5, $\frac{v}{u}$ is Hölder continuous in $\overline{Q_{r_1}}(Y)$ for all $Y \in C_r(Y_0) \cap \partial_x Q$ and small $r_1 > 0$. Combining these two facts, we get the Hölder continuity of $\frac{v}{u}$ in $\overline{Q_r}(Y_0)$.

In the general case, as in [FS], we represent u in the form $u = u_0 + u_1$, where

$$\begin{aligned} Lu_0 &= Lu_1 = 0 \quad \text{in} \quad Q, \\ u_0 &= 0, \quad u_1 = u \quad \text{on} \quad \partial_x Q, \\ u_0 &= u, \quad u_1 = 0 \quad \text{on} \quad \partial_t Q = \overline{\Omega} \times \{0\}. \end{aligned}$$

Here without loss of generality we may assume $u > 0$ on $\Omega \times \{0\}$; otherwise we replace t by $t + \text{const}$. Then $u_0 > 0$ in Q, and the previous arguments show that $\frac{v}{u_0}$ and $\frac{u_1}{u_0}$ are Hölder continuous in $\overline{Q_r}\,(Y_0)$. Hence the same holds true for

$$\frac{v}{u} = \frac{v}{u_0} \cdot \frac{1}{1 + \frac{u_1}{u_0}}.$$

This completes the proof. □

Remark. The similar result for divergence case in [FS] (Theorem 8) was obtained by employing Green's function. Our proof of Theorem 4.4 also works for the divergence case, since the boundary backward Harnack inequality is available (Theorem 4 in [FS]).

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School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455-0100

E-mail address: `safonov@math.umn.edu`

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455-0100

Current address: Department of Mathematics, University of Texas at Austin, Austin, Texas 78712

E-mail address: `yyuan@math.utexas.edu`