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A Bernstein problem for special Lagrangian equations

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1. Introduction

In this paper we derive a Bernstein type result for the special Lagrangian equation

$$F\left(D^2u\right) = \arctan\lambda_1 + \cdots + \arctan\lambda_n = c, \tag{1.1}$$

where $\lambda_i s$ are the eigenvalues of the Hessian D^2u. Namely, any global convex solution to (1.1) in R^n must be a quadratic polynomial. Recall the classical result, any global convex solution in R^n to the Laplace equation $\triangle u = \lambda_1 + \cdots + \lambda_n = c$ or the Monge-Ampère equation $\log\det D^2u = \log\lambda_1 + \cdots + \log\lambda_n = c$ must be quadratic.

Equation (1.1) originates from special Lagrangian geometry [HL]. The (Lagrangian) graph $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is constant c or u satisfies (1.1), and it is special if and only if $(x, \nabla u(x))$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17].

In terms of minimal surface, our result is the following

Theorem 1.1. *Suppose $M = (x, \nabla u)$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ and u is a smooth convex function in $\mathbb{R}^n$. Then M is a plane.*

In fact, we have stronger results.

Theorem 1.2. *Suppose $M = (x, \nabla u)$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ and u is a smooth function in $\mathbb{R}^n$ whose Hessian satisfies $D^2u \geq -\epsilon(n)I$, where $\varepsilon(n)$ is a small dimensional constant. Then M is a plane.*

Theorem 1.3. *Suppose $M = (x, \nabla u)$ is a minimal surface in $\mathbb{R}^3 \times \mathbb{R}^3$ and u is a smooth function in $\mathbb{R}^3$ whose Hessian satisfies $D^2 u \geq -CI$. Then M is a plane.*

The lower bound on the Hessian D^2u is necessary for Theorem 1.3, as one sees from the following example. Let u be a harmonic function in $\mathbb{R}^2$, say, $u = x_1^3 - 3x_1x_2^2$, then $(x, \nabla u(x))$ is a minimal surface in $\mathbb{R}^4$, which is not a plane.

Borisenko [Bo] proved Theorem 1.1 under the additional assumption that u grows linearly at ∞ and $\arctan\lambda_1 + \cdots + \arctan\lambda_n = k\pi$. For $c = k\pi$, the special Lagrangian equation (1.1) in R^3 also takes the form

$$\triangle u = \det D^2 u. \tag{1.2}$$

It was proved in [BCGJ] that any strictly convex solution to (1.2) in $\mathbb{R}^3$ with quadratic growth at ∞ must be quadratic.

Fu [F] showed that any global minimal surface $(x, \nabla u(x)) \subset \mathbb{R}^2 \times \mathbb{R}^2$ is either a plane or the potential u is harmonic. This result also follows from Theorem 1.3 easily. We may assume $c \geq 0$ in the special Lagrangian equation $\arctan\lambda_1 + \arctan\lambda_2 = c$. Then either $c = 0$, that is $\triangle u = 0$, or $\left(D^2u\right) > -\frac{1}{\tan c} I$, which in turn implies that u is quadratic by Theorem 1.3.

The heuristic idea of the proof of Theorem 1.1 is to find a subharmonic function S in terms of the Hessian D^2u such that S achieves its maximum at a finite point in R^n. By the strong maximum principle, S is constant. Consequently, D^2u is a constant matrix. The right function S is the one associated to the volume form of M in R^{2n}, $\det\left(I + D^2uD^2u\right)$, see Lemma 2.1. However the nonnegative Hessian D^2u has no upper bound. We make a (Lewy) rotation of the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to $\bar{x} = (x + y)/\sqrt{2}$, $\bar{y} = (-x + y)/\sqrt{2}$. The special Lagrangian property of M is invariant, and M has a new representation $\left(\bar{x}, \nabla\bar{u}\left(\bar{x}\right)\right)$ with the potential function $\bar{u}$ satisfying $-I \leq \left(D^2\bar{u}\right) = \left(I + D^2u\right)^{-1}\left(-I + D^2u\right) \leq I$. To make the whole idea work, we need the machinery from geometric measure theory, see Sect. 3.

Note that the special Lagrangian feature of the minimal surface $M = \left(\bar{x}, \nabla\bar{u}\left(\bar{x}\right)\right)$ is essential in finding a subharmonic function. The function $\det\left(I + D^2\bar{u}D^2\bar{u}\right)$ is subharmonic as long as $-I \leq \left(D^2\bar{u}\right) \leq I$, in which case $\det\left(I + D^2\bar{u}D^2\bar{u}\right) \leq 2^n$. In contrast, for general minimal surface $M = (x, f(x)) \subset R^n \times R^k$ with high co-dimension $k \geq 2$, assuming that

$$\det\left[I + (\nabla f)^t(\nabla f)\right] \leq K < \left[\cos\left(\pi/\left(2\sqrt{2p}\right)\right)\right]^{-2p}$$

with $p = \min\{n, k\}$, Fischer-Colbrie [F-C] and Hildebrandt, Jost, and Widman [HJW] were able to show that the composition of the square of the distance function on the Grassmanian manifold $G(n, k)$ with the harmonic map from M to $G(n, k)$ is subharmonic. Later on, Jost and Xin [JX] proved the same thing under the assumption that $\det\left[I+(\nabla f)^t(\nabla f)\right]\le K<4$. As a consequence, Bernstein type results were obtained in all these papers.

Theorem 1.2 is just a consequence of Allard's ε-regularity theory, once Theorem 1.1 is available.

Theorem 1.3 relies on the well-known result that any non-parametric minimal cone of dimension three must be flat, see [F-C] and [B]. A quick "PDE" proof of this fact was found in a recent paper [HNY]. Whether Theorem 1.3 holds true in higher dimensional case remains an issue to us.

Notation. $\partial_i=\frac{\partial}{\partial x_i},\ \partial_{ij}=\frac{\partial^2}{\partial x_i\partial x_j},\ u_i=\partial_i u,\ u_{ji}=\partial_{ij}u$, etc.

2. Preliminary computations

Let $(x,\nabla u(x))\subset\mathbb{R}^n\times\mathbb{R}^n$ be a non-parametric minimal surface, then we have

$$\triangle_g(x,\nabla u(x))=0, \tag{2.1}$$

where $\triangle_g=\sum_{i,j=1}^n\frac{1}{\sqrt{\det g}}\partial_i\left(\sqrt{\det g}\,g^{ij}\partial_j\right)$ is the Laplace-Beltrami operator of the induced metric $g=\left(g_{ij}\right)=\left(I+D^2uD^2u\right)$ with $\left(g^{ij}\right)=\left(g_{ij}\right)^{-1}$. Notice that $\triangle_g x=0$, $\triangle_g$ also takes the form

$$\triangle_g=\sum_{i,j=1}^n g^{ij}\partial_{ij}. \tag{2.2}$$

Lemma 2.1. *Let $(x,\nabla u(x))\subset\mathbb{R}^n\times\mathbb{R}^n$ be a minimal surface. Suppose the Hessian D^2u of the smooth function u is diagonalized at p, $D^2u(p)=diag[\lambda_1,\cdots,\lambda_n]$. Then*

(2.3)

$$\triangle_g\log\det g=\sum_{i,j=1}^n g^{ij}\partial_{ij}\log\det g\overset{p}{=}\sum_{a,b,c=1}^n 2g^{aa}g^{bb}g^{cc}u_{abc}^2\left(1+\lambda_b\lambda_c\right).$$

Proof. As preparation, we first compute the first and second order derivatives of the metric g.

$$\begin{aligned}\partial_j g_{ab}&=\sum_{k=1}^n\left(u_{akj}u_{kb}+u_{ak}u_{kbj}\right)\\&\overset{p}{=}u_{abj}\left(\lambda_a+\lambda_b\right).\end{aligned} \tag{2.4}$$

$$\begin{aligned}\partial_i g^{ab}&=\sum_{k=1}^n-g^{ak}\partial_i g_{kl}g^{lb}\\&\overset{p}{=}-g^{aa}\partial_i g_{ab}g^{bb}\\&\overset{p}{=}-g^{aa}g^{bb}u_{abi}\left(\lambda_a+\lambda_b\right).\end{aligned} \tag{2.5}$$

$$\partial_{ij} g_{ab} = \sum_{k=1}^{n} \left(u_{akji} u_{kb} + u_{akj} u_{kbi} + u_{aki} u_{kbj} + u_{ak} u_{kbji} \right)$$
$$\overset{p}{=} u_{abji} \left(\lambda_a + \lambda_b \right) + \sum_{k=1}^{n} \left(u_{akj} u_{kbi} + u_{aki} u_{kbj} \right).$$

We need to substitute the 4^{th} order derivative of u with lower order derivatives, we use the minimal surface equation (2.1) with (2.2),

$$\triangle_g u_a = \sum_{i,j=1}^{n} g^{ij} \partial_{ij} u_a = 0.$$

Take the derivative with respect to x_b, we have

$$\sum_{i,j=1}^{n} \left(g^{ij} \partial_{ij} u_{ab} + \partial_b g^{ij} \partial_{ij} u_a \right) = 0.$$

Then

$$\sum_{i,j=1}^{n} g^{ij} \partial_{ij} u_{ab} \overset{p}{=} \sum_{i,j=1}^{n} g^{ii} g^{jj} u_{ijb} \left(\lambda_i + \lambda_j \right) u_{aji}$$

and

(2.6)
$$\sum_{i,j=1}^{n} g^{ij} \partial_{ij} g_{ab} \overset{p}{=} \sum_{i,j=1}^{n} g^{ii} g^{jj} u_{ijb} u_{aji} \left(\lambda_i + \lambda_j \right) \left(\lambda_a + \lambda_b \right) + \sum_{i,k=1}^{n} 2 g^{ii} u_{aki} u_{kbi}.$$

Relying on (2.4) (2.5) (2.6), we arrive at

$$\sum_{i,j=1}^{n} g^{ij} \partial_{ij} \log \det g$$
$$= \sum_{i,j,a,b=1}^{n} g^{ij} \partial_i \left(g^{ab} \partial_j g_{ab} \right)$$
$$= \sum_{i,j,a,b=1}^{n} \left(g^{ij} \partial_i g^{ab} \partial_j g_{ab} + g^{ij} g^{ab} \partial_{ij} g_{ab} \right)$$
$$\overset{p}{=} \sum_{i,a,b=1}^{n} -g^{ii} g^{aa} g^{bb} u_{abi}^2 \left(\lambda_a + \lambda_b \right)^2 + \sum_{i,j,a=1}^{n} 2 g^{aa} g^{ii} g^{jj} u_{aji}^2 \left(\lambda_i + \lambda_j \right) \lambda_a$$
$$+ \sum_{i,k,a=1}^{n} 2 g^{aa} g^{ii} u_{aki}^2$$

$$\overset{p}{=} \sum_{a,b,c=1}^{n} -g^{aa}g^{bb}g^{cc}u_{abc}^2 (\lambda_a+\lambda_b)^2 + \sum_{a,b,c=1}^{n} 2g^{aa}g^{bb}g^{cc}u_{abc}^2 (\lambda_b+\lambda_c)\lambda_a + \sum_{a,b,c=1}^{n} 2g^{aa}g^{cc}u_{abc}^2$$

$$\overset{p}{=} \sum_{a,b,c=1}^{n} -2g^{aa}g^{bb}g^{cc}u_{abc}^2 \left(\lambda_b^2+\lambda_a\lambda_b\right) + \sum_{a,b,c=1}^{n} 4g^{aa}g^{bb}g^{cc}u_{abc}^2\lambda_a\lambda_b + \sum_{a,b,c=1}^{n} 2g^{aa}g^{cc}u_{abc}^2$$

$$\overset{p}{=} \sum_{a,b,c=1}^{n} 2g^{aa}g^{cc}u_{abc}^2 \left(-g^{bb}\lambda_b^2+1\right) + \sum_{a,b,c=1}^{n} 2g^{aa}g^{bb}g^{cc}u_{abc}^2\lambda_a\lambda_b$$

$$\overset{p}{=} \sum_{a,b,c=1}^{n} 2g^{aa}g^{bb}g^{cc}u_{abc}^2 (1+\lambda_a\lambda_b)$$

$$\overset{p}{=} \sum_{a,b,c=1}^{n} 2g^{aa}g^{bb}g^{cc}u_{abc}^2 (1+\lambda_b\lambda_c),$$

where we use $g^{bb} \overset{p}{=} \frac{1}{1+\lambda_b^2}$. This finishes the proof of Lemma 2.1. □

Proposition 2.1. *Let $C = (x, \nabla u(x)) \subset \mathbb{R}^{2n}$ be a minimal cone, smooth away from the origin. Suppose the Hessian D^2u satisfies $-I \leq \left(D^2u\right) \leq I$. Then C is a plane.*

Proof. Since $(x, \nabla u(x))$ is cone, $\nabla u(x)$ is homogeneous degree one and $D^2u(x)$ is homogeneous degree zero. It follows that $\log\det g = \log\det\left(I + D^2uD^2u\right)$ takes its maximum at a finite point (away from 0) in $\mathbb{R}^n$. By the assumption $-I \leq \left(D^2u\right) \leq I$, it follows from Lemma 2.1 that

$$\sum_{i,j=1}^{n} g^{ij}\partial_{ij}\log\det g \geq 0.$$

By the strong maximum principle, we see that $\log\det g \equiv const$. Applying Lemma 2.1 again, we obtain

$$0 \overset{p}{=} \sum_{a,b,c=1}^{n} 2g^{aa}g^{bb}g^{cc}u_{abc}^2 (1+\lambda_b\lambda_c) \geq 0.$$

Then

$$u_{abc}^2(1+\lambda_a\lambda_b) = u_{abc}^2(1+\lambda_b\lambda_c) = u_{abc}^2(1+\lambda_c\lambda_a) = 0.$$

Observe that one of $\lambda_a\lambda_b$, $\lambda_b\lambda_c$, and $\lambda_c\lambda_a$ must be nonnegative, we get $u_{abc}(p) = 0$. Since the point p in Lemma 2.1 can be arbitrary, we conclude that $D^3 u \equiv 0$. Consequently, u is a quadratic function and the cone $(x, \nabla u(x))$ is a plane. □

3. Proof of theorems

Proof of Theorem 1.1. Step A. We first seek a better representation of M via Lewy transformation. We rotate the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to $(\bar{x}, \bar{y})$ by $\pi/4$, namely, set $\bar{x} = (x + y)/\sqrt{2}$, $\bar{y} = (-x + y)/\sqrt{2}$. Then M has a new parametrization

$$\begin{cases} \bar{x} = \frac{1}{\sqrt{2}}(x + \nabla u(x)) \\ \bar{y} = \frac{1}{\sqrt{2}}(-x + \nabla u(x)) \end{cases}.$$

Since u is convex, we have

$$\begin{aligned} \left|\bar{x}^2 - \bar{x}^1\right|^2 &= \frac{1}{2}\Big[\left|x^2 - x^1\right|^2 + 2\left(x^2 - x^1\right)\cdot\left(\nabla u\left(x^2\right) - \nabla u\left(x^1\right)\right) \\ &\qquad + \left|\nabla u\left(x^2\right) - \nabla u\left(x^1\right)\right|^2\Big] \\ &\geq \frac{1}{2}\left|x^2 - x^1\right|^2. \end{aligned}$$

It follows that M is still a graph over the whole $\bar{x}$ space $\mathbb{R}^n$. Further M is still a Lagrangian graph over $\bar{x}$, that means M has the representation $(\bar{x}, \nabla\bar{u}(\bar{x}))$ with a potential function $\bar{u} \in C^\infty(\mathbb{R}^n)$ (cf. [HL, Lemma 2.2]).

Note that any tangent vector to M takes the form

$$\frac{1}{\sqrt{2}}\left(\left(I + D^2 u(x)\right)e, \left(-I + D^2 u(x)\right)e\right),$$

where $e \in \mathbb{R}^n$. It follows that

$$D^2\bar{u}(\bar{x}) = \left(I + D^2 u(x)\right)^{-1}\left(-I + D^2 u(x)\right).$$

By the convexity of u, we have

$$-I \leq \left(D^2\bar{u}\right) \leq I.$$

Step B. The remaining proof is routine. We "blow down" M at ∞. Without loss of generality, we assume $\bar{u}(0) = 0$, $\nabla\bar{u}(0) = 0$. Set $M_k = (\bar{x}, \nabla\bar{u}_k)$, where

$$\bar{u}_k(\bar{x}) = \frac{\bar{u}(k\bar{x})}{k^2}, \quad k = 1, 2, 3, \cdots.$$

We see that M_k is still a minimal surface and $-I \le \left(D^2 \bar{u}_k\right) \le I$. Then there exists a subsequence, still denoted by $\left\{\bar{u}_k\right\}$ and $v \in C^{1,1}\left(R^n\right)$ such that

$$\bar{u}_k \to v \ \text{ in } C^{1,\alpha}_{loc}\left(R^n\right)$$

and

$$-I \le \left(D^2 v\right) \le I.$$

We apply the compactness theorem (cf. [S, Theorem 34.5] to conclude that $M_v = \left(\bar{x}, \nabla v\left(\bar{x}\right)\right)$ is a minimal surface, By the monotonicity formula (cf. [S, p. 84]) and Theorem 19.3 in [S], we know that M_v is a minimal cone.

We claim that M_v is smooth away from the vertex. Suppose M_v is singular at P away from the vertex. We blow up M_v at P to get a tangent cone, which is a lower dimensional special Lagrangian cone cross a line, repeat the procedure if the resulting cone is still singular away from the vertex. Finally we get a special Lagrangian cone which is smooth away from the vertex, and the eigenvalues of the Hessian of the potential function are bounded between -1 and 1. By Proposition 2.1, the cone is flat. This is a contradiction to Allard's regularity result (cf. [S, Theorem 24.2]).

Applying Proposition 2.1 to M_v, we see that M_v is flat.

Step C. By our blow-down procedure and the monotonicity formula, we see that

$$\lim_{r\to+\infty} \frac{\mu\left(\mathfrak{B}_r(0,0)\cap M\right)}{|B_r|} = 1,$$

where B_r is the ball with radius r in $\mathbb{R}^n$, $\mathfrak{B}_r(0,0)$ is the ball with radius r and center $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$, and $\mu\left(\mathfrak{B}_r(0,0)\cap M\right)$ is the area of M inside $\mathfrak{B}_r(0,0)$. Since M is smooth, we have

$$\lim_{r\to 0} \frac{\mu\left(\mathfrak{B}_r(0,0)\cap M\right)}{|B_r|} = 1.$$

Consequently, for $r_2 > r_1 > 0$, the monotonicity formula reads

$$0 = \frac{\mu\left(\mathfrak{B}_{r_2}(0,0)\cap M\right)}{\left|B_{r_2}\right|} - \frac{\mu\left(\mathfrak{B}_{r_1}(0,0)\cap M\right)}{\left|B_{r_1}\right|} = \int_{\mathfrak{B}_{r_2}\setminus\mathfrak{B}_{r_1}} \frac{\left|D^{\perp} r\right|^2}{r^n} d\mu,$$

where $r = |(x,y)|$, $D^{\perp} r$ is the orthogonal projection of Dr to the normal space of M, and $d\mu$ is the area form on M. Therefore, we see that M is a plane. □

Remark. In Step B, we use the heavy compactness result (cf. [S, Theorem 34.5]) just for a short presentation of the proof. One can also take advantage of the special Lagrangian equation (1.1), use the compactness result for viscosity solution to derive that $M_v = \left(\bar{x}, \nabla v\left(\bar{x}\right)\right)$ is a minimal surface, see Lemma 2.2 in [Y].

Proposition 3.1. *There exist a dimensional constant $\varepsilon'(n) > 0$ such that any minimal surface $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ with $-\left(1+\varepsilon'(n)\right) I \leq \left(D^2 u\right) \leq \left(1+\varepsilon'(n)\right) I$ for $x \in \mathbb{R}^n$, must be a plane.*

Proof. Suppose not. Then there exists a sequence of minimal surface $M_k = (x, \nabla u_k) \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $-\left(1+\frac{1}{k}\right) I \leq \left(D^2 u_k\right) \leq \left(1+\frac{1}{k}\right) I$ and M_k is not a plane. By Allard's regularity result (cf. [S, Theorem 24.2]) the density D_k for M_k satisfies

$$D_k \geq 1+\delta(n),$$

where $\delta(n) > 0$ is a dimensional constant and

$$D_k = \lim_{r \to +\infty} \frac{\mu\left(\mathfrak{B}_r \cap M_k\right)}{\left|B_r\right|}.$$

By a similar argument as Step B in the proof of Theorem 1.1, we extract a subsequence of $\{v_k\}$ converging to V_∞ in $C^{1,\alpha}_{loc}(R^n)$ such that $M_\infty = (x, \nabla V_\infty(x))$ is a smooth minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ with $-I \leq \left(D^2 u_\infty\right) \leq I$ and $D_\infty \geq 1+\delta(n)$. By our Theorem 1.1, M_∞ is a plane and $D_\infty = 1$. This contradiction finishes the proof of the proposition. □

Proof of Theorem 1.2. We repeat the rotation argument in Step A of the proof of Theorem 1.1 to get a new representation for M, $\left(\bar{x}, \nabla \bar{u}\left(\bar{x}\right)\right)$ with

$$-\left(1+\frac{2\varepsilon(n)}{1-\varepsilon(n)}\right) I \leq \left(D^2 \bar{u}\right) \leq I.$$

We choose $\varepsilon(n) = \frac{\varepsilon'(n)}{2+\varepsilon'(n)}$ and apply Proposition 3.1. Then Theorem 1.2 follows. □

Proof of Theorem 1.3. The strategy is similar to the proof of Theorem 1.1.

Step A. We first make a different rotation of the coordinate system to get a better representation of M. Set $\bar{x} = \frac{1}{\sqrt{1+4C^2}}(2Cx+y)$, $\bar{y} = \frac{1}{\sqrt{1+4C^2}}(-x+2Cy)$. Then M has a new parametrization

$$\begin{cases} \bar{x} = \frac{1}{\sqrt{1+4C^2}}(2Cx+\nabla u(x)) \\ \bar{y} = \frac{1}{\sqrt{1+4C^2}}(-x+2C\nabla u(x)) \end{cases}.$$

Since $u+\frac{1}{2}C|x|^2$ is convex, we have

$$\begin{aligned} \left|\bar{x}^2-\bar{x}^1\right|^2 &= \frac{1}{1+4C^2}\begin{bmatrix} C^2\left|x^2-x^1\right|^2+2C\left(x^2-x^1\right) \\ \cdot\left(\nabla u\left(x^2\right)+Cx^2-\nabla u\left(x^1\right)-Cx^1\right) \\ +\left|\nabla u\left(x^2\right)+Cx^2-\nabla u\left(x^1\right)-Cx^1\right|^2 \end{bmatrix} \\ &\geq \frac{1}{1+4C^2}C^2\left|x^2-x^1\right|^2. \end{aligned}$$

As in the proof of Theorem 1.1, we get a new representation for $M = (\bar{x}, \nabla \bar{u}(\bar{x}))$ and

$$D^2 \bar{u}(\bar{x}) = \left(2CI + D^2 u\right)^{-1} \left(-I + 2CD^2 u(x)\right).$$

From $D^2 u \geq -CI$, we see that

$$-\frac{1+2C^2}{C} I \leq \left(D^2 \bar{u}\right) \leq 2CI.$$

Step B. As Step B in the proof of Theorem 1.1, any tangent cone of M at ∞ is flat. The only difference is that, instead of relying on Proposition 2.1, we use the fact that any non-parametric minimal cone of dimension three must be flat, see [F-C, Theorem 2.3], [B, Theorem]. For a quick PDE proof of this fact, see [HNY, p. 2].

Step C is exactly as in the proof of Theorem 1.1.

Therefore, we conclude Theorem 1.3. □

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