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Minimal Cones with Isotropic Links

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We show that any closed oriented immersed isotropic minimal surface Σ with genus g_Σ in $S^5 \subset \mathbb{C}^3$ is (1) Legendrian (and totally geodesic) if $g_\Sigma = 0$; (2) either Legendrian or with exactly $2g_\Sigma - 2$ Legendrian points if $g_\Sigma \geq 1$. In general, any compact oriented immersed isotropic minimal submanifold $L^{n-1} \subset S^{2n-1} \subset \mathbb{C}^n$ must be Legendrian if its first Betti number is zero. Corresponding results for nonorientable links are also provided.

1 Introduction

In this paper we study the problem of when a minimal cone $C(L)$ with isotropic link L on S^{2n-1} in $\mathbb{C}^n$ becomes special Lagrangian. Equivalently, given a minimal isotropic submanifold $L \subset S^{2n-1} \subset \mathbb{C}^n$, when is L Legendrian?

Following Harvey and Lawson [5, page 110], the isotropy condition for submanifold M in $\mathbb{C}^n$ means

$$J(TM) \perp TM, \tag{1.1}$$

where J is the complex structure in $\mathbb{C}^n$, or the standard symplectic 2-form on $\mathbb{R}^{2n}$ vanishes on M. The dimension of an isotropic submanifold is at most the half-dimension n, and when it is n the submanifold is Lagrangian. For an immersed $(n-1)$-dimensional submanifold L in S^{2n-1}, let $u : L \to S^{2n-1}$ be the restriction of the coordinate functions in

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$\mathbb{R}^{2n}$ to L. A point $u \in L$ is Legendrian if T_uL is isotropic in $\mathbb{R}^{2n}$ and

$$J(T_uL) \perp u. \tag{1.2}$$

L is Legendrian if all the points u are Legendrian. This is equivalent to that L is an $(n-1)$-dimensional integral submanifold of the standard contact distribution on S^{2n-1}. All links Σ and L are assumed to be connected in this paper.

Theorem 1.1. Let Σ be a closed oriented immersed isotropic minimal surface with genus g_Σ in $S^5 \subset \mathbb{C}^3$. Then (1) if $g_\Sigma = 0$, Σ is Legendrian (and totally geodesic); (2) if $g_\Sigma \geq 1$, Σ is either Legendrian or has exactly $2g_\Sigma - 2$ Legendrian points counting the multiplicity. □

It is known that the immersed minimal Legendrian sphere ($g_\Sigma = 0$) must be a great two sphere in S^5 (cf. [6, Theorem 2.7]). Bryant's classification [3, page 269] of minimal surfaces with constant curvature in spheres provides examples of flat Legendrian minimal tori as well as flat non-Legendrian isotropic minimal tori ($g_\Sigma = 1$) in S^5. The constructions of Haskins [6] and Haskins and Kapouleas [7] show that there are infinitely many immersed (embedded if $g_\Sigma = 1$) minimal Legendrian surfaces for each odd genus in S^5.

For a preliminary study of the problem of when a minimal cone with isotropic link becomes isotropic in general dimensions and codimensions, we have the following.

Theorem 1.2. Let L^m be a compact isotropic immersed oriented minimal submanifold in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. If the first Betti number of L^m is 0, then the minimal cone $C(L^m)$ is isotropic, in particular, $C(L^{n-1})$ is Lagrangian (or equivalently L^{n-1} is Legendrian) when m is the top dimension $n-1$. □

Certain Bochner conditions would imply the vanishing of the first Betti number, such as the following. The Ricci curvature of L^m is nonnegative (e.g., when the norm of the second fundamental form of L^m in S^{2n-1} is less than $\sqrt{m-1}$), and is positive somewhere or the Euler characteristic of L^m is nonzero. A direct approach from the above geometric intrinsic conditions to the isotropy conclusion is also included in Section 3.

When the dimension m of the link is 2, Theorem 1.2 also implies Theorem 1.1(1). Even if the first Betti number is not zero ($g_{L^2} > 0$) in Theorem 1.2, we can still conclude that the cone $C(L^2)$ is isotropic either everywhere or along exactly $2g_{L^2} - 2 = -\chi(L^2)$ lines. This generalization of Theorem 1.1(2) to higher codimensions can be proved in the same way as for Theorem 1.1(2).

All of our results, Theorems 1.2, 1.1 (except the totally geodesic part) and their generalization to higher codimensions are valid for nonorientable links (note that $\chi(\Sigma) = 2 - g_\Sigma$ for a compact nonorientable surface Σ), see Remarks 2.2 and 3.1. In particular, the nonorientable version of Theorem 1.2 implies that one cannot immerse a compact nonorientable L^{n-1} with 0 first Betti number minimally and isotropically into $S^{2n-1} \subset \mathbb{C}^n$. Otherwise, the cone $C(L^{n-1})$ would be a special Lagrangian cone, then $C(L^{n-1})$ would be orientable, and L^{n-1} would also be orientable. This is a contradiction. As a simple example, there exists no isotropic minimal immersion of a real projective sphere $\mathbb{R}P^2$ into $S^5 \subset \mathbb{C}^3$.

Our local example in Section 4, of non-Lagrangian minimal cones with isotropic links in $\mathbb{R}^5 \subset \mathbb{C}^3$ suggests there *might* be non-Legendrian yet isotropic minimal Riemann surfaces of genus $g_\Sigma > 1$ in $S^5 \subset \mathbb{C}^3$.

It is not clear to us whether there exists an isotropic minimal surface in S^5 with exactly $2g_\Sigma - 2$ Legendrian points for $g_\Sigma > 1$, or some necessary condition such as the moment condition explored by Fu in [4] to force those isotropic minimal surfaces to be Legendrian.

2 Hopf differentials and proof of Theorem 1.1

To measure how far the isotropic Σ is away from being Legendrian, or the deviation of the corresponding cone from being Lagrangian, we project Ju onto the tangent space of Σ in $\mathbb{C}^3$, where J is the complex structure in $\mathbb{C}^3$. Denote the length of the projection by

$$f = \left| \Pr Ju \right|^2. \tag{2.1}$$

To compute the length, we need some preparation. Locally, take an isothermal coordinate system (t^1, t^2) on the isotropic minimal surface

$$u : \Sigma \longrightarrow S^5 \subset \mathbb{C}^3. \tag{2.2}$$

Set the complex variable

$$z = t^1 + \sqrt{-1}t^2. \tag{2.3}$$

Then the induced metric has the local expression with the conformal factor φ

$$g = \varphi^2 \left[\left(dt^1\right)^2 + \left(dt^2\right)^2 \right] = \varphi^2 \, dz \, d\bar{z}. \tag{2.4}$$

We project Ju to each of the orthonormal bases $\varphi^{-1}u_1, \varphi^{-1}u_2$ with $u_i = \partial u/\partial t^i$. Then the sum of the squares of each projection is

$$f = \frac{\left|\langle Ju, u_1\rangle\right|^2 + \left|\langle Ju, u_2\rangle\right|^2}{\varphi^2} = \frac{4\left|\langle Ju, u_z\rangle\right|^2}{\varphi^2}, \tag{2.5}$$

where $u_z = \partial u/\partial z$ and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^6$, and in particular $0 \leq f \leq 1$. In fact, f is the square of the norm of the symplectic form ω in $\mathbb{C}^3$ restricted on the cone $C(\Sigma)$ with link Σ:

$$\omega\big|_{C(\Sigma)} \wedge *\omega\big|_{C(\Sigma)} = f \cdot \text{ volume form of } C(\Sigma). \tag{2.6}$$

The minimality condition

$$\Delta_g u = \frac{4}{\varphi^2} u_{z\bar{z}} = -2u \tag{2.7}$$

and the isotropy condition

$$\langle Ju_i, u_j\rangle = 0 \tag{2.8}$$

imply the holomorphic condition

$$\langle Ju, u_z\rangle_{\bar{z}} = \langle Ju_{\bar{z}}, u_z\rangle + \langle Ju, u_{z\bar{z}}\rangle = \left\langle Ju, -\frac{\varphi^2}{2}u\right\rangle = 0. \tag{2.9}$$

The induced metric g yields a compatible conformal structure on the oriented surface Σ which makes Σ a Riemann surface. We will consider two cases according to the genus g_Σ.

(1) $g_\Sigma = 0$: by the uniformization theorem for Riemann surfaces (cf. [1, pages 125, 181]), there exists a holomorphic covering map

$$\Phi : \left(S^2, g_{\text{canonical}}\right) \longrightarrow (\Sigma, g) \tag{2.10}$$

or locally

$$\Phi : \left(\mathbb{C}^1, \frac{1}{\left(1+w^2\right)^2}\, dw\, d\overline{w}\right) \longrightarrow (\Sigma, g). \tag{2.11}$$

For $z = \Phi(w)$ one has

$$\frac{1}{\left(1+w^2\right)^2}\, dw\, d\overline{w} = \Phi^*\left(\psi^2 g\right) = \Phi^*\left(\psi^2\varphi^2\, dz\, d\bar{z}\right) = \psi^2\varphi^2\left|z_w\right|^2 dw\, d\overline{w}, \tag{2.12}$$

where ψ is a positive (real analytic) function on Σ. In particular

$$|z_w|^2 = \frac{1}{\psi^2\varphi^2(1+w^2)^2}. \tag{2.13}$$

Note that

$$\langle Ju, u_w\rangle = \langle Ju, u_z\rangle z_w = \langle Ju, u_z\rangle \frac{1}{w_z} \tag{2.14}$$

is a holomorphic function of z, in turn it is a holomorphic function of w. Also $\langle Ju, u_w\rangle$ is bounded, approaching 0 as w goes to ∞, because

$$|\langle Ju, u_w\rangle|^2 = \frac{|\langle Ju, u_z\rangle|^2}{\varphi^2}\frac{1}{\psi^2(1+w^2)^2}. \tag{2.15}$$

So $\langle Ju, u_w\rangle \equiv 0$. Therefore $f \equiv 0$ and Σ is Legendrian.

For completeness, we present a simple direct proof of the known result that any immersed minimal Legendrian $S^2 \subset S^5 \subset \mathbb{C}^3$ must be totally geodesic. We still use the above isothermal coordinates now on the Legendrian minimal sphere. Note that Ju, Ju_1, and Ju_2 are orthogonal vectors in the normal space of S^2 in $S^5 \subset \mathbb{C}^3$. The second fundamental form of the minimal Legendrian S^2 in $S^5 \subset \mathbb{C}^3$ can be represented as

$$\begin{aligned}\mathbb{I}(i,j) &= \langle u_{ij}, Ju\rangle Ju + \langle u_{ij}, Ju_1\rangle \frac{Ju_1}{\varphi^2} + \langle u_{ij}, Ju_2\rangle \frac{Ju_2}{\varphi^2}\\ &= \langle u_{ij}, Ju_1\rangle \frac{Ju_1}{\varphi^2} + \langle u_{ij}, Ju_2\rangle \frac{Ju_2}{\varphi^2},\end{aligned} \tag{2.16}$$

where we use $\langle u_{ij}, Ju\rangle = \langle u_i, Ju\rangle_j - \langle u_i, Ju_j\rangle = 0$, because S^2 is Legendrian.

$$\begin{aligned}|\mathbb{I}|^2 &= \frac{1}{\varphi^6}\sum_{i,j=1}^{2}\left(\langle u_{ij}, Ju_1\rangle^2 + \langle u_{ij}, Ju_2\rangle^2\right)\\ &= \frac{4}{\varphi^6}\sum_{i,j=1}^{2}|\langle u_{ij}, Ju_z\rangle|^2\\ &= \frac{4}{\varphi^6}\begin{pmatrix} |\langle u_{zz}+u_{\overline{zz}}+2u_{z\bar{z}}, Ju_z\rangle|^2 + 2|\sqrt{-1}\langle u_{zz}-u_{\overline{zz}}, Ju_z\rangle|^2 \\ +|-\langle u_{zz}+u_{\overline{zz}}-2u_{z\bar{z}}, Ju_z\rangle|^2 \end{pmatrix}\\ &= \frac{4}{\varphi^6}\left(|\langle u_{zz}, Ju_z\rangle|^2 + 2|\langle u_{zz}, Ju_z\rangle|^2 + |\langle u_{zz}, Ju_z\rangle|^2\right)\\ &= \frac{16}{\varphi^6}|\langle u_{zz}, Ju_z\rangle|^2,\end{aligned} \tag{2.17}$$

where in the third line we convert u_{ij} from real coordinates to the complex ones and then we use the identities induced from the Legendrian and minimal conditions

$$\begin{aligned}\langle u_{\bar z\bar z}, Ju_z\rangle &= \langle u_{\bar z}, Ju_z\rangle_{\bar z} - \langle u_{\bar z}, Ju_{z\bar z}\rangle = -\left\langle u_{\bar z}, J\left(\frac{-\varphi^2}{2}u\right)\right\rangle = 0,\\ \langle u_{z\bar z}, Ju_z\rangle &= \left\langle \frac{-\varphi^2}{2}u, Ju_z\right\rangle = 0.\end{aligned} \tag{2.18}$$

Next we claim the Hopf cubic differential $\langle u_{zz}, Ju_z\rangle dz\,dz\,dz$ is holomorphic and globally defined on S^2. Again the Legendrian and minimal conditions lead to

$$\begin{aligned}\langle u_{zz}, Ju_z\rangle_{\bar z} &= \langle u_{zz\bar z}, Ju_z\rangle + \langle u_{zz}, Ju_{z\bar z}\rangle\\ &= \left\langle \left(\frac{-\varphi^2}{2}u\right)_z, Ju_z\right\rangle + \left\langle u_{zz}, J\left(\frac{-\varphi^2}{2}u\right)\right\rangle\\ &= \left\langle \left(\frac{-\varphi^2}{2}\right)_z u - \frac{\varphi^2}{2}u_z, Ju_z\right\rangle - \frac{\varphi^2}{2}\langle u_{zz}, Ju\rangle\\ &= -\frac{\varphi^2}{2}\left(\langle u_z, Ju\rangle_z - \langle u_z, Ju_z\rangle\right) = 0.\end{aligned} \tag{2.19}$$

For any change of holomorphic variables $z = z(w)$ on S^2 we have

$$\begin{aligned}\langle u_{ww}, Ju_w\rangle dw\,dw\,dw &= \langle u_{zz}z_w z_w + u_z(z_w)_w, J(u_z z_w)\rangle (w_z)^3\,dz\,dz\,dz\\ &= \langle u_{zz}z_w z_w + u_z(z_w)_w, Ju_z\rangle z_w (w_z)^3\,dz\,dz\,dz\\ &= \left(z_w z_w\langle u_{zz}, Ju_z\rangle + (z_w)_w\langle u_z, Ju_z\rangle\right) z_w (w_z)^3\,dz\,dz\,dz\\ &= \langle u_{zz}, Ju_z\rangle (z_w)^3 (w_z)^3\,dz\,dz\,dz\\ &= \langle u_{zz}, Ju_z\rangle dz\,dz\,dz.\end{aligned} \tag{2.20}$$

This shows that the cubic holomorphic differential is independent of the choice of local holomorphic coordinates, hence it is defined on entire S^2.

Finally one repeats the above argument of showing that any holomorphic Hopf one differential on S^2 must vanish to conclude that the Hopf cubic differential

$$\langle u_{zz}, Ju_z\rangle dz\,dz\,dz \tag{2.21}$$

on S^2 must also vanish everywhere.

Therefore the second fundamental form of the immersed minimal Legendrian S^2 on S^5 vanishes everywhere and then the immersed minimal Legendrian S^2 is totally geodesic.

Remark 2.1. For an isotropic minimal immersed cone $C(S^2) \subset \mathbb{C}^n$ with immersed S^2 link on S^{2n-1}, the above argument leads to the vanishing of the second fundamental form of the cone along the normal subspace $J(T(C(S^2)))$.

(2) $g_\Sigma \geq 1$: as in the $g_\Sigma = 0$ case, the isotropic and minimal condition gives us a local holomorphic function $\langle Ju, u_z \rangle$ and global holomorphic Hopf one differential $\langle Ju, u_z \rangle dz$. We only consider the case that $\langle Ju, u_z \rangle dz$ is not identically 0. The zeros of $\langle Ju, u_z \rangle$ are therefore isolated and near each of the zeros, we can write

$$\langle Ju, u_z \rangle = h(z) z^k, \tag{2.22}$$

where h is a local holomorphic function nonvanishing at the zero point $z = 0$ and k is a positive integer. One can also view

$$\langle Ju, u_z \rangle = \frac{1}{2}(\langle Ju, u_1 \rangle - \sqrt{-1}\langle Ju, u_2 \rangle) \tag{2.23}$$

as the tangent vector

$$\frac{1}{2}\langle Ju, u_1 \rangle u_1 - \frac{1}{2}\langle Ju, u_2 \rangle u_2 = \frac{1}{2}\langle Ju, u_1 \rangle \partial_1 - \frac{1}{2}\langle Ju, u_2 \rangle \partial_2 \tag{2.24}$$

along the tangent space $T\Sigma$, where $\partial_i = \partial u / \partial t^i$. The projection $\mathrm{Pr}\, Ju$ on the tangent space of $T\Sigma$ is locally represented as

$$\mathrm{Pr}\, Ju = \frac{\langle Ju, u_1 \rangle \partial_1 + \langle Ju, u_2 \rangle \partial_2}{\varphi^2}. \tag{2.25}$$

The index of the globally defined vector field $\mathrm{Pr}\, Ju$ at each of its singular points, that is, $\mathrm{Pr}\, Ju = 0$ is the negative of that for the vector field $(1/2)\langle Ju, u_1 \rangle \partial_1 - (1/2)\langle Ju, u_2 \rangle \partial_2$. Note that the index of the latter is k.

From the Poincaré-Hopf index theorem, for any vector field V with isolated singularities on Σ, one has

$$\sum_{V=0} \mathrm{index}(V) = \chi(\Sigma) = 2 - 2g_\Sigma \leq 0. \tag{2.26}$$

The zeros of $\mathrm{Pr}\, Ju$ are just the Legendrian points on Σ. So we conclude that the number of Legendrian points is $2g_\Sigma - 2$ counting the multiplicity. This completes the proof of Theorem 1.1.

Remark 2.2. As mentioned in the introduction, Theorem 1.1 (except the totally geodesic part) and its generalization to higher codimensions can be extended for the nonorientable links. This can be seen as follows. The Poincaré-Hopf index theorem holds on compact nonorientable surfaces, our count of the indices of the still globally defined $\mathrm{Pr}\, Ju$ via *local* holomorphic functions is valid too, and the index of a singular point of a vector field is independent of local orientations. Moreover, this index counting argument yields an alternative proof for Theorem 1.1(1) (except the totally geodesic part) and its generalization.

3 Harmonic forms and proof of Theorem 1.2

Consider an immersed isotropic minimal submanifold in S^{2n-1},

$$u : L^m \longrightarrow S^{2n-1} \subset \mathbb{C}^n, \tag{3.1}$$

the minimality condition reads

$$\Delta_g u = -mu \tag{3.2}$$

with g being the induced metric. The isotropy condition reads for any local coordinates $(t^1, \dots, t^m)$ on L^m,

$$\langle Ju_i, u_j \rangle = 0 \tag{3.3}$$

with J being the complex structure of $\mathbb{C}^n$ and $u_i = \partial u / \partial t^i$.

Again, to measure the deviation of the corresponding cone $C(u(L^m))$ from being isotropic, we project Ju onto the tangent space of $u(L^m)$ in $\mathbb{C}^n$. Note that the projection is the vector

$$\mathrm{Pr}\, Ju = \sum_{i,j=1}^{m} g^{ij} \langle Ju, u_i \rangle u_j, \tag{3.4}$$

where $g_{ij} = \langle u_i, u_j \rangle$, $1 \le i, j \le m$. The corresponding one form

$$\alpha = \sum_{i=1}^{m} \langle Ju, u_i \rangle dt^i \tag{3.5}$$

is of course globally defined on L^m. In fact it is a harmonic one form, because α is closed and coclosed:

$$
\begin{aligned}
d\alpha &= \sum_{i,j=1}^{m} \langle Ju, u_i\rangle_j dt^j \wedge dt^i = \sum_{i,j=1}^{m} \left(\langle Ju_j, u_i\rangle + \langle Ju, u_{ij}\rangle\right) dt^j \wedge dt^i \\
&= \sum_{i,j=1}^{m} \langle Ju, u_{ij}\rangle dt^j \wedge dt^i = 0, \\
\delta\alpha &= (-1)^{m\cdot 1+m+1} * d * \alpha \\
&= - * d\left(\sum_{i,j=1}^{m} (-1)^{j+1} \sqrt{g} g^{ij} \langle Ju, u_i\rangle dt^1 \wedge \cdots \wedge \widehat{dt^j} \wedge \cdots \wedge dt^m \right) \\
&= - * \sum_{i,j=1}^{m} \partial_j \left(\sqrt{g} g^{ij} \langle Ju, u_i\rangle\right) dt^1 \wedge \cdots \wedge dt^j \wedge \cdots \wedge dt^m \\
&= -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{m} \partial_j \left(\sqrt{g} g^{ij} \langle Ju, u_i\rangle\right) \\
&= -\sum_{i,j=1}^{m} \left(\langle Ju_j, g^{ij} u_i\rangle + \left\langle Ju, \frac{1}{\sqrt{g}} \partial_j \left(\sqrt{g} g^{ij} u_i\right) \right\rangle \right) \\
&= -\langle Ju, -mu\rangle = 0,
\end{aligned} \tag{3.6}
$$

where we use the isotropy and minimality conditions.

Now the Hodge-de Rham theorem implies that the harmonic one form α must vanish because of the first Betti number zero assumption. It follows that $\mathrm{Pr}\, Ju$ must vanish. Therefore, the cone $C(L^m)$ is isotropic. The proof of Theorem 1.2 is complete.

Remark 3.1. As the projection $\mathrm{Pr}\, Ju$ and the adjoint operator δ is independent of the local orientations and the Hodge-de Rham theorem holds for compact nonorientable manifolds (cf. [9, pages 125-126]), we see that Theorem 1.2 remains true for nonorientable links L^m.

Corollary 3.2. Let L^m be a compact immersed isotropic minimal submanifold in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. If the Ricci curvature of L^m is nonnegative, and it is positive somewhere, or the Euler characteristic $\chi(L^m)$ is not zero, then the minimal cone $C(L^m)$ is isotropic, in particular, $C(L^{n-1})$ is Lagrangian (or equivalently L^{n-1} is Legendrian) when m is the top-dimension $n-1$. □

Under the above condition, from Bochner [2, page 381], it follows immediately that the first Betti number of L^m is zero. Then Theorem 1.2 and its nonorientable version imply the corollary.

Still we present a direct proof of the corollary based on the Bochner-type computation. Denote the square of the length of the projection $\Pr Ju$ by f, so

$$f = \sum_{i,j=1}^{m} g^{ij} \langle Ju, u_i \rangle \langle Ju, u_j \rangle. \tag{3.7}$$

Note that f is the square of the norm of the restriction on the cone $C(L^m)$ of the standard symplectic form ω on $\mathbb{C}^n$.

If $\mathrm{Ric}(L^m) \geq 0$, the Bochner-type formula $(*)$ in the following proposition implies that f is subharmonic, hence constant because L^m is compact. If $\mathrm{Ric}(L^m) > 0$ at a point p, then $\langle Ju, u_i \rangle = 0$ at p for $i = 1, \dots, m$. It follows that f is identically zero, so L^m is Legendrian. If $\chi(L^m)$ is nonzero, then every vector field on L^m must vanish somewhere (cf. [8, page 133]). In particular, the globally defined vector field $\Pr Ju$, the projection of Ju on TL^m, has zeros. This means that f is zero somewhere, hence f is zero everywhere. So $C(L^m)$ is isotropic. The direct approach to the corollary is complete, pending to the following.

Proposition 3.3 (Bochner-type formula for f)**.** Let $u : L^m \to S^{2n-1}$ be an isotropic minimal immersion and let $f = |\Pr Ju|^2$. Then at the center p of a normal coordinate chart on L^m,

$$\frac{1}{2}\Delta_g f = \sum_{i,j=1}^{m} R_{ij} \langle Ju, u_i \rangle \langle Ju, u_j \rangle + \sum_{i,j=1}^{m} \langle Ju, u_{ij} \rangle^2, \tag{$*$}$$

where R_{ij} is the Ricci curvature of L^m. □

Proof. We will use the summation convention for repeated indices and all computations will be at the center p of a normal coordinate chart. First, we have

$$\begin{aligned} (\Delta_g u)_\alpha &= \partial_\alpha \left(g^{ij} \partial_i \partial_j u - g^{ij} \Gamma_{ij}^k \partial_k u \right) \\ &= u_{\alpha ii} - \left(g_{ki,i\alpha} - \frac{1}{2} g_{ii,k\alpha} \right) u_k. \end{aligned} \tag{3.8}$$

Then using isotropy condition $\langle Ju_i, u_j\rangle = 0$ and the minimality assumption $\Delta_g u = -mu$, we have

$$\begin{aligned}
\Delta_g f &= g^{ij}\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right)\left(g^{\alpha\beta}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle\right)\\
&= \partial_i\left(\partial_i g^{\alpha\beta}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2\partial_i\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\alpha\right\rangle\right)\\
&= \partial^2_{ii} g^{\alpha\beta}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2\partial^2_{ii}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\alpha\right\rangle + 2\partial_i\left\langle Ju, u_\alpha\right\rangle\partial_i\left\langle Ju, u_\alpha\right\rangle\\
&= -\left(\partial^2_{ii} g_{\alpha\beta}\right)\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2\partial_i\left(\left\langle Ju_i, u_\alpha\right\rangle + \left\langle Ju, u_{\alpha i}\right\rangle\right)\left\langle Ju, u_\alpha\right\rangle\\
&\quad + 2\left(\left\langle Ju_i, u_\alpha\right\rangle + \left\langle Ju, u_{\alpha i}\right\rangle\right)\left(\left\langle Ju_i, u_\alpha\right\rangle + \left\langle Ju, u_{\alpha i}\right\rangle\right)\\
&= -g_{\alpha\beta,ii}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2\left(\left\langle Ju_i, u_{\alpha i}\right\rangle + \left\langle Ju, u_{\alpha ii}\right\rangle\right)\left\langle Ju, u_\alpha\right\rangle + 2\left\langle Ju, u_{\alpha i}\right\rangle^2\\
&= -g_{\alpha\beta,ii}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle\\
&\quad + 2\left(\partial_i\left\langle Ju_i, u_\alpha\right\rangle - \left\langle Ju_{ii}, u_\alpha\right\rangle + \left\langle Ju, u_{\alpha ii}\right\rangle\right)\left\langle Ju, u_\alpha\right\rangle + 2\left\langle Ju, u_{\alpha i}\right\rangle^2\\
&= -g_{\alpha\beta,ii}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle - 2\left\langle J\Delta_g u, u_\alpha\right\rangle\left\langle Ju, u_\alpha\right\rangle\\
&\quad + 2\left\langle Ju, \left(\Delta_g u\right)_\alpha + \left(g_{\beta i,i\alpha} - \frac{1}{2}g_{ii,\beta\alpha}\right)u_\beta\right\rangle\left\langle Ju, u_\alpha\right\rangle + 2\left\langle Ju, u_{\alpha i}\right\rangle^2\\
&= \left(2g_{i\beta,i\alpha} - g_{\alpha\beta,ii} - g_{ii,\alpha\beta}\right)\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2(m)\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\alpha\right\rangle\\
&\quad - 2(m)\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\alpha\right\rangle + 2\left\langle Ju, u_{\alpha i}\right\rangle^2\\
&= 2R_{\alpha i\beta i}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2\left\langle Ju, u_{\alpha i}\right\rangle^2\\
&= 2R_{\alpha\beta}\left\langle Ju, u_\alpha\right\rangle\left\langle Ju, u_\beta\right\rangle + 2\left\langle Ju, u_{\alpha i}\right\rangle^2.
\end{aligned} \tag{3.9}$$

This completes the proof of the Bochner-type formula for f. ■

4 Local example

In this section, we construct non-Lagrangian minimal cones in $\mathbb{C}^3$ with isotropic link in $\mathbb{R}^5$. The local example illustrates there *might* be non-Lagrangian minimal cones in $\mathbb{C}^3$ with isotropic link on S^5 with genus $g_\Sigma > 1$.

Consider the cone in $\mathbb{R}^6$

$$X = \left(x^1, x^2, x^3, x^1, x^2, x^3 h\left(\frac{x^1, x^2}{x^3}\right)\right), \tag{4.1}$$

where $h : \mathbb{R}^2 \to \mathbb{R}$ is a scalar function. For the homogeneous order-one function

$$H(x^1, x^2, x^3) = x^3 h\left(\frac{x^1}{x^3}, \frac{x^2}{x^3}\right), \tag{4.2}$$

one has

$$\begin{aligned} \nabla H(x) &= \nabla H\left(\frac{x^1}{x^3}, \frac{x^2}{x^3}, 1\right) \\ &= \left(h_1\left(\frac{x'}{x^3}\right), h_2\left(\frac{x'}{x^3}\right), h\left(\frac{x'}{x^3}\right) - \frac{x^1}{x^3} h_1\left(\frac{x'}{x^3}\right) - \frac{x^2}{x^3} h_2\left(\frac{x'}{x^3}\right)\right), \end{aligned} \tag{4.3}$$

where $x' = (x^1, x^2)$. Take the complex structure on $\mathbb{R}^3 \times \mathbb{R}^3$ and let ω be the standard symplectic form on $\mathbb{C}^3$. Then

$$\begin{aligned} \omega|_X &= dx^3 \wedge dH \\ &= h_1\left(\frac{x'}{x^3}\right) dx^3 \wedge dx^1 + h_2\left(\frac{x'}{x^3}\right) dx^3 \wedge dx^2. \end{aligned} \tag{4.4}$$

In particular, $\omega|_X = 0$ when $x^3 \equiv 1$, that is, the cone has an isotropic link in $\mathbb{R}^5 = \{x^3 = 1\}$.

The induced metric on the cone in coordinates (x^1, x^2, x^3) is

$$g(x) = 2\left(dx^1\right)^2 + 2\left(dx^2\right)^2 + \left(dx^3\right)^2 + (dH)^2 \tag{4.5}$$

and satisfies

$$g_{ij}(x) = g_{ij}\left(\frac{x'}{x^3}, 1\right). \tag{4.6}$$

The Hessian of $H(x)$ satisfies

$$\begin{aligned} D^2 H(x) &= \frac{1}{x^3} D^2 H\left(\frac{x'}{x^3}, 1\right) \\ &= \frac{1}{x^3} \begin{bmatrix} I_{2\times 2} & 0 \\ \frac{-x'}{x^3} & 1 \end{bmatrix} \begin{bmatrix} D^2 h & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{2\times 2} & \frac{-x'}{x^3} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{4.7}$$

The minimal surface system is

$$\sum_{i,j=1}^{3} g^{ij} \partial_{ij} X = 0. \tag{4.8}$$

In the above setting, the minimal surface system reduces to

$$\sum_{i,j=1}^{3} g^{ij}(x)\partial_{ij}H(x) = \frac{1}{x^3}\sum_{i,j=1}^{3} g^{ij}\left(\frac{x'}{x^3},1\right)H_{ij}\left(\frac{x'}{x^3},1\right) = 0, \tag{4.9}$$

or

$$\sum_{i,j=1}^{3} g^{ij}\left(\frac{x'}{x^3},1\right)H_{ij}\left(\frac{x'}{x^3},1\right) = 0, \tag{4.10}$$

or even the equation with $x^3 = 1$:

$$\begin{aligned} 0 &= \mathrm{Tr}\left(g^{-1}(x',1)D^2H(x',1)\right) \\ &= \mathrm{Tr}\left(\begin{bmatrix} I_{2\times 2} & -x' \\ 0 & 1 \end{bmatrix} g^{-1}(x',1)\begin{bmatrix} I_{2\times 2} & 0 \\ -x' & 1 \end{bmatrix}\begin{bmatrix} D^2h & 0 \\ 0 & 0 \end{bmatrix}\right. \\ &\qquad\qquad \left. = F\big(h_{11},h_{12},h_{22},h_1,h_2,h,x^1,x^2\big)\right). \end{aligned} \tag{4.11}$$

Now that $F(h_{11},h_{12},h_{22},h_1,h_2,h,x^1,x^2) = 0$ is an elliptic equation, by the Cauchy-Kowalewski theorem one can solve the Cauchy problem with analytic data:

$$\begin{aligned} &F\big(h_{11},h_{12},h_{22},h_1,h_2,h,x^1,x^2\big) = 0, \\ &h\big(x^1,0\big) = \varphi\big(x^1\big), \\ &h_2\big(x^1,0\big) = \psi\big(x^1\big). \end{aligned} \tag{4.12}$$

Conclusion. One can prescribe the Cauchy data for the minimal cone with isotropic link in $\mathbb{R}^5$ so that

$$h_1 = \varphi' \neq 0, \qquad h_2 = \psi \neq 0, \tag{4.13}$$

in turn, we have constructed the non-Lagrangian minimal cone

$$\omega|_X = h_1\left(\frac{x^1,x^2}{x^3}\right)dx^3\wedge dx^1 + h_2\left(\frac{x^1,x^2}{x^3}\right)dx^3\wedge dx^2 \neq 0. \tag{4.14}$$

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