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GLOBAL SOLUTIONS TO SPECIAL LAGRANGIAN EQUATIONS

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ABSTRACT. We show that any global solution to the special Lagrangian equations with the phase larger than a critical value must be quadratic.

1. INTRODUCTION

In this note, we show that any global solution u in $\mathbb{R}^n$ to the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i = c \tag{1.1}$$

with phase $|c| > \frac{\pi}{2}(n-2)$ must be a quadratic polynomial, which states the λ_i's are the eigenvalues of the Hessian D^2u. Recall the Bernstein theorem, where any global solution to the minimal surface equation $div\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$ in $\mathbb{R}^7$ must be a linear function.

Equation (1.1) stems from the special Lagrangian geometry [HL]. The Lagrangian graph $(x, \nabla u(x)) \subset \mathbb{R}^n \times \mathbb{R}^n$ is called special when the phase or the argument of the complex number $\left(1+\sqrt{-1}\lambda_1\right)\cdots\left(1+\sqrt{-1}\lambda_n\right)$ is constant c, that is, u satisfies equation (1.1), and it is special if and only if $(x, \nabla u(x))$ is a minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL, Theorem 2.3, Proposition 2.17]. To be precise, we state

Theorem 1.1. *Let u be a smooth solution in $\mathbb{R}^n$ to (1.1) with $|c| > \frac{\pi}{2}(n-2)$. Then u is quadratic.*

Fu [F] proved Theorem 1.1 in the two-dimensional case. Indeed (1.1) with $c = \frac{\pi}{2}$ in the 2-d case also has the Monge-Ampère form $\det D^2u = 1$, and Jörgens already showed Theorem 1.1 in this special case earlier on (cf. [N]).

Other Bernstein-Liouville type results concerning (1.1) are in order. Borishenko [B] showed that any convex solution with linear growth to (1.1) with $c = k\pi$ is linear. The author [Y] proved that any convex solution to (1.1) must be quadratic. For $c = k\pi$ in $n = 3$ case, (1.1) has another form

$$\triangle u = \det D^2 u. \tag{1.2}$$

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It was proved in [BCGJ] that any strictly convex solution to (1.2) with quadratic growth must be quadratic. Under the assumption that the Hessian is bounded and $\lambda_i \lambda_j \geq -\frac{3}{2}$, it was also shown in [TW] that any global solution to (1.1) is quadratic.

The heuristic idea of the proof of Theorem 1.1 is to find a better graph representation of $(x, \nabla u(x))$ so that the Hessian of the new potential is bounded, and the new potential function satisfies a convex uniformly elliptic equation. By Krylov-Evan's [K], [E] interior Hölder estimates on the Hessian, we draw the conclusion.

As there are nontrivial global harmonic solutions to (1.1) with $c = 0$ in the case $n = 2$, we guess (1.1) with $c = \frac{\pi}{2}(n-2)$ also has nontrivial global solutions in the higher-dimensional case. Observe that in the case $n = 3$ and $c = \frac{\pi}{2}$, (1.1) also takes the interesting form $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1$.

2. Proof

Step 1. We first find a better graph representation of M through Lewy rotation (cf. [N]) so that the Hessian of the potential function is bounded. By symmetry we only consider the case $c > \frac{\pi}{2}(n-2)$. Let $\sum_{i=1}^n \theta_i = \frac{\pi}{2}(n-2) + \delta$, where $\theta_i = \arctan \lambda_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\delta \in (0, \pi)$. Note that

$$-\frac{\pi}{2} + \frac{(n-1)}{n}\delta < \theta_i - \frac{\delta}{n} < \frac{\pi}{2} - \frac{\delta}{n}. \tag{2.1}$$

The first inequality follows from $\frac{\pi}{2}(n-2) + \delta < \theta_i + \frac{\pi}{2}(n-1)$. We rotate the $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ coordinate system to $(\bar{x}, \bar{y})$ by $\frac{\delta}{n}$, namely, $\bar{x} = x\cos\frac{\delta}{n} + y\sin\frac{\delta}{n}$, $\bar{y} = -x\sin\frac{\delta}{n} + y\cos\frac{\delta}{n}$. In terms of complex variables $z = x + \sqrt{-1}y$, that is, we identify $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$, the rotation takes the form $\bar{z} = e^{-\sqrt{-1}\delta/n}z$. Then M has a new parametrization

$$\begin{cases} \bar{x} = x\cos\frac{\delta}{n} + \nabla u(x)\sin\frac{\delta}{n}, \\ \bar{y} = -x\sin\frac{\delta}{n} + \nabla u(x)\cos\frac{\delta}{n}. \end{cases}$$

By (2.1), $M = (x, \nabla u(x))$ is still a graph over the whole $\bar{x}$ space $\mathbb{R}^n$. Further the rotation belongs to $U(n)$. Then M is also a special Lagrangian graph $(\bar{x}, \nabla\bar{u}(\bar{x}))$, where $\bar{u}$ is a smooth function [HL, p. 87, Proposition 2.17]. Let $\bar{\lambda}_i$ be the eigenvalues of $D^2\bar{u}$. Then $\bar{\theta}_i = \arctan\bar{\lambda}_i = \theta_i - \frac{\delta}{n} \in \left(-\frac{\pi}{2} + \frac{(n-1)}{n}\delta, \frac{\pi}{2} - \frac{\delta}{n}\right)$. That is,

$$\left|D^2\bar{u}\right| \leq C(\delta).$$

Finally $\bar{u}$ satisfies the equation

$$\sum_{i=1}^n \arctan\bar{\lambda}_i = \frac{\pi}{2}(n-2). \tag{2.2}$$

Step 2. We proceed with the following lemma, which is Lemma 8.1 in [CNS] when n is even and $c = \frac{\pi}{2}(n-2)$.

Lemma 2.1. *Let $f(\lambda_1, \cdots, \lambda_n) = \sum_{i=1}^n \arctan\lambda_i$ and let $\Gamma = \{\lambda | f(\lambda) = c\}$ with $|c| \geq \frac{\pi}{2}(n-2)$. Then Γ is convex.*

Proof. We skip the case $n = 1$. By symmetry we just consider the case $c \geq 0$. Set $c = \frac{\pi}{2}(n-2) + \delta$ with $\delta \in [0, \pi)$. We may assume that $\theta_i = \arctan\lambda_i \geq 0$ for

$i = 1, \cdots, n-1$. The normal of Γ is $\nabla f = \left(\cos^2\theta_1, \cdots, \cos^2\theta_n\right)$. Let

$$A \triangleq -\frac{1}{2}D^2 f = \begin{bmatrix} \tan\theta_1\cos^4\theta_1 & & \\ & \cdots & \\ & & \tan\theta_n\cos^4\theta_n \end{bmatrix}.$$

Take any tangent vector $T = (t_1, \cdots, t_n) \in T_\lambda\Gamma$, that is,

$$\sum_{i=1}^{n} t_i \cos^2\theta_i = 0.$$

We show that $A(T,T) \geq 0$.

Case a) $\theta_n \geq 0$. Certainly it is true.

Case b) $\theta_n < 0$. First we know that $\theta_i > 0$ for $i = 1, \cdots, n-1$ and $\delta < \frac{\pi}{2}$. Next we have

$$A(T,T) = \sum_{i=1}^{n-1} \tan\theta_i \cos^4\theta_i t_i^2 + \tan\theta_n \cos^4\theta_n t_n^2.$$

Now we use the trick in [CNS, p. 299],

$$\begin{aligned}(-t_n\cos\theta_n)^2 &= \left(\sum_{i=1}^{n-1} t_i\cos^2\theta_i\right)^2 \\ &\leq \left(\sum_{i=1}^{n-1} t_i^2\cos^4\theta_i\tan\theta_i\right)\left(\sum_{i=1}^{n-1}\cot\theta_i\right).\end{aligned}$$

Then

$$\tan\theta_n\cos^4\theta_n t_n^2 \geq \left(\sum_{i=1}^{n-1} t_i^2\cos^4\theta_i\tan\theta_i\right)\left(\sum_{i=1}^{n-1}\cot\theta_i\right)\tan\theta_n$$

and

$$\begin{aligned}A(T,T) &\geq \left(\sum_{i=1}^{n-1} t_i^2\cos^4\theta_i\tan\theta_i\right)\left[1 + \left(\sum_{i=1}^{n-1}\cot\theta_i\right)\tan\theta_n\right] \\ &= \left(\sum_{i=1}^{n-1} t_i^2\cos^4\theta_i\tan\theta_i\right)\left(\sum_{i=1}^{n}\cot\theta_i\right)\tan\theta_n.\end{aligned}$$

Let $\alpha_i = \frac{\pi}{2} - \theta_i$. We have

$$\begin{gathered}\frac{\pi}{2} < \pi - \delta = \alpha_1 + \cdots + \alpha_n < \pi, \\ 0 < \alpha_1, \cdots, \alpha_{n-1} < \frac{\pi}{2} < \alpha_n,\end{gathered}$$

and

$$\sum_{i=1}^{n}\cot\theta_i = \sum_{i=1}^{n-1}\tan\alpha_i + \tan\alpha_n.$$

It follows that $\tan\alpha_n < 0$ and

$$\frac{\tan(\alpha_1 + \cdots + \alpha_{n-1}) + \tan\alpha_n}{1 - \tan(\alpha_1 + \cdots + \alpha_{n-1})\tan\alpha_n} = \tan(\alpha_1 + \cdots + \alpha_n) < 0.$$

Then $\tan(\alpha_1 + \cdots + \alpha_{n-1}) + \tan\alpha_n < 0$. Note that $\alpha_1 + \cdots + \alpha_{n-1} = \pi - \delta - \alpha_n < \frac{\pi}{2}$; we have

$$\begin{aligned} \tan(\alpha_1 + \cdots + \alpha_{n-1}) &\geq \tan\alpha_1 + \tan(\alpha_2 + \cdots + \alpha_{n-1}) \\ &\cdots \\ &\geq \tan\alpha_1 + \tan\alpha_2 + \cdots + \tan\alpha_{n-1}. \end{aligned}$$

So $\sum_{i=1}^n \cot\theta_i = \sum_{i=1}^n \tan\alpha_i < 0$ and $A(T,T) \geq 0$. Therefore Γ is convex (w.r.t. the normal ∇f). □

Remark 1. The level set Γ is no longer convex or concave when $|c| < \frac{\pi}{2}(n-2)$.

Step 3. The final argument is standard. We now have global solution $\bar{u}$ with bounded Hessian on the convex level set Γ, more precisely a convex level set in the symmetric matrix space (cf. [CNS, p. 276]). In another word, $\bar{u}$ satisfies (2.2), which is now uniformly elliptic. By Krylov-Evans theorem ([K],[E])

$$\left[D^2\bar{u}\right]_{C^\beta(B_r)} \leq C(n,\delta)\frac{\left\|D^2\bar{u}\right\|_{L^\infty(B_{2r})}}{r^\beta} \leq \frac{C(n,\delta)}{r^\beta},$$

where $\beta = \beta(n,\delta) \in (0,1)$. Let r go to $+\infty$; we see that $D^2\bar{u}$ is a constant matrix. Thus $(\bar{x}, \nabla\bar{u})$ is a plane, and consequently u is quadratic.

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