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Rigidity of Complete Entire Self-Shrinking Solutions to Kähler–Ricci Flow

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This article is dedicated to Professor Wei-Yue Ding on the occasion of his seventieth birthday.

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We show that every complete entire self-shrinking solution on complex Euclidean space to the Kähler–Ricci flow must be generated from a quadratic potential.

1 Introduction

In this note, we prove the following result.

Theorem 1.1. Suppose that u is an entire smooth pluri-subharmonic solution on $\mathbb{C}^m$ to the complex Monge–Ampère equation

$$\ln\det(u_{\alpha\bar{\beta}}) = \frac{1}{2}x \cdot Du - u. \tag{1}$$

Assume that the corresponding Kähler metric $g = (u_{\alpha\bar{\beta}})$ is complete. Then, u is quadratic. □

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Any solution to (1) leads to a self-shrinking solution $v(x,t) = -tu(x/\sqrt{-t})$ to a parabolic complex Monge–Ampère equation

$$v_t = \ln\det(v_{\alpha\bar{\beta}})$$

in $\mathbb{C}^m \times (-\infty, 0)$, where $z^\alpha = x^\alpha + \sqrt{-1}x^{m+\alpha}$. Note that the above equation of v is the potential equation of the Kähler–Ricci flow $\partial_t g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}$. In fact, the corresponding metric $(u_{\alpha\bar{\beta}})$ is a shrinking Kähler–Ricci (non-gradient) soliton.

Assuming a certain inverse quadratic decay of the metric—a specific completeness assumption—Theorem 1.1 has been proved in [2]. Similar rigidity results for self-shrinking solutions to Lagrangian mean curvature flows were obtained in [2, 7, 8].

The idea of our argument, as in [2], is still to force the phase $\ln\det(u_{\alpha\bar{\beta}})$ in Equation (1) to attain its global maximum at a finite point. As this phase satisfies an elliptic equation without the zeroth order terms (see (3)), the strong maximum principle implies the constancy of the phase. Consequently, the homogeneity of the self-similar terms on the right-hand side of Equation (1) leads to the quadratic conclusion for the solution.

However, the difficulty of the above argument lies in the first step: Here, we cannot construct a barrier as in [2], which requires the specific inverse quadratic decay of the metric, to show the phase achieves its maximum at a finite point. The new observation is that the radial derivative of the phase, which is the negative of the scalar curvature of the metric (4), is in fact non-positive; hence, the phase value at the origin is its global maximum. The non-negativity of the scalar curvature is a result of Chen [3], as the induced metric $g(x,t) = (u_{\alpha\bar{\beta}}(x/\sqrt{-t}))$ is a complete ancient solution to the (Kähler–)Ricci flow. Here, we provide a direct elliptic argument for the non-negativity of the scalar curvature for the complete self-shrinking solutions (in Section 3, after necessary preparation in Section 2, where a pointwise approach to Perelman's upper bound of the Laplacian of the distance [9] is also included). Heuristically, one sees the minimum of the scalar curvature is non-negative from its inequality (6); it is definitely so if the minimum is attained at a finite point. Note that a thorough study of the lower bound of scalar curvatures of the gradient Ricci solitons is presented in [5, Chapter 27].

2 Preliminary Results

For the potential u of the Kähler metric $g = (g_{\alpha\bar{\beta}}) = (u_{\alpha\bar{\beta}})$ on $\mathbb{C}^m$, we denote the phase by $\Phi = \ln\det(u_{\alpha\bar{\beta}})$. Then, the Ricci curvature is given by $R_{\alpha\bar{\beta}} = -\frac{\partial^2\Phi}{\partial z^\alpha \partial \bar{z}^\beta}$. The "complex" scalar curvature is $R = g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$ (R is one-half of the usual "real" scalar curvature). Let $\rho(x)$ denote the Riemannian distance from x to 0 in $(\mathbb{C}^m, g)$. For a solution u of (1), we derive the following equations and inequalities for those geometric quantities.

2.1 Equation for phase Φ

Since u is a solution of (1), the phase satisfies the equation $\Phi = \frac{1}{2} x \cdot Du - u$. Taking two derivatives,

$$- R_{\alpha\bar{\beta}} = \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{2} x \cdot Du_{\alpha\bar{\beta}}. \tag{2}$$

Differentiating $\Phi = \ln \det(u_{\alpha\bar{\beta}})$,

$$D\Phi = g^{\alpha\bar{\beta}} Du_{\alpha\bar{\beta}}.$$

Combining these equations, we obtain

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{2} x \cdot D\Phi. \tag{3}$$

In particular, we have the important relation

$$R = -\tfrac{1}{2} x \cdot D\Phi. \tag{4}$$

2.2 Inequality for scalar curvature R

Differentiating $R = -\frac{1}{2} x \cdot D\Phi$ twice and using $R_{\alpha\bar{\beta}} = -\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta}$,

$$\frac{\partial^2 R}{\partial z^\alpha \partial \bar{z}^\beta} = -\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{1}{2} x \cdot D \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = R_{\alpha\bar{\beta}} + \frac{1}{2} x \cdot DR_{\alpha\bar{\beta}}. \tag{5}$$

Also, differentiating $R = g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$,

$$DR = -g^{\alpha\bar{\gamma}} Du_{\bar{\gamma}\delta} g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} DR_{\alpha\bar{\beta}}.$$

Hence, by (2),

$$\begin{aligned} \tfrac{1}{2} x \cdot DR &= -g^{\alpha\bar{\gamma}} (\tfrac{1}{2} x \cdot Du_{\bar{\gamma}\delta}) g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} \tfrac{1}{2} x \cdot DR_{\alpha\bar{\beta}} \\ &= g^{\alpha\bar{\gamma}} (R_{\bar{\gamma}\delta}) g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} \tfrac{1}{2} x \cdot DR_{\alpha\bar{\beta}}. \end{aligned}$$

Coupled with (5), we obtain

$$g^{\alpha\bar{\beta}}\frac{\partial^2 R}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{1}{2}x \cdot DR = R - g^{\alpha\bar{\gamma}} g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} R_{\bar{\gamma}\delta} \leq R - \frac{1}{m}R^2,$$

or equivalently

$$g^{\alpha\bar{\beta}}\frac{\partial^2 R}{\partial z^\alpha \partial \bar{z}^\beta} \leq \frac{1}{2}x \cdot DR + R - \frac{1}{m}R^2. \tag{6}$$

2.3 Inequality for distance ρ

Fix a point $x \in \mathbb{C}^m$, and let $\rho = \rho(x)$. We assume that x is not in the cut locus of 0. Since $(\mathbb{C}^m, g)$ is complete, there is a (unique) unit speed minimizing geodesic $\chi : [0, \rho] \to \mathbb{C}^m$ from 0 to x. We introduce a vector field $X(\tau)$ along $\chi(\tau)$ defined by $X = \chi^\alpha \frac{\partial}{\partial z^\alpha} + \chi^{\bar{\beta}} \frac{\partial}{\partial \bar{z}^\beta}$, where we regard $\chi \in \mathbb{C}^m$ as a tangent vector. Note that $X(0) = 0$ and $X(\rho) = x^i \frac{\partial}{\partial x^i}$.

We proceed to compute the directional derivative $x \cdot D\rho(x)$ using the metric g:

$$\begin{aligned} x \cdot D\rho(x) &= \langle X(\rho), \nabla_g \rho \rangle_g = \langle X(\rho), \dot{\chi}(\rho) \rangle \\ &= \int_0^\rho \frac{\mathrm{d}}{\mathrm{d}\tau} \langle X(\tau), \dot{\chi}(\tau) \rangle \,\mathrm{d}\tau = \int_0^\rho \langle \nabla_\tau X(\tau), \dot{\chi}(\tau) \rangle \,\mathrm{d}\tau, \end{aligned}$$

where the tangent vector $\dot{\chi}(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\chi$ and for simplicity of notation we have dropped the subscript g in the inner product $\langle \, , \, \rangle_g$. To calculate the above integrand, we first compute the covariant derivative of X along χ:

$$\begin{aligned} \nabla_\tau X &= \dot{\chi}^\alpha \frac{\partial}{\partial z^\alpha} + \dot{\chi}^{\bar{\beta}} \frac{\partial}{\partial \bar{z}^\beta} + \chi^\alpha \nabla_{\dot{\chi}} \frac{\partial}{\partial z^\alpha} + \chi^{\bar{\beta}} \nabla_{\dot{\chi}} \frac{\partial}{\partial \bar{z}^\beta} \\ &= \dot{\chi} + \chi^\alpha \Gamma^\mu_{\gamma\alpha} \dot{\chi}^\gamma \frac{\partial}{\partial z^\mu} + \chi^{\bar{\beta}} \Gamma^{\bar{\nu}}_{\bar{\delta}\bar{\beta}} \dot{\chi}^{\bar{\delta}} \frac{\partial}{\partial \bar{z}^\nu}. \end{aligned}$$

Then, using the identity $\Gamma^\mu_{\gamma\alpha} g_{\mu\bar{\beta}} = u_{\gamma\alpha\bar{\beta}}$ (for a Kähler potential) and (2), we have

$$\langle \nabla_\tau X, \dot{\chi} \rangle = 1 + X \cdot Du_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}} = 1 - 2R_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}}.$$

Therefore, we have the formula

$$x \cdot D\rho(x) = \rho(x) - \int_0^\rho 2R_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}} \,\mathrm{d}\tau. \tag{7}$$

We have the following estimate for the Laplacian of the distance function ρ.

Lemma 2.1. Suppose $\mathrm{Ric} \le K$ on $B_g(0, \rho_0)$ for $\rho_0 > 0$. If $\rho(x) > \rho_0$ and x is not in the cut locus of 0, then

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \rho}{\partial z^\alpha \partial \bar{z}^\beta}(x) \le \left[\frac{2m-1}{2\rho_0} + \frac{1}{3}K\rho_0\right] + \frac{1}{2}x \cdot D\rho(x) - \frac{1}{2}\rho(x). \tag{8}$$

□

Proof. The mean curvature H of the geodesic sphere $\partial B_g(0, \rho)$ with respect to the normal $\nabla_g \rho$ equals $\frac{-1}{2m-1}\Delta_g \rho$. As calculated in [1, p. 52], H satisfies the following differential inequality:

$$H_\rho \ge H^2 + \frac{1}{2m-1}\mathrm{Ric}(\nabla_g \rho, \nabla_g \rho).$$

Let $H = \frac{-1}{\rho} + b$. Since the Riemannian metric g is asymptotically Euclidean as $\rho \to 0$, we know b is bounded for small ρ (in fact $O(\rho)$). We then have a corresponding inequality for b:

$$\frac{1}{\rho^2} + b_\rho \ge \frac{1}{\rho^2} - 2\frac{1}{\rho}b + b^2 + \frac{1}{2m-1}\mathrm{Ric}(\nabla_g \rho, \nabla_g \rho),$$

and consequently

$$(\rho^2 b)_\rho \ge \rho^2 b^2 + \frac{\rho^2}{2m-1}\mathrm{Ric}(\nabla_g \rho, \nabla_g \rho) \ge \frac{\rho^2}{2m-1}\mathrm{Ric}(\nabla_g \rho, \nabla_g \rho).$$

Integrating along $\chi(\tau)$, we arrive at

$$b(\chi(\rho_0)) \ge \frac{1}{\rho_0^2}\int_0^{\rho_0} \frac{\tau^2}{2m-1}\mathrm{Ric}(\dot{\chi}, \dot{\chi})\,\mathrm{d}\tau.$$

Then, for $\rho \ge \rho_0$,

$$\begin{aligned} H(\chi(\rho)) &= H(\chi(\rho_0)) + \int_{\rho_0}^{\rho} H_\rho \,\mathrm{d}\tau \\ &\ge H(\chi(\rho_0)) + \int_{\rho_0}^{\rho} \frac{1}{2m-1}\mathrm{Ric}(\dot{\chi}, \dot{\chi})\,\mathrm{d}\tau \\ &\ge \frac{-1}{\rho_0} + \frac{1}{\rho_0^2}\int_0^{\rho_0} \frac{\tau^2}{2m-1}\mathrm{Ric}(\dot{\chi}, \dot{\chi})\,\mathrm{d}\tau + \int_{\rho_0}^{\rho} \frac{1}{2m-1}\mathrm{Ric}(\dot{\chi}, \dot{\chi})\,\mathrm{d}\tau. \end{aligned}$$

Substituting back $\Delta_g \rho = -(2m-1)H$ and recalling $x = \chi(\rho)$, we obtain

$$\Delta_g \rho \le \frac{2m-1}{\rho_0} - \int_0^{\rho} \mathrm{Ric}(\dot{\chi}, \dot{\chi})\,\mathrm{d}\tau + \int_0^{\rho_0} \left(1 - \frac{\tau^2}{\rho_0^2}\right)\mathrm{Ric}(\dot{\chi}, \dot{\chi})\,\mathrm{d}\tau, \tag{9}$$

when $\rho \geq \rho_0$. In fact, this estimate was first derived in [9, Section 8] (by a second variation argument).

Note that the Riemannian Laplacian $\Delta_g = 2g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$ and $\mathrm{Ric}(\dot{\chi}, \dot{\chi}) = 2R_{\alpha\bar{\beta}}\dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}}$. Then, (8) follows from combining (7) with (9). ■

To prove Theorem 1.1, we will also need an inequality for $\rho^y(x) =$ distance from x to y in $(\mathbb{C}^m, g)$. Following the previous argument and using $(X - y^i \frac{\partial}{\partial x^i})$ instead of X for the vector field along χ, we have the following lemma.

Lemma 2.2. Suppose $\mathrm{Ric} \leq K$ on $B_g(y, \rho_0)$. Fix $A > \rho_0$, and let $x \in B_g(y, A)$. If $\rho^y(x) > \rho_0$ and x is not in the cut locus of y, then

$$g^{\alpha\bar{\beta}}\frac{\partial^2 \rho^y}{\partial z^\alpha \partial \bar{z}^\beta}(x) \leq \left[\frac{2m-1}{2\rho_0} + \frac{1}{3}K\rho_0\right] + \frac{1}{2}x \cdot D\rho^y(x) - \frac{1}{2}\rho^y(x) + C_0 A|y|, \tag{10}$$

where the constant C_0 only depends on the "Euclidean" norms of $Du_{\alpha\bar{\beta}}$ and g^{-1} in $B_g(y, A)$. □

Proof. Arguing as above, let χ be the unit speed minimizing geodesic from y to x. Using $(X - y^i \frac{\partial}{\partial x^i})$ for the vector field along χ (note that $X(0) = y^i \frac{\partial}{\partial x^i}$), we have

$$(x - y) \cdot D\rho^y(x) = \rho^y(x) - \int_0^{\rho^y(x)} 2R_{\alpha\bar{\beta}}\dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}}\, d\tau - \int_0^{\rho^y(x)} y \cdot Du_{\alpha\bar{\beta}}\dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}}\, d\tau.$$

The conclusion of the lemma follows, as above, from combining this equation with (9). ■

3 Proof of Theorem 1.1

First, we prove the scalar curvature $R \geq 0$ on complete $(\mathbb{C}^m, g)$. Choose a cutoff function ϕ such that $\phi \equiv 1$ on $[0, 1]$, $\phi \equiv 0$ on $[2, \infty)$, $\phi' \leq 0$, $|\phi'| \leq C_1\phi^{1/2}$, and $|\phi'' - 2(\phi')^2/\phi| \leq C_2\phi^{1/2}$. For any small $\rho_0 > 0$, $K = K(\rho_0)$ can be chosen so that $Ric \leq K$ on $B_g(0, 2\rho_0)$. Fix $A > \rho_0$, we derive an effective negative lower bound (11) for R on $B_g(0, A)$. Set $\tilde{R} = \phi(\rho/A)R$. If $R < 0$ at some point in $B_g(0, 2A)$, then $\tilde{R}$ achieves a negative minimum at some point $p \in B_g(0, 2A)$, as $\overline{B_g(0, 2A)}$ is compact for each $A > 0$ by the completeness of $(\mathbb{C}^m, g)$. We consider two cases.

Case 1: p is not in the cut locus of 0. Then, ρ is smooth near p, and we have

$$\Delta_g \tilde{R} = \left(\frac{\phi''}{A^2} + \frac{\phi'}{A}\Delta_g \rho\right) R + \phi \Delta_g R + 2\langle \nabla\phi, \nabla R\rangle,$$

where we have used $|\nabla\rho| = 1$. In order to have a linear differential inequality for $\tilde{R}$ with smooth coefficients (even for the Lipschitz function ρ), we rewrite

$$\begin{aligned}
\langle \nabla\phi, \nabla R\rangle &= \left\langle \nabla\phi, \frac{\nabla\tilde{R}}{\phi} - \frac{\nabla\phi}{\phi^2}\tilde{R}\right\rangle = \left\langle \nabla\frac{\tilde{R}}{R}, \frac{\nabla\tilde{R}}{\phi}\right\rangle - \frac{|\nabla\phi|^2}{\phi^2}\tilde{R},\\
&\overset{|\nabla\rho|=1}{=} \frac{|\nabla\tilde{R}|^2}{\tilde{R}} - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R}\right\rangle - \frac{(\phi')^2}{A^2\phi}R,\\
&\overset{\tilde{R}<0}{\leq} -\left\langle \frac{\nabla R}{R}, \nabla\tilde{R}\right\rangle - \frac{(\phi')^2}{A^2\phi}R.
\end{aligned}$$

Using $\Delta_g = 2g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$, the inequalities (8) and (6) for ρ and R, and the inequalities for ϕ, we obtain

$$\begin{aligned}
g^{\alpha\bar{\beta}}\frac{\partial^2\tilde{R}}{\partial z^\alpha \partial\bar{z}^\beta} &\overset{\phi' R>0}{\leq} \frac{1}{2A^2}\left[\phi'' - \frac{2(\phi')^2}{\phi}\right]R + \frac{\phi' R}{A}\left[\left(\frac{2m-1}{2\rho_0} + \frac{1}{3}K\rho_0\right) + \frac{1}{2}x\cdot D\rho(x)\right]\\
&\quad + \phi\left(\frac{1}{2}x\cdot DR + R - \frac{1}{m}R^2\right) - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R}\right\rangle\\
&\leq \frac{C_2}{2A^2}\phi^{1/2}|R| + \frac{C_1}{A}\left(\frac{2m-1}{2\rho_0} + \frac{1}{3}K\rho_0\right)\phi^{1/2}|R| - \frac{\phi R^2}{m}\\
&\quad + \tilde{R} + \frac{1}{2}x\cdot D\tilde{R} - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R}\right\rangle\\
&\leq \frac{C(m,\rho_0)}{A^2} + \tilde{R} + b(x)\cdot D\tilde{R},
\end{aligned}$$

where $b(x)$ is a smooth function and $C(m,\rho_0)$ is a constant that depends only on m, ρ_0, C_1, and C_2. Since $\tilde{R}$ achieves its minimum at p, we have $\tilde{R}(p) \geq -\frac{C(m,\rho_0)}{A^2}$ and $R \geq -\frac{C(m,\rho_0)}{A^2}$ on $B_g(0, A)$.

Case 2: p is in the cut locus of 0. Then, ρ is not smooth at p, and we argue using Calabi's trick [1, p. 53] of approximating ρ from above by smooth functions (cf. [4, pp. 453–456]). For completeness, we include the argument here. Let χ be a unit speed geodesic from 0 to p that minimizes length, and define $\rho_\varepsilon = \rho^{\chi(\varepsilon)} + \varepsilon$, where $\rho^{\chi(\varepsilon)}$ is the distance to $\chi(\varepsilon)$. Then, $\rho_\varepsilon(p) = \rho(p)$ and $\rho_\varepsilon \geq \rho$ near p. Since p is not in the cut locus of $\chi(\varepsilon)$, we know that ρ_ε is smooth near p. Let $\tilde{R}_\varepsilon = \phi(\rho_\varepsilon/A)R$. Then, $\tilde{R}_\varepsilon$ is smooth near p.

Furthermore, since ϕ is decreasing and $R<0$ near p, the above properties of ρ_ε show that $\tilde{R}_\varepsilon(p)=\tilde{R}(p)$ and $\tilde{R}_\varepsilon \geq \tilde{R}$ near p. It follows that $\tilde{R}_\varepsilon$ has a local minimum at p. Arguing as we did in Case 1, and using Lemma 2.2 to estimate ρ_ε, we have

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \tilde{R}_\varepsilon}{\partial z^\alpha \partial \bar{z}^\beta} \leq \frac{C(m,\rho_0)}{A^2} + \tilde{R}_\varepsilon + b(x)\cdot D\tilde{R}_\varepsilon + \frac{\phi' R}{A}[C_0 2A|\chi(\varepsilon)|],$$

where $b(x)$ is a smooth function, $C(m,\rho_0)$ is a constant that depends only on m, ρ_0, C_1, and C_2, and C_0 is a constant depending on the "Euclidean" norms of $Du_{\alpha\bar{\beta}}$ and g^{-1} in $\overline{B_g(0,2A)}$. Note that we may choose C_0 independent of (small) ε. At p, we have

$$\tilde{R}_\varepsilon(p) \geq -\frac{C(m,\rho_0)}{A^2} - \frac{\phi' R}{A}(p)[C_0 2A|\chi(\varepsilon)|].$$

Taking $\varepsilon \to 0$, we arrive at the inequality $\tilde{R}(p) \geq -\frac{C(m,\rho_0)}{A^2}$, which shows $R \geq -\frac{C(m,\rho_0)}{A^2}$ on $B_g(0,A)$.

Combining Case 1 and Case 2, we have shown that

$$R \geq -\frac{C(m,\rho_0)}{A^2} \quad \text{on } B_g(0,A). \tag{11}$$

Taking $A\to\infty$, we arrive at $R\geq 0$ on $\mathbb{C}^m$.

Now, we finish the proof of Theorem 1.1. Since $R\geq 0$, it follows from the equation $R=-g^{\alpha\bar{\beta}}\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = -\frac{1}{2}x\cdot D\Phi$ that Φ achieves its global maximum at the origin. Applying the strong maximum principle to Equation (3), we conclude that Φ is constant. Using $\frac{1}{2}x\cdot Du - u = \Phi$, we have

$$\tfrac{1}{2}x\cdot D[u+\Phi(0)] = u+\Phi(0).$$

Finally, it follows from Euler's homogeneous function theorem that smooth $u+\Phi(0)$ is a homogeneous order 2 polynomial.

Remark. In fact, one sees that the Lipschitz function $\tilde{R}$ (on the set where $\tilde{R}<0$) is a subsolution to

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \tilde{R}}{\partial z^\alpha \partial \bar{z}^\beta} \leq \frac{C(m,\rho_0)}{A^2} + \tilde{R} + b(x)\cdot D\tilde{R}$$

in the viscosity sense by using the same trick of Calabi. It follows from the comparison principle (cf. [6, p. 18]) that the same negative lower bounds for $\tilde{R}$ and hence R can be derived. □

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