Maximal Hypersurfaces over Exterior Domains

GUANGHAO HONG
Xi'an Jiaotong University

YU YUAN
University of Washington

Dedicated to Luis A. Caffarelli on the occasion of his 70th birthday

Abstract

Exterior problems for the maximal surface equation are studied. We obtain the precise asymptotic behavior of the exterior solution at infinity. We also prove that the exterior Dirichlet problem is uniquely solvable for admissible boundary data and prescribed asymptotic behavior at infinity. © 2020 Wiley Periodicals LLC.

1 Introduction

The maximal surface equation is

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \tag{1.1}$$

or equivalently in the nondivergence form

$$\triangle u+\frac{(Du)' D^2uDu}{1-|Du|^2}=0. \tag{1.2}$$

This equation arises as the Euler equation of the variational problem that maximizes the area functional $\int \sqrt{1-|Du|^2}$ among the spacelike hypersurfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$ (see the definitions in Section 2). The graph of a solution to (1.1) is called a maximal hypersurface and the graph of a solution to the variational problem is called an area-maximizing hypersurface.

Calabi [4] ($n \leq 4$) and Cheng-Yau [5] (all dimensions) proved that every entire maximal hypersurface in $\mathbb{L}^{n+1}$ or every global solution u to the maximal surface equation (1.1) with $|Du(x)| < 1$ on $\mathbb{R}^n$ must be linear.

The Dirichlet problem for bounded domains was studied by Bartnik-Simon [2] and the isolated singularity problem was studied by Ecker [6]. The exterior problem is a "complementary" one for elliptic equations; see, for example, [1, 14] for minimal hypersurfaces, [3] for the Monge-Ampere equation, [11] for the special Lagrangian equation as well as other fully nonlinear elliptic equations, and [10] for infinity harmonic functions, besides the classic works such as [7] for linear

Communications on Pure and Applied Mathematics, Vol. LXXIV, 0589–0614 (2021)

ones. We study the exterior problems for the maximal surface equation in this paper. We obtain the precise asymptotic behavior of the exterior solution at infinity, and we prove that the exterior Dirichlet problem is uniquely solvable.

Throughout the paper, we assume $A \subset \mathbb{R}^n$ is a bounded closed set. We say that u is an exterior solution in $\mathbb{R}^n \backslash A$ if $u \in C^2(\mathbb{R}^n \backslash A)$ with $|Du(x)| < 1$ solves the equation (1.1) in $\mathbb{R}^n \backslash A$. Given an exterior solution u, for any bounded C^1 domain $U \supset A$, the integral

$$\operatorname{Res}[u] := \int_{\partial U} \frac{\partial u / \partial \vec{n}}{\sqrt{1 - |Du|^2}}\, d\sigma$$

is independent of the choices of U because of the divergence structure of the equation. The number $\operatorname{Res}[u]$ can be regarded as the residue of the exterior solution u.

THEOREM 1.1. *Let u be a smooth exterior solution in $\mathbb{R}^n \backslash A$ with A being bounded. Then there exists a vector $a \in B_1$ and a constant $c \in \mathbb{R}$ such that for $n = 2$*

$$\begin{aligned} u(x) = {} & a \cdot x + (1 - |a|^2) \operatorname{Res}[u] \ln \sqrt{|x|^2 - (a \cdot x)^2} + c \\ & + \operatorname{Res}[u] |a| \sqrt{1 - |a|^2} \frac{|x|(a \cdot x)}{|x|^2 - (a \cdot x)^2} \cdot \frac{\ln |x|}{|x|} + O_k(|x|^{-1}), \end{aligned} \tag{1.3}$$

and for $n \geq 3$

$$u(x) = a \cdot x + c - (1 - |a|^2) \operatorname{Res}[u] \Big(\sqrt{|x|^2 - (a \cdot x)^2} \Big)^{2-n} + O_k(|x|^{1-n}) \tag{1.4}$$

as $|x| \to \infty$ for all $k = 0, 1, \ldots$. The notation $\varphi(x) = O_k(|x|^m)$ means that $|D^k \varphi(x)| = O(|x|^{m-k})$.

On the other hand, for any bounded closed set A, given an admissible boundary value function $g : \partial A \to \mathbb{R}$ and prescribed asymptotic behavior at infinity, the exterior Dirichlet problem for maximal surface equation is uniquely solvable. We say g is admissible if g is bounded and there exists a spacelike function ψ in $\mathbb{R}^n \backslash A$ such that $\psi = g$ on ∂A in the sense of [2, (1.1)] (see Remark 2.2 in Section 2).

THEOREM 1.2. *Let $A \subset \mathbb{R}^n$ be a bounded closed set and $g : \partial A \to \mathbb{R}$ be an admissible boundary value function. Then*:

(1) *$n = 2$, given any $a \in B_1$ and $d \in \mathbb{R}$, there exists a unique smooth solution u of maximal surface equation on $\mathbb{R}^2 \backslash A$ such that $u = g$ on ∂A and*

$$u(x) = a \cdot x + d \ln \sqrt{|x|^2 - (a \cdot x)^2} + O(1) \quad \text{as } x \to \infty;$$

(2) *$n \geq 3$, given any $a \in B_1$ and $c \in \mathbb{R}$, there exists a unique smooth solution u of maximal surface equation on $\mathbb{R}^n \backslash A$ such that $u = g$ on ∂A and*

$$u(x) = a \cdot x + c + o(1) \quad \text{as } x \to \infty.$$

Of course, the function u enjoys finer asymptotic properties and the relation $d = (1 - |a|^2) \operatorname{Res}[u]$ holds by Theorem 1.1.

Variational solutions over exterior domains has been studied in [9]. At present, the related results for variational solutions are far from complete; see [9] for details.

The article is organized as follows. In Section 2, we set up some notations and definitions, and we collect some results from [2, 5, 6] that are needed in the proofs of the later sections. In Section 3, we prove that a spacelike function over an exterior domain can be spacelike extended to the whole $\mathbb{R}^n$. This is the starting point of our work. Interestingly there is a striking similarity between our argument and the argument in [3, pp. 571–572] where Caffarelli and Li prove the locally convex solution of $\det D^2u = 1$ over an exterior domain can be extended to a global convex function (after finitely enlarging the bounded complementary domain in both cases). In Section 4, we prove a growth control theorem for the exterior solution u at infinity. This is the key content of this paper. Inspired by Ecker's proof in [6] and relying on his results there, our argument involves compactness, blowdown analysis, and the comparison principle. In Section 5, we prove the gradient estimate for u based on the growth control theorem and Cheng-Yau's estimate on the second fundamental form. In Section 6, we prove Theorem 1.1. Since the equation (1.1) becomes uniformly elliptic by the gradient estimate of the previous section, the standard tools such as the Harnack inequality and the Schauder estimate apply. The known radially symmetric solutions play a key role in the proof. In Section 7, we prove Theorem 1.2. We solve the equation in a series of bigger and bigger ring-shaped domains and use the compactness method to get an exterior solution. We use the Lorentz transformations of radially symmetric solutions as barrier functions to guarantee the prescribed asymptotic behavior of the exterior solution near ∞. The uniqueness of solutions follows from the comparison principle.

2 Notations and Preliminary Results

We denote the Lorentz-Minkowski space by $\mathbb{L}^{n+1} = \{X = (x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}\}$, with the flat metric $\sum_{i=1}^n dx_i^2 - dt^2$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{L}^{n+1}$ with the signature $(+, \dots, +, -)$.

The light cone at $X_0 = (x_0, t_0) \in \mathbb{L}^{n+1}$ is defined by

$$C_{X_0} = \{X \in \mathbb{L}^{n+1} : \langle X - X_0, X - X_0 \rangle = 0\}.$$

The upper and lower light cones will be denoted by $C^+_{X_0}$ and $C^-_{X_0}$ respectively.

The Lorentz balls are defined by

$$L_R(X_0) = \{X \in \mathbb{L}^{n+1} : \langle X - X_0, X - X_0 \rangle < R^2\}.$$

Let M be an n-dimensional hypersurface in $\mathbb{L}^{n+1}$ that can be represented as the graph of $u \in C^{0,1}(\Omega)$, where Ω is a open set in $\mathbb{R}^n$. We say that M (or u) is

- *weakly spacelike* if $|Du| \le 1$ a.e. in Ω,
- *spacelike* if $|u(x) - u(y)| < |x - y|$ whenever $x, y \in \Omega$, $x \neq y$, and the line segment $\overline{xy} \subset \Omega$, and
- *strictly spacelike* if $u \in C^1(\Omega)$ and $|Du| < 1$ in Ω.

If M (or u) is strictly spacelike and $u \in C^2(\Omega)$, the Lorentz metric on $\mathbb{L}^{n+1}$ induces a Riemannian metric g on M. Under the coordinates $(x_1, \dots, x_n) \in \Omega$,

$$g_{ij} = \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j} \right\rangle = \delta_{ij} - u_i u_j,$$

where $X = (x, u(x))$ is the position vector on the graph of u, and $u_k = u_{x_k} = \frac{\partial u}{\partial x_k}$ for $k = 1, \dots, n$. So $g = \mathrm{I} - Du(Du)'$, $\det g = 1 - |Du|^2$,

$$g^{-1} = \mathrm{I} + \frac{Du(Du)'}{1 - |Du|^2}, \quad \text{and} \quad g^{ij} = \delta_{ij} + \frac{u_i u_j}{1 - |Du|^2}.$$

The second fundamental form is

$$\mathrm{II}_{ij} = \frac{u_{ij}}{\sqrt{\det g}} \quad \text{and so} \quad |\mathrm{II}|^2 = \frac{g^{ij} g^{kl} u_{ik} u_{jl}}{\det g}$$

(see [2, (2.3)]) where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$ and the summation convention on repeated indices is used. Note that $|D^2 u| \le |\mathrm{II}|$.

The following fundamental results were achieved by Bartnik and Simon in [2].

THEOREM 2.1 (Solvability of variational problem on bounded domains [2, prop. 1.1]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $\varphi : \partial\Omega \to R$ be a bounded function. Then the variational problem*

$$\sup_{v \in K} \int_\Omega \sqrt{1 - |Dv|^2} \tag{2.1}$$

where $K = \{v \in C^{0,1}(\Omega) : |Dv| \le 1$ a.e. in Ω and $v = \varphi$ on $\partial\Omega\}$ has a unique solution u if and only if the set K is nonempty.

Remark 2.2. In above theorem, $v = \varphi$ on $\partial\Omega$ means that, for every $x_0 \in \partial\Omega$ and every open straight line segment l contained in Ω and with endpoint x_0,

$$\lim_{x \to x_0,\, x \in l} v(x) = \varphi(x_0).$$

Regarding this definition and the existence of a weakly spacelike extension of φ, we refer the readers to the discussion in [2, p. 133, pp. 148–149] .

DEFINITION 2.3 (Area-maximizing hypersurface). A weakly spacelike function $u \in C(\Omega)$ ($\Omega \subset \mathbb{R}^n$ is not necessarily bounded) is called *area maximizing* if it solves the variational problem (2.1) with respect to its own boundary values for every bounded subdomain in Ω. The graph of u is called an *area-maximizing hypersurface*.

LEMMA 2.4 (Closedness of variational solutions [2, lemma 1.3]). *If $\{u_k\}$ is a sequence of area-maximizing functions in Ω and $u_k \to u$ in Ω locally uniformly, then u is also an area-maximizing function.*

One key result in [2, theorem 3.2] is that if an area-maximizing hypersurface contains a segment of light ray, then it contains the whole of the ray extended all the way to the boundary or to infinity. This implies the following conclusion.

THEOREM 2.5 (The relationship between the variational solutions and the solutions of maximal surface equation). *The solution u of* (2.1) *is smooth and solves equation* (1.1) *in*

$$\operatorname{reg} u := \Omega \setminus \operatorname{sing} u$$

where

$$\operatorname{sing} u := \{\overline{xy} : x, y \in \partial\Omega, x \neq y, \overline{xy} \subset \Omega, \textit{and } |\varphi(x) - \varphi(y)| = |x - y|\}.$$

Furthermore,

$$u(tx + (1-t)y) = t\varphi(x) + (1-t)\varphi(y), \quad 0 < t < 1,$$

where $x, y \in \partial\Omega$ are such that $\overline{xy} \subset \Omega$ and $|\varphi(x) - \varphi(y)| = |x - y|$.

Remark 2.6 (Solvability of maximal surface equation on bounded domains). If the boundary data φ admits a weakly spacelike extension and satisfies $|\varphi(x) - \varphi(y)| < |x - y|$ for all $x, y \in \partial\Omega$ with $\overline{xy} \subset \Omega$ and $x \neq y$, then $\operatorname{sing} u = \varnothing$ and hence smooth u solves the equation (1.1) in Ω.

Bartnik proved the following:

THEOREM 2.7 (Bernstein theorem for variational solutions [6, theorem F]). *Entire area-maximizing hypersurfaces in L^{n+1} are weakly spacelike hyperplanes.*

DEFINITION 2.8 (Isolated singularity [6, p. 382]). A weakly spacelike hypersurface M in L^{n+1} containing 0 is called an area-maximizing hypersurface with an isolated singularity at 0 if $M \setminus \{0\}$ is area maximizing but M cannot be extended as an area-maximizing hypersurface into 0.

For a weakly spacelike entire or exterior hypersurface M (i.e., u is defined on $\mathbb{R}^n$ or an exterior domain $\mathbb{R}^n \setminus A$ with A bounded), we define $M_r = r^{-1}M$ with $r > 0$ to be the graph of $u_r(x) = r^{-1}(rx)$. If for some $r_j \to +\infty$, $u_{r_j}(x)$ converges locally uniformly to a function $u_\infty(x)$ on $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$, then u_∞ (its graph M_∞) is called a blowdown of u (M). Note that by weakly spacelikeness, the Arzelà-Ascoli theorem always ensures the existence of blowdowns. By Lemma 2.4, $u_\infty(x)$ (M_∞) is area maximizing on $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$ and $u_\infty(0) = 0$.

Ecker proved that the isolated singularities of area-maximizing hypersurface are light-cone like [6, theorem 1.5]. The following lemma will also be used in our proof of Theorem 1.1.

LEMMA 2.9 ([6, lemma 1.10]). *Let M be an entire area-maximizing hypersurface with an isolated sigularity at* 0 *and assume that some blowdown of M also has an isolated singularity at* 0. *Then M has to be either C_0^+ or C_0^-.*

We also need the following radial, catenoid-like solutions to the maximal surface equation of (1.1) in $\mathbb{R}^n \setminus \{0\}$, used as barriers in [2,6]. For $\lambda \in \mathbb{R}$, set

$$w_\lambda(x) := \int_0^{|x|} \frac{\lambda}{\sqrt{t^{2(n-1)} + \lambda^2}}\, dt. \tag{2.2}$$

For $n > 2$, the integral $\int_0^{+\infty} \frac{\lambda}{\sqrt{t^{2(n-1)}+\lambda^2}}\, dt$ is bounded, and we denote this value as $M(\lambda, n)$. More precisely, by computation

$$\int_0^r \frac{\lambda}{\sqrt{t^{2(n-1)} + \lambda^2}} dt = M(\lambda, n) - \frac{\lambda}{n-2} r^{2-n} + O(r^{4-3n}) \tag{2.3}$$

for large r. It is obvious that $M(\lambda, n) = \operatorname{sign}(\lambda)|\lambda|^{\frac{1}{n-1}} M(1, n) \to \pm\infty$ as $\lambda \to \pm\infty$ and $M(\lambda, n) \to 0$ as $\lambda \to 0$. For $n = 2$, the integral $\int_0^{+\infty} \frac{\lambda}{\sqrt{t^2+\lambda^2}}\, dt$ is infinite and by computation

$$\int_0^r \frac{\lambda}{\sqrt{t^2 + \lambda^2}} dt = m(\lambda) + \lambda \ln r + O(r^{-2}) \tag{2.4}$$

for large r, where $m(\lambda) = \int_0^1 \frac{\lambda}{\sqrt{t^2+\lambda^2}}\, dt + \int_1^{+\infty} \big(\frac{\lambda}{\sqrt{t^2+\lambda^2}} - \frac{\lambda}{t}\big) dt$.

DEFINITION 2.10 (Lorentz transformations, the speed of light is normalized to 1). For a parameter $\kappa \in (-1, 1)$, the Lorentz transformation $L_\kappa : \mathbb{L}^{n+1} \to \mathbb{L}^{n+1}$ is defined as

$$L_\kappa : (x', x_n, t) \to \left(x', \frac{x_n + \kappa t}{\sqrt{1-\kappa^2}}, \frac{\kappa x_n + t}{\sqrt{1-\kappa^2}} \right)$$

where $x' = (x_1, \ldots, x_{n-1})$.

The Lorentz transformations are isometries of $\mathbb{L}^{n+1}$. L_κ maps spacelike (weakly spacelike) surfaces to spacelike (weakly spacelike) surfaces and it maps maximal surfaces (area-maximizing surfaces) to maximal surfaces (area-maximizing surfaces). Geometrically L_κ can be seen as a hyperbolic rotation. It maps the light cone $\{(x, t) \in \mathbb{L}^{n+1} : t = |x|\}$ to itself, and it maps the horizontal hyperplanes to the hyperplanes with slope κ:

$$L_\kappa(\{(x, t) \in \mathbb{L}^{n+1} : t = T\}) = \left\{(x, t) \in \mathbb{L}^{n+1} : t = \sqrt{1-\kappa^2}\, T + \kappa x_n\right\}$$

for $T \in (-\infty, +\infty)$.

More generally, for any vector $a \in B_1$ we define $L_a := T_a L_{|a|} T_a^{-1}$ where T_a is a rotation that keeps the t-axis fixed and transforms e_n to $\frac{a}{|a|}$ in $\mathbb{R}^n$ (in case of $a = 0$ we just define $T_0 := \mathrm{id}$).

3 Extension of Spacelike Hypersurface with Hole

We start our proofs for the two main theorems by extending any spacelike function over an exterior domain to a global spacelike function after finitely enlarging the bounded complementary domain.

THEOREM 3.1. *Let u be a spacelike function in $\mathbb{R}^n \setminus A$ with A being bounded. Then there exists $R^* > 0$ such that $|u(x) - u(y)| < |x - y|$ for all $x, y \in \mathbb{R}^n \setminus B_{R^*}$.*

PROOF.

Step 1. We first show that there exists a ball $B_{R_0}(x_0) \supset A$ such that on the boundary $osc_{x \in \partial B_{R_0}(x_0)} u(x) < 2R_0$. Without loss of generality we assume $A \subset B_1$. We suppose $\operatorname{osc}_{\partial B_{100}} u(x) \geq 200$ with $\max_{\partial B_{100}} u(x) = u(100e_1)$, and we will show that $\operatorname{osc}_{\partial B_{200}(100e_2)} u(x) < 400$.

Suppose $\max_{\partial B_{100}} u(x) = -\min_{\partial B_{100}} u(x)$ because otherwise we can consider $u - (\max_{\partial B_{100}} u + \min_{\partial B_{100}} u)/2$ in place of u. First, one can see that $\operatorname{osc}_{\partial B_{100}} u(x) \leq 202$ from the Lipschitz condition on u and the geometry of $\bar{B}_{100} \setminus B_1$. So $100 \leq u(100e_1) \leq 101$ and $\min_{\partial B_{100}} u(x) \in (-101, -100)$. Suppose $u(x_1) = \min_{\partial B_{100}} u(x)$ for some $x_1 \in \partial B_{100}$. Then $|x_1 - (-100e_1)| \leq 3$ because $u(x) > u(100e_1) - |100e_1 - x| > 100 - 200 = -100$ for any $x \in \partial B_{100} \setminus B_3(-100e_1)$. Thus $u(-100e_1) \in (-104, -97)$. Therefore

$$u(100e_2) > u(100e_1) - |100e_1 - 100e_2| \geq 100 - 100\sqrt{2} > -42$$

and

$$u(100e_2) < u(-100e_1) + |-100e_1 - 100e_2| < -97 + 100\sqrt{2} < 45.$$

In the same way, $u(-100e_2) \in (-42, 45)$. Denote $u(100e_2) = M$. Then

$$u(x) \in (M - 90, M + 90) \quad \text{for all } x \in B_3(-100e_2).$$

Let $\max_{x \in \partial B_{200}(100e_2) \setminus B_3(-100e_2)} |u(x) - M| := Q < 200$. Therefore

$$\operatorname{osc}_{\partial B_{200}(100e_2)} u(x) \leq 2 \max(Q, 90) < 400.$$

(See Figure 3.1.)

Step 2. We show that there exists $R_1 > R_0$ such that for all $R \geq R_1$ we have $|u(x) - u(y)| < |x - y|$ for all $x, y \in \partial B_R(x_0)$ with $x \neq y$. By making a suitable transformation, we may assume $x_0 = 0$, $R_0 = 1$, and $\max_{\partial B_1} u = -\min_{\partial B_1} u = 1 - \epsilon_0$ for some $0 < \epsilon_0 < 1$. Then for $R > 1$, $\max_{\partial B_R} |u| \leq R - \epsilon_0$. For $x, y \in \partial B_R$ with $x \neq y$, if the line segment $\overline{xy} \subset \bar{B}_R \setminus B_1$, then $|u(x) - u(y)| < |x - y|$. Otherwise, $\operatorname{dist}(0, \overline{xy}) < 1$ and $|x - y| > 2\sqrt{R^2 - 1}$. If

$$R \geq \frac{1 + \epsilon_0^2}{2\epsilon_0},$$

then $|u(x) - u(y)| \leq 2(R - \epsilon_0) \leq 2\sqrt{R^2 - 1} < |x - y|$.

Step 3. Set $R^* := |x_0| + R_1$. Suppose the line segment $\overline{xy} \cap \partial B_{R_1}(x_0) = \{p, q\}$ and p is closer to x than q. Then

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(p)| + |u(p) - u(q)| + |u(q) - u(y)| \\ &< |x - p| + |p - q| + |q - y| = |x - y|. \end{aligned}$$

If $p = q$, the conclusion is also true. We have $|u(x) - u(y)| < |x - y|$ directly if $\overline{xy} \cap \partial B_{R_1}(x_0) = \varnothing$. □

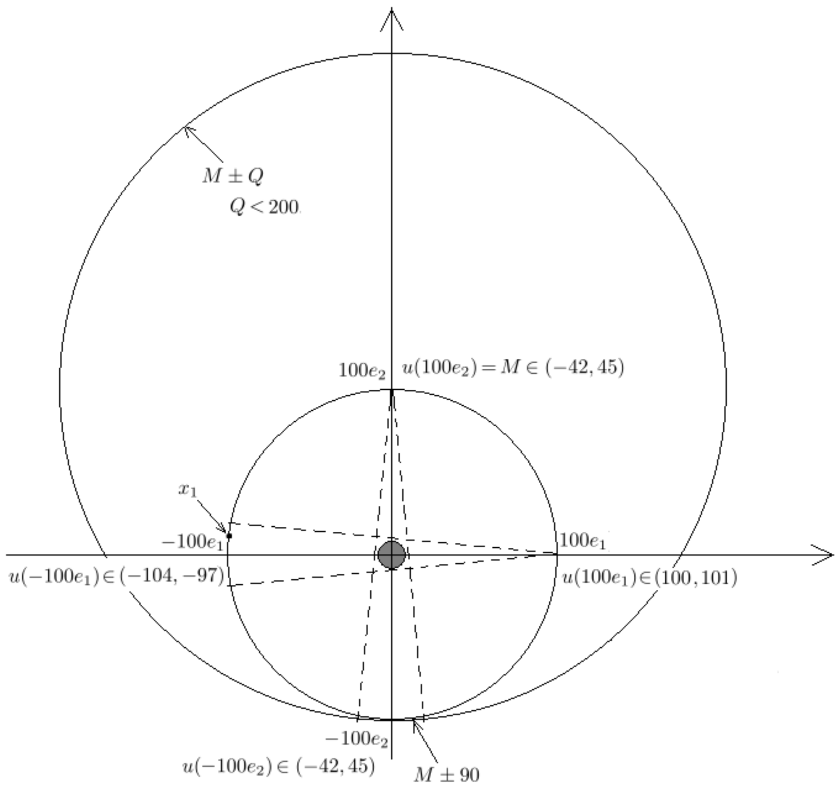

FIGURE 3.1. Shift the ball to hide the shadow.

For completeness, we include the promised full spacelike extension result here, which is not needed in the proofs of our two main theorems.

THEOREM 3.2. *Let u be a spacelike function in $\mathbb{R}^n \backslash A$ with A being bounded. Then there exists $R^* > 0$ such that $|u(x) - u(y)| < |x - y|$ for all $x, y \in \mathbb{R}^n \backslash B_{R^*}$. Moreover, there exists a spacelike function $\widetilde{u}$ in $\mathbb{R}^n$ such that $\widetilde{u} = u$ in $\mathbb{R}^n \backslash B_{R^*}$.*

PROOF. We only need to prove the second part of the theorem. By Remark 2.6, there exists a spacelike function w in B_{R^*} such that $w = u$ on ∂B_{R^*}. Define $\widetilde{u} := w$ in B_{R^*} and $\widetilde{u} := u$ in $\mathbb{R}^n \backslash B_{R^*}$. For $x, y \in \mathbb{R}^n$ with $x \neq y$, if both x and y are in $\overline{B}_{R^*}$ or $\mathbb{R}^n \backslash B_{R^*}$ then $|u(x) - u(y)| < |x - y|$. Otherwise, let $\{z\} = \overline{xy} \cap \partial B_{R^*}$, then

$$|u(x) - u(y)| \leq |u(x) - u(z)| + |u(z) - u(y)| < |x - z| + |z - y| = |x - y|.$$

If we assume the spacelike function u is also strictly spacelike $|Du(x)| < 1$ (to exclude spacelike functions such as $\arctan \lambda$), we can get a spacelike extension inside B_{R^*} directly, without relying on the singularity analysis of variational solutions to the maximal surface equations of [2] contained in Remark 2.6.

In fact (cf. [12, p. 61]), for $x \in \overline{B}_{R^*}$ set

$$w(x) = \inf_{b \in \partial B_{R^*}} \{u(b) + m|x - b|\}$$

with $m = \|Du\|_{L^\infty(\partial B_{R^*})} < 1$. Then $w(x) = u(x)$ for $x \in \partial B_{R^*}$, and for $x, y \in \overline{B}_{R^*}$

$$\begin{aligned} w(y) &= \inf_{b \in \partial B_{R^*}} \{u(b) + m|y - b|\} \\ &\le \inf_{b \in \partial B_{R^*}} \{u(b) + m|x - b| + m|y - x|\} \\ &\le w(x) + m|y - x|. \end{aligned}$$

Symmetrically $w(x) \le w(y) + m|x - y|$. Hence w is spacelike inside B_{R^*}, $|u(x) - u(y)| < m|x - y| < |x - y|$.

There is another differential way to do this extension inside B_{R^*}. Without loss of generality, we assume $R^* = 1$; then $\operatorname{osc}_{|x|=1} u(x) < 2$. For $x \in \overline{B}_1$ set

$$w(x) = |x|[u(x/|x|) - m] + m$$

with $m = \frac{1}{2}[\max_{|x|=1} u(x) + \min_{|x|=1} u(x)]$. Then $w(x) = u(x)$ on ∂B_1. For $x \in B_1 \setminus \{0\}$,

$$|Dw(x)| = |Du(x/|x|)| < 1.$$

The Lipschitz norm of w at $x = 0$ is also less than 1 because

$$|u(x/|x|) - m| \le \frac{\max_{|x|=1} u(x) - \min_{|x|=1} u(x)}{2} < 1.$$

We also reach the same spacelike conclusion of w inside B_1. □

4 Growth Control of *u* at Infinity

In this section, we show that the linear growth rate of an exterior solution u at infinity is uniformly less than one, that is to say, u is controlled not only by the light cone but by a cone with slop less than one. Meanwhile we prove that the blowdown of u is unique and is a linear function with slope less than one. We also prove that the graph of u is supported by a hyperplane either from below or from above.

THEOREM 4.1. *Let u be an exterior solution in $\mathbb{R}^n \setminus A$ with A being bounded. Then there exist $B_R \supset A$, $0 < \epsilon < 1$, and $c_0 \in \mathbb{R}$ such that*

$$-(1 - \epsilon)|x| \le u(x) - c_0 \le (1 - \epsilon)|x|$$

in $\mathbb{R}^n \setminus B_R$. Moreover, there exists a vector $a \in \overline{B}_{1-\epsilon}$ such that

$$\lim_{r \to \infty} \frac{u(rx)}{r} = a \cdot x \quad \text{locally uniformly in } \mathbb{R}^n \setminus \{0\}.$$

The function u also enjoys the property that either for some $c \in \mathbb{R}$, $u(x) \ge a \cdot x + c$ in $\mathbb{R}^n \setminus B_R$ and $u(y) = a \cdot y + c$ at some point $y \in \partial B_R$, or for some $c \in \mathbb{R}$, $u(x) \le a \cdot x + c$ in $\mathbb{R}^n \setminus B_R$ and $u(y) = a \cdot y + c$ at some point $y \in \partial B_R$.

PROOF. We apply Theorem 3.1. For simplicity of notation, we assume $R^* = 1$. So we have $|u(x) - u(y)| < |x - y|$ for any $x, y \in \mathbb{R}^n \setminus B_1$ with $x \neq y$. We also assume $\max_{\partial B_1} u = -\min_{\partial B_1} u = 1 - \epsilon_1$ for some $0 < \epsilon_1 < 1$. We will show that $-(1-\epsilon)|x| \leq u(x) \leq (1-\epsilon)|x|$ in $\mathbb{R}^n \setminus B_1$ for some $0 < \epsilon < 1$.

It is easy to see that $-|x| + \epsilon_1 \leq u(x) \leq |x| - \epsilon_1$ in $\mathbb{R}^n \setminus B_1$. So there are four possibilities for u:

(a) There is $0 < \epsilon < 1$ such that $u(x) \geq -(1-\epsilon)|x|$ in $\mathbb{R}^n \setminus B_1$ and there is a sequence of points $\{x_j\}$ with $1 < |x_j| := R_j \to +\infty$ such that $u(x_j) > (1 - \frac{1}{j})|x_j|$.

(b) The function $-u$ satisfies (a).

(c) There are two sequences of points $\{x_j^{\pm}\}$ with $1 < |x_j^{\pm}| := R_j^{\pm} \to +\infty$ such that $u(x_j^+) > (1 - \frac{1}{j})|x_j^+|$ and $u(x_j^-) < -(1 - \frac{1}{j})|x_j^-|$.

(d) There is $0 < \epsilon < 1$ such that $-(1-\epsilon)|x| \leq u(x) \leq (1-\epsilon)|x|$ in $\mathbb{R}^n \setminus B_1$.

We will show that the cases (a)–(c) cannot happen.

Suppose that u satisfies (a). Let

$$\hat{x} := \lim_{k\to\infty} \frac{x_{j_k}}{R_{j_k}} \in \partial B_1$$

for some subsequence $\{j_k\}$. We assume $\hat{x} = e_n$ and consider $\{j_k\}$ as $\{j\}$. Define $v_j(x) := \frac{u(R_j x)}{R_j}$. A subsequence of $v_j(x)$ (still denoted as $v_j(x)$) converges locally uniformly to a function $V(x)$ in $\mathbb{R}^n \setminus \{0\}$. By Lemma 2.4, $V(x)$ is *area maximizing* in $\mathbb{R}^n \setminus \{0\}$. It is obvious that $V(0) = 0$, $V(e_n) = 1$ and $V(x) \geq -(1-\epsilon)|x|$. By weakly spacelikeness of u, we know that $V(te_n) = t$ for $t \in [0,1]$. Theorem 2.5 says that once the null line (*sing u*) appears, it can not stops at an interior point. So we have $V(te_n) = t$ for $t \in [0, +\infty)$. If 0 is a removable singularity for V, then V is a plane by Theorem 2.7 and V has to be $V(x) = x_n$ that contradicts $V(x) \geq -(1-\epsilon)|x|$. So 0 is an isolated singularity for V. Let V_∞ be a blowdown of V, then $V_\infty(te_n) = t$ for $t \in (0, +\infty)$ and $V_\infty(x) \geq -(1-\epsilon)|x|$. So 0 is an isolated singularity for V_∞. By Lemma 2.9, we have $V(x) = |x|$.

Let $z \in \partial B_1$ be such that $u(z) = \min_{\partial B_1} u := \lambda$. For small $\delta > 0$, consider $w(x) := \lambda - 1 + \delta + (1-\delta)|x|$. Since $\lim_{j\to\infty} \frac{u(R_j x)}{R_j} = |x|$ uniformly on ∂B_1, $u(x) \geq w(x)$ on ∂B_{R_j} for $j \geq j_0(\delta)$. But $u(x) \geq \lambda = w(x)$ on ∂B_1 and $w(x)$ is a subsolution to (1.1) in $\mathbb{R}^n \setminus \bar{B}_1$, so $u(x) \geq w(x)$ in $\mathbb{R}^n \setminus \bar{B}_1$. Let $\delta \to 0$, we get $u(x) \geq \lambda - 1 + |x|$ in $\mathbb{R}^n \setminus \bar{B}_1$. Especially, $u(2z) \geq \lambda - 1 + |2z| = \lambda + 1$ and hence $u(2z) - u(z) \geq 1 = |2z - z|$. This contradicts the obvious fact that u is spacelike "in $\mathbb{R}^n \setminus B_1$", included in Theorem 3.1.

The case (b) cannot happen for the same reason.

Now we suppose u satisfies (c). For each j, let w_j be the solution of (1.1) in B_j with $w_j = u$ on ∂B_j. The existence of w_j is due to Remark 2.6. For each j, either $\max_{\partial B_1}(w_j - u) \geq 0$ or $\min_{\partial B_1}(w_j - u) \leq 0$ (or both). Thus

$\max_{\partial B_1}(w_j - u) \geq 0$ or $\min_{\partial B_1}(w_j - u) \leq 0$ happens for infinitely many j. We assume $\max_{\partial B_1}(w_j - u) := \lambda_j \geq 0$ happens for infinitely many j. Let $z_j \in \partial B_1$ be such that $w_j(z_j) - u(z_j) = \lambda_j$ and consider $\widetilde{w}_j = w_j - \lambda_j$ for these j. So $\widetilde{w}_j \leq u$ in $B_j \setminus B_1$ and $\widetilde{w}_j(z_j) = u(z_j)$. Note that $|\widetilde{w}_j(0)| \leq |\widetilde{w}_j(z_j)| + 1 = |u(z_j)| + 1 \leq 2$ for all these j. Therefore, by Arzelà-Ascoli a subsequence $\widetilde{w}_{j_k}$ converges locally uniformly to a function W in $\mathbb{R}^n$. By Lemma 2.4, W is an area maximizing surface. So it is a plane with slope less than or equal to 1 by Theorem 2.7. Furthermore, we know $W \leq u$ in $\mathbb{R}^n \setminus B_1$ and $W(z) = u(z)$ by continuity, where z is an accumulating point of $\{z_{j_k}\}$.

By assumption of (c), there are $\{x_j^-\}$ with $|x_j^-| \to +\infty$ such that $W(x_j^-) \leq u(x_j^-) < -(1 - \frac{1}{j})|x_j^-|$. Thus W has to be a plane with slope 1. We assume $DW(x) = e_n$, so $W(x) = x_n + u(z) - z_n$. If $z_n < 0$, then denote $\widetilde{z} = (z', -z_n) \in \partial B_1$ and we have $u(\widetilde{z}) \geq W(\widetilde{z}) = -2z_n + u(z) = u(z) + |\widetilde{z} - z|$. This contradicts the fact that $|u(x) - u(y)| < |x - y|$ for any $x, y \in \partial B_1$ with $x \neq y$, proved in Theorem 3.1. If $z_n \geq 0$, then consider the point $z + e_n \in \mathbb{R}^n \setminus \bar{B}_1$ and we have $u(z + e_n) \geq W(z + e_n) = u(z) + 1 = u(z) + |(z + e_n) - z|$. This contradicts the obvious fact that u is spacelike "in $\mathbb{R}^n \setminus B_1$", included in Theorem 3.1.

If it is the case that $\min_{\partial B_1}\big(w_j - u\big) \leq 0$ happens for infinitely many j, we move up w_j by $-\min_{\partial B_1}\big(w_j - u\big)$ and get a plane $\widehat{W}$ above u by the same process. This time by the assumption that there are $\{x_j^+\}$ with $|x_j^+| \to +\infty$ such that $\widehat{W}(x_j^+) \geq u(x_j^+) > (1 - \frac{1}{j})|x_j^+|$, we also know the slope of $\widehat{W}$ is one. Furthermore, $\widehat{W}$ also touches u at some point of ∂B_1. Again, this contradicts the obvious fact that u is spacelike "in $\mathbb{R}^n \setminus B_1$", included in Theorem 3.1.

Therefore only the case (d) can (and must) happens and we have proved the first part of the theorem. In this case we can also construct the plane W in the same way just as we did in the first paragraph when we proved the impossibility of case (c). That is to say, we can place a plane (with slope less than or equal to $1 - \epsilon$) either below or above the graph of u in $\mathbb{R}^n \setminus B_1$ and the plane touches u at some point of ∂B_1. This property implies that the blowdown of u must be unique and equal to the blowdown of W. We show this as follows. Assume that $W(x) = c + a \cdot x \leq u$ in $\mathbb{R}^n \setminus B_1$ where $|a| \leq 1 - \epsilon$. Let V be any blowdown of u, then $a \cdot x \leq V(x) \leq (1 - \epsilon)|x|$ in $\mathbb{R}^n$, which implies that 0 is a removable singularity of V by the fact that the isolated singularities of an area-maximizing hypersurface are light-cone-like [6, theorem 1.5]. Then V is an entire solution and must be a plane. The only possible situation is $V(x) = a \cdot x$. □

5 Gradient Estimate

With the strong growth control achieved in the previous section and the known curvature estimate, we can establish the gradient estimate and ascertain $Du(\infty)$ in this section. We state the curvature estimate of Cheng-Yau [5] in the following improved extrinsic form carried out by Schoen [6, theorem 2.2]).

THEOREM 5.1. *Let $M = (x, u(x))$ be a maximal hypersurface, $x_0 \in M$, and assume that for some $\rho > 0$, $L_{2\rho}(x_0) \cap M \Subset M$. Then we have for all $x \in L_\rho(x_0)$*

$$|\mathrm{II}|^2(x) \leq \frac{c(n)\rho^2}{(\rho^2 - l_{x_0}^2(x))^2} \tag{5.1}$$

where $c(n)$ is a constant depending only on the dimension n and

$$l_{x_0}(x) = (|x - x_0|^2 - |u(x) - u(x_0)|^2)^{\frac{1}{2}}.$$

If M is an entire maximal hypersurface, then ρ in (5.1) can be chosen to be arbitrarily large by properness of the function $l_{x_0}(x)$ on M [5, prop. 1], so $|\mathrm{II}| \equiv 0$ and hence the Bernstein theorem follows. But the following corollary is what we need.

COROLLARY 5.2. *For any $0 < \epsilon < 1$, there exists a positive constant $C(\epsilon, n)$ such that if u solves the equation (1.1) in $\mathbb{R}^n \setminus \bar{B}_1$ and satisfies $-(1-\epsilon)|x| \leq u(x) \leq (1-\epsilon)|x|$ in $\mathbb{R}^n \setminus B_1$, then*

$$|\mathrm{II}|(x) \leq \frac{C(\epsilon, n)}{|x|} \quad \text{for } |x| \geq \frac{8}{\epsilon}.$$

PROOF. Fix a point $x \in \mathbb{R}^n$ with $|x| \geq \frac{8}{\epsilon}$; for any $y \in \partial B_1$

$$|u(x) - u(y)| \leq |u(x)| + |u(y)| \leq (1-\epsilon)|x| + (1-\epsilon)$$

and

$$|x - y| \geq |x| - 1.$$

So

$$l_x(y) = (|y - x|^2 - |u(y) - u(x)|^2)^{\frac{1}{2}} > \sqrt{\frac{\epsilon}{2}}|x|.$$

Similarly, we also have

$$l_x(z) > \sqrt{\frac{\epsilon}{2}}|z| > \sqrt{\frac{\epsilon}{2}}|x| \quad \text{for } |z| \geq \frac{8}{\epsilon}|x|.$$

This means that

$$L_{\sqrt{\frac{\epsilon}{2}}|x|}(x) \cap M \Subset M$$

or the former is compactly contained in the latter. So by letting $x = x_0$ and $2\rho = \sqrt{\frac{\epsilon}{2}}|x|$ in (5.1) we get

$$|\mathrm{II}|(x) \leq \frac{\sqrt{c(n)}}{\sqrt{\frac{\epsilon}{8}}|x|} = \frac{C(\epsilon, n)}{|x|}. \qquad \square$$

THEOREM 5.3. *Let u be an exterior solution in $\mathbb{R}^n \setminus A$. For any open set $U \supset A$ there is $\theta > 0$ such that $|Du| \le 1 - \theta$ in $\mathbb{R}^n \setminus U$. Moreover,* $\lim_{x\to\infty} Du(x) = a$, *where a is given by Theorem* 4.1.

PROOF. Assume $A \subset B_1$ and $-(1-\epsilon)|x| \le u(x) \le (1-\epsilon)|x|$ in $\mathbb{R}^n \setminus B_1$. Denote $\widehat{R} := \frac{10}{\epsilon}$. Since $|Du(x)| < 1$ for $x \in \mathbb{R}^n \setminus U$, if $|Du| \le 1-\theta$ is not true, then there is a sequence of points $\{x_j\}$ such that $|Du(x_j)| > 1 - \frac{1}{j}$ and $|x_j| \to +\infty$. Define $R_j := \widehat{R}^{-1}|x_j|$ and $v_j(x) := \frac{u(R_j x)}{R_j}$ (assume $R_j > 1$). Then by Theorem 4.1, we have $v_j(x) \to V(x) = a \cdot x$.

On the other hand, by Corollary 5.2, the curvature $|\mathrm{II}|$ is uniformly bounded for all $v_j(x)$ on the compact set $\overline{B}_{\widehat{R}+1} \setminus B_{\widehat{R}-1}$, so is $|D^2 v_j(x)|$. This means $Dv_j(x) \to DV(x) = a$ in $\overline{B}_{\widehat{R}+1} \setminus B_{\widehat{R}-1}$. Denote

$$\lim_{k\to\infty} \frac{\widehat{R} x_{j_k}}{|x_{j_k}|} = \hat{x} \in \partial B_{\widehat{R}}$$

for some subsequence j_k. Then

$$DV(\hat{x}) = \lim_{k\to\infty} Dv_{j_k}\left(\frac{\widehat{R} x_{j_k}}{|x_{j_k}|}\right).$$

But

$$\left| Dv_{j_k}\left(\frac{\widehat{R} x_{j_k}}{|x_{j_k}|}\right)\right| = |Du(x_{j_k})| > 1 - \frac{1}{j_k}$$

and then the last inequality implies $|DV(\hat{x})| = 1$. This is a contradiction.

The conclusion $\lim_{x\to\infty} Du(x) = a$ can be proved in the same compactness way as above.

There is another Harnack way to show the existence of $Du(\infty)$, once $|Du|$ is uniformly bounded away from 1, $|Du| \le 1-\theta$. Indeed, each bounded component u_m of Du satisfies a uniformly elliptic equation

$$\partial_{x_i}\left[F_{p_i p_j}(Du)\partial_{x_j} u_m\right] = 0 \ \text{ in } \mathbb{R}^n \setminus A$$

with $F(p) = \sqrt{1-|p|^2}$. By Moser's Harnack [13, theorem 5], $\lim_{x\to\infty} Du(x)$ exists. □

Because we will use Moser's results again in the next section, we state them here in the needed form for convenience.

THEOREM 5.4 (Harnack inequality [13, theorem 1]). *Let w be a nonnegative solution of*

$$(a_{ij}(x) w_j)_i = 0 \tag{5.2}$$

in $\mathbb{R}^n \setminus B_{R_0}$, where $\Lambda^{-1}\mathrm{I} \le (a_{ij}(x)) \le \Lambda \mathrm{I}$ for a constant $\Lambda \in [1,\infty)$. Then for any $R \ge 10R_0$

$$\sup_{\partial B_R} w \le \Gamma \inf_{\partial B_R} w \tag{5.3}$$

for $\Gamma = \Gamma(n, \Lambda)$.

THEOREM 5.5 (Behavior at ∞ [13, theorem 5]). *Let w be a bounded solution to the uniformly elliptic equation* (5.2) *in* $\mathbb{R}^n \setminus B_1$. *Then* $\lim_{|x|\to\infty} w(x)$ *exists.*

6 Asymptotic Behavior: Proof of Theorem 1.1

Now we are ready to prove Theorem 1.1. We present the proof in the following four subsections. We first treat the special case $Du(\infty) = a = 0$. The general case can be transformed to this special case by a suitable hyperbolic rotation (Lorentz transformation).

6.1 Case $a = 0, n = 2$

Step 1. $|u(x)| \le c + d \ln|x|$ for large c and d.

We still assume $R = 1$ in Theorem 4.1. By Theorem 4.1 and Theorem 5.3, we known that $\lim_{r\to\infty} \frac{u(rx)}{r} = 0$ and $\lim_{x\to\infty} |Du(x)| = 0$. Moreover, we have either $u(x) \ge c$ for some $c \in \mathbb{R}$ in $\mathbb{R}^n \setminus B_1$ and $u(y) = c$ at some point $y \in \partial B_1$, or $u(x) \le c$ for some $c \in \mathbb{R}$ in $\mathbb{R}^n \setminus B_1$ and $u(y) = c$ at some point $y \in \partial B_1$. We assume the former case happens and $c = 0$, $y = e_1$. That is, $u(x) \ge 0$ in $\mathbb{R}^n \setminus B_1$ and $u(e_1) = 0$. Recall the radial barrier w_λ in (2.2). Set $\phi_\lambda(x) := w_\lambda(x) - w_\lambda(e_1)$ and $\psi_\lambda(x) := \phi_\lambda(x) + \max_{\partial B_1} u$. As the first step of the proof, we want to show that $u(x) \le \psi_\lambda(x)$ in $\mathbb{R}^n \setminus B_1$ for sufficiently large λ.

We observe that as long as λ is large enough, $\phi_\lambda(2e_1)$ can be arbitrarily close to 1. Since $u(2e_1) < 1$, we can choose λ_0 such that $\phi_{\lambda_0}(2e_1) > u(2e_1)$. Now we claim that $u(x) \le \psi_{\lambda_1}(x)$ in $\mathbb{R}^n \setminus B_1$, where $\lambda_1 := (\Gamma + 1)\lambda_0$ and the constant Γ is from Theorem 5.4 for u. It is easy to see that $\psi_{\lambda_0}(x) > \Gamma\phi_{\lambda_0}(x)$ in $\mathbb{R}^n \setminus B_R$ for some $R = R(\lambda_0, \Gamma)$ large enough. If $u(x) \le \psi_{\lambda_1}(x)$ in $\mathbb{R}^n \setminus B_R$, then $u(x) \le \psi_{\lambda_1}(x)$ in $\mathbb{R}^n \setminus B_1$ by the comparison principle since $u \le \max_{\partial B_1} u = \psi_{\lambda_1}$ on ∂B_1. Suppose $u(z) > \psi_{\lambda_1}(z)$ at some point $z \in \mathbb{R}^n \setminus B_R$; then $u > \phi_{\lambda_0}$ on $\partial B_{|z|}$ by Theorem 5.4. Since $u \ge 0 = \phi_{\lambda_0}$ on ∂B_1, we have $u \ge \phi_{\lambda_0}$ in $B_{|z|} \setminus B_1$ by the comparison principle, especially $u(2e_1) \ge \phi_{\lambda_0}(2e_1)$. This is a contradiction. So we proved that $u(x) \le \psi_{\lambda_1}(x)$ in $\mathbb{R}^n \setminus B_1$.

Step 2. $u(x) = c + d \ln|x| + o(1)$ for some c and d.

We still assume $u \ge 0$ as above. Denote

$$\lambda^* := \inf\{\lambda \ge 0 : u \le \psi_\lambda \text{ in } \mathbb{R}^n \setminus B_1\}.$$

By continuity, $u \le \psi_{\lambda^*}$ in $\mathbb{R}^n \setminus B_1$. If $\lambda^* = 0$, then $0 \le u \le \max_{\partial B_1} u$ in $\mathbb{R}^n \setminus B_1$. By Theorem 5.5, u has a limit at infinity. Now we assume $\lambda^* > 0$ and our aim is to show that also $u \ge \phi_{\lambda^*}$ in $\mathbb{R}^n \setminus B_1$.

For all positive integers $k > \max\{10, \frac{2}{\lambda^*}\}$, there exist y^k such that $|y^k| \ge e^{k^2}$, $|y^{k+1}| > |y^k|$, and $u(y^k) > \psi_{\lambda^*-(1/k)}(y^k)$. By (2.4), there exists $\widehat{k}$ such that for all $k \ge \widehat{k}$, we have

$$\psi_{\lambda^*}(y^k) - u(y^k) < \psi_{\lambda^*}(y^k) - \psi_{\lambda^*-\frac{1}{k}}(y^k) < \frac{2}{k} \ln|y^k|.$$

The function $w(x) := \psi_{\lambda^*}(x) - u(x)$ satisfies equation (5.2) with

$$a_{ij}(x) = \int_0^1 \frac{\delta_{ij}}{\sqrt{1-|Dw^t|^2}} + \frac{w_i^t w_j^t}{(\sqrt{1-|Dw^t|^2})^3}\, dt \tag{6.1}$$

where $w^t := (1-t)u + t\psi_{\lambda^*}$. By Theorem 5.4, we have $\psi_{\lambda^*}(x) - u(x) < \frac{2\cdot\Gamma}{k} \ln|x|$ on $\partial B_{|y^k|}$ for all $k \geq \widehat{k}$. Fix any small $\delta > 0$. Note that $\psi_{\lambda^*}(x) - \phi_{\lambda^*-\delta}(x) > \frac{\delta}{2}\ln|x|$ outside some ball. So there exists $\widetilde{k}$ such that $u(x) > \phi_{\lambda^*-\delta}(x)$ on $\partial B_{|y^k|}$ for all $k \geq \widetilde{k}$. Thus $u \geq \phi_{\lambda^*-\delta}$ in $\mathbb{R}^n \setminus B_1$ by the comparison principle. By continuity, we have $u \geq \phi_{\lambda^*}$ in $\mathbb{R}^n \setminus B_1$.

Now we have established that $\phi_{\lambda^*} \leq u \leq \psi_{\lambda^*}(x)$ in $\mathbb{R}^n \setminus B_1$. That is $0 \leq \psi_{\lambda^*}(x) - u \leq \max_{\partial B_1} u$. So by Theorem 5.5, $\psi_{\lambda^*}(x) - u$ has a limit at infinity. Denote this $\lambda^* = d$, then we have

$$u(x) = c + d\ln|x| + o(1)$$

as $|x| \to \infty$ for some constant c. Since we assumed u is bounded below, the constant $d \geq 0$. If u is bounded above, then we have $u(x) = c + d\ln|x| + o(1)$ with $d \leq 0$.

Step 3. Improve $o(1)$ to $O(|x|^{-1})$.

We still assume $u \geq 0$ as above. Suppose $d > 0$. Choose $R_0 > 10$ such that $|Du(x)| < \frac{1}{10}$ and $u(x) < 2d\ln|x|$ when $|x| \geq R_0$. For any point x with $|x| := 2R \geq 2R_0$, define $v(y) := \frac{u(x+Ry)}{R}$. Since u satisfies the nondivergence form equation (1.2), $v(y)$ satisfies the equation $a_{ij}(y)v_{ij}(y) = 0$ for $y \in B_1$ with $a_{ij}(y) = \delta_{ij} + \frac{v_i v_j}{1-|Dv|^2}$. By Morrey-Nirenberg's $C^{1,\alpha}$ estimate for the two-dimensional uniformly elliptic nondivergence form equation [8, theorem 12.4], for some $\alpha > 0$ we have

$$\|v\|_{C^{1,\alpha}(B_{1/2})} \leq C\|v\|_{L^\infty(B_1)} \leq \frac{C\ln|x|}{|x|} \tag{6.2}$$

where the first C is a universal constant independent of u, the second (and hereafter) C depends on the residue $\text{Res}[u] = d$.

$$|Du(x)| = |Dv(0)| \leq \frac{C\ln|x|}{|x|} \quad \text{for } t|x| \geq 2R_0. \tag{6.3}$$

Let e be any unit vector, then v_e satisfies the equation $(a_{ij}(y)(v_e)_j)_i = 0$ in B_1, with $a_{ij} = \frac{\delta_{ij}}{\sqrt{1-|Dv|^2}} + \frac{v_i v_j}{(\sqrt{1-|Dv|^2})^3}$. By (6.2), $\|a_{ij}\|_{C^\alpha(B_{1/2})}$ is bounded by a universal constant. By Schauder estimate [8, theorem 8.32],

$$|Dv_e(0)| \leq C\|v_e\|_{L^\infty(B_{\frac{1}{2}})} \leq \frac{C\ln|x|}{|x|}.$$

Note that $Ru_{ee}(x) = v_{ee}(0)$, so we have

$$|D^2u(x)| \le \frac{C \ln|x|}{|x|^2} \quad \text{for } |x| \ge 2R_0. \tag{6.4}$$

In fact, using a bootstrap argument, we have

$$|D^k u(x)| \le \frac{C \ln|x|}{|x|^k} \quad \text{for } |x| \ge 2R_0, \tag{6.5}$$

for all $k = 1, 2, \dots$.

We write equation (1.2) as

$$\triangle u = \frac{-(Du)' D^2 u Du}{1 - |Du|^2} := f(x) \quad \text{in } \mathbb{R}^n \setminus B_{2R_0}.$$

Then $|f(x)| \le \frac{C(\ln|x|)^3}{|x|^4}$ by (6.3) and (6.4). Define $K[u](x) := u(\frac{x}{|x|^2})$ for $x \in B_{1/(2R_0)} \setminus \{0\}$. Then

$$\triangle K[u] = |x|^{-4} f\left(\frac{x}{|x|^2}\right) := g(x) \quad \text{in } B_{1/(2R_0)} \setminus \{0\}$$

with $|g(x)| \le C(-\ln|x|)^3$. Let $N[g]$ be the Newtonian potential of g in $B_{1/(2R_0)}$. Since g is in $L^p(B_{1/(2R_0)})$ for any $p > 0$, $N[g]$ is in $W^{2,p}$ for any p and hence is in $C^{1,\alpha}$ for any $0 < \alpha < 1$. Now $K[u] - N[g]$ is harmonic in $B_{1/(2R_0)} \setminus \{0\}$. Notice that $|K[u](x)| \le -2d \ln|x| + C$ in $B_{1/(2R_0)} \setminus \{0\}$, so $|K[u] - N[g]| \le -2d \ln|x| + C$ in $B_{1/(2R_0)} \setminus \{0\}$. Therefore $K[u] - N[g]$ is the sum of $c_1 \ln|x|$ (for some constant c_1) and a harmonic function in $B_{1/(2R_0)}$. So $K[u](x)$ is the sum of $c_1 \ln|x|$ and a $C^{1,\alpha}$ function in $B_{1/(2R_0)}$. Fix an $\alpha \in (0, 1)$; for some affine function $c_2 + b \cdot x$, we have $|K[u](x) - (c_1 \ln|x| + c_2 + b \cdot x)| \le C|x|^{1+\alpha}$ in $B_{1/(2R_0)} \setminus \{0\}$. Go back to u and we have $|u(x) - (-c_1 \ln|x| + c_2 + b \cdot \frac{x}{|x|^2})| \le C|x|^{-1-\alpha}$ for $|x| \ge 2R_0$. From the result of Step 2, we must have $-c_1 = d$ and $c_2 = c$. Thus

$$u(x) = c + d \ln|x| + O(|x|^{-1}).$$

The same (but easier) argument also applies to the case $d = 0$.

Step 4. Improve $O(|x|^{-1})$ to $O_k(|x|^{-1})$.

Since $\psi_d(x) = \tilde{c} + d \ln|x| + O_k(|x|^{-1})$ for some $\tilde{c}$, we consider $w(x) := \psi_d(x) - u(x) - \tilde{c} + c = O(|x|^{-1})$. The function w satisfies $(a_{ij} w_j)_i = 0$ with a_{ij} given by (6.1). In view of (6.5) and $|D^k \psi_d(x)| \le \frac{C}{|x|^k}$, we have

$$|D^k w^t(x)| \le \frac{C \ln|x|}{|x|^k}.$$

By differentiating (6.1) directly one sees that for any $k \ge 1$ there exists $R_k \ge 4(R_0 + 1)$ such that $|D^k a_{ij}(x)| \le |x|^{-(k+1)}$ for $x \in \mathbb{R}^n \setminus B_{R_k}$. We assume

R_k is nondecreasing with respect to k. Fix k, let $x \in \mathbb{R}^n \setminus B_{2R_k}$ and $4R = |x|$, and define $v(y) := \frac{w(Ry+x)}{R}$ for $y \in B_1$. The function v satisfies the equation $(\widetilde{a}_{ij} v_j)_i = 0$ in B_1 with $\widetilde{a}_{ij}(y) = a_{ij}(x + Ry)$. We have $\|D^l \widetilde{a}_{ij}\|_{C^0(B_1)} = R^l \|D^l a_{ij}\|_{C^0(B_R(x))} \leq C(k)$ for $l = 0, 1, \dots, k$ and so $\|\widetilde{a}_{ij}\|_{C^k(B_1)} \leq C(k)$. Then by the Schauder estimate,

$$R^{k-1}|D^k w(x)| = |D^k v(0)| \leq C(k)\|v\|_{L^\infty(B_1)} \leq \frac{C(k)}{|x|^2},$$

and hence $|D^k w(x)| \leq \frac{C(k)}{|x|^{k+1}}$ for $|x| \geq 2R_k$. This means $w(x) = O_k(|x|^{-1})$ and hence

$$u(x) = c + d \ln |x| + O_k(|x|^{-1}).$$

Step 5. Ascertain the value of d.

$$\begin{aligned}\mathrm{Res}[u] &= \frac{1}{2\pi} \int_{\partial B_r} \frac{\partial u/\partial \vec{n}}{\sqrt{1 - |Du|^2}}\, ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{d}{r} + O(r^{-2})\right) r\, d\theta = d + O(r^{-1}).\end{aligned}$$

Letting $r \to \infty$, we have $d = \mathrm{Res}[u]$.

6.2 Case $a = 0, n \geq 3$

Step 1. $|u(x)| \leq c$ for large c.

We still assume $u \geq 0$ and define ϕ_λ and ψ_λ as above. Using the same method, we can prove $u(x) \leq \psi_\lambda(x)$ for some large λ in $\mathbb{R}^n \setminus B_1$. But in the dimensions $n \geq 3$, ψ_λ is bounded.

Step 2. $u(x) = u_\infty + O(|x|^{2-n})$.

Since u is bounded, applying Theorem 5.5 directly to u, we have $u(x) = u_\infty + o(1)$ where $u_\infty := \lim_{x\to\infty} u(x)$. Define $\phi_\lambda(x) := w_\lambda(x) - w_\lambda(e_1) + \min_{\partial B_1} u$ and $\psi_\lambda(x) := w_\lambda(x) - w_\lambda(e_1) + \max_{\partial B_1} u$ for $\lambda \in (-\infty, +\infty)$. We can choose λ_1 and λ_2 such that

$$\lim_{x\to\infty} \phi_{\lambda_1}(x) = u_\infty = \lim_{x\to\infty} \psi_{\lambda_2}(x).$$

By the comparison principle,

$$\phi_{\lambda_1}(x) \leq u(x) \leq \psi_{\lambda_2}(x),$$

and this means that

$$u(x) = u_\infty + O(|x|^{2-n}).$$

Step 3. $u(x) = u_\infty - d|x|^{2-n} + O(|x|^{1-n})$ for some d.

We adopt the same strategy as in the step 3 of above subsection: establish the decay rate of $|Du(x)|$ and $|D^2u(x)|$, apply the Kelvin transform to $u(x) - u_\infty$, and estimate the Newtonian potential of the right-hand side. The only difference is that when we estimate the decay rate of $|Du(x)|$, we cannot use Morrey's $C^{1,\alpha}$

estimate, which is only true for two dimensions; alternatively, the first-order derivatives of u (also $v(y) := \frac{u(x+Ry)-u_\infty}{R}$) satisfy a uniformly elliptic divergence form equation and thus we can apply De Giorgi–Nash's theorem (see [8, chap. 8]) to Dv.

$$\begin{aligned}\|Dv\|_{C^\alpha(B_{\frac{1}{2}})} &\le C\|Dv\|_{L^2(B_{\frac{3}{4}})} \\ &\le C\|v\|_{L^2(B_1)} \qquad \text{(Caccioppoli)} \\ &\le C\|v\|_{L^\infty(B_1)} \le C|x|^{1-n}.\end{aligned}$$

This treatment also fits the two-dimensional case. We leave the remaining details to the readers.

Step 4. Improve $O(|x|^{1-n})$ to $O_k(|x|^{1-n})$.

Do the same thing to $u(x) - u_\infty$ as in step 4 of above subsection.

Step 5. Ascertain the value of d.

$$\begin{aligned}\mathrm{Res}[u] &= \frac{1}{(n-2)|\partial B_1|}\int_{\partial B_r} \frac{\partial u/\partial \vec{n}}{\sqrt{1-|Du|^2}}\,d\sigma \\ &= \frac{1}{(n-2)|\partial B_1|}\int_{\partial B_1}\left(\frac{(n-2)d}{r^{n-1}} + O(r^{-n})\right) r^{n-1}\,dS^{n-1} = d + O(r^{-1}).\end{aligned}$$

Letting $r \to \infty$, we have $d = \mathrm{Res}[u]$.

6.3 Case $|a| > 0, n = 2$

By a rotation, we can assume $a = \eta e_n$ with $\eta \in (0,1)$. Make the Lorentz transformation $L_{-\eta} : \mathbb{L}^{2+1} \to \mathbb{L}^{2+1}$,

$$L_{-\eta} : (x_1, x_2, t) \to \left(x_1, \frac{x_2 - \eta t}{\sqrt{1-\eta^2}}, \frac{-\eta x_2 + t}{\sqrt{1-\eta^2}}\right) := (\widetilde{x}_1, \widetilde{x}_2, \widetilde{t}).$$

Then the plane $\{t = \eta x_2\}$ is transformed to the plane $\{\widetilde{t} = 0\}$ and the graph of u over $\mathbb{R}^2 \setminus A$ is transformed to another maximal hypersurface that is the graph of some function (say $\widetilde{u}$) defined on $\mathbb{R}^2 \setminus \widetilde{A}$ for some bounded closed set $\widetilde{A}$. The blowdown of $\widetilde{u}$ is the 0 function. So $\widetilde{u}$ has the asymptotic expansion

$$\widetilde{u}(\widetilde{x}) = \widetilde{c} + \widetilde{d}\ln|\widetilde{x}| + O(|\widetilde{x}|^{-1}). \tag{6.6}$$

Transforming back and making some direct computations, we can establish the asymptotic expansion of u. The details are as follows.

The Lorentz transformation

$$\begin{aligned}L_\eta : (\widetilde{x}_1, \widetilde{x}_2, \widetilde{u}(\widetilde{x}_1, \widetilde{x}_2)) &\to \left(\widetilde{x}_1, \frac{\widetilde{x}_2 + \eta\widetilde{u}}{\sqrt{1-\eta^2}}, \frac{\eta\widetilde{x}_2 + \widetilde{u}}{\sqrt{1-\eta^2}}\right) \\ &= (x_1, x_2, u(x_1, x_2)).\end{aligned} \tag{6.7}$$

Using the polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ and substituting (6.6) into (6.7), we get

$$\begin{aligned}\tilde{x}_2 + \eta\tilde{c} + \frac{\eta\tilde{d}}{2}\ln\bigl(r^2\cos^2\theta + \tilde{x}_2^2\bigr) + O\left(\frac{1}{\sqrt{r^2\cos^2\theta + \tilde{x}_2^2}}\right)& \\ = r\sin\theta\sqrt{1-\eta^2}.&\end{aligned} \tag{6.8}$$

We want to solve $\tilde{x}$ in (6.8) and substitute it into (6.6) and the third equality of (6.7); then we will get the expansion of u. We need to solve $\tilde{x}$ three times iteratively.

First, we assume $\sin\theta \neq 0$. From (6.8) we can see

$$\tilde{x}_2 = r\sin\theta\sqrt{1-\eta^2}\left(1 + O\left(\frac{\ln r}{r}\right)\right) \quad \text{as } r \to +\infty.$$

Then

$$r^2\cos^2\theta + \tilde{x}_2^2 = r^2(1-\eta^2\sin^2\theta)\left(1 + O\left(\frac{\ln r}{r}\right)\right),$$

and hence

$$\ln(r^2\cos^2\theta + \tilde{x}_2^2) = 2\ln\Bigl(r\sqrt{1-\eta^2\sin^2\theta}\Bigr) + O\left(\frac{\ln r}{r}\right) \tag{6.9}$$

where $O(\ln r/r)$ is independent of small $\sin\theta$. Substitute (6.9) into (6.8) and solve $\tilde{x}_2$ again,

$$\tilde{x}_2 = r\sin\theta\sqrt{1-\eta^2} - \eta\tilde{c} - \eta\tilde{d}\ln\Bigl(r\sqrt{1-\eta^2\sin^2\theta}\Bigr) + O\left(\frac{\ln r}{r}\right). \tag{6.10}$$

Now we have

$$r^2\cos^2\theta + \tilde{x}_2^2 = r^2(1-\eta^2\sin^2\theta)\left(1 - \frac{2\eta\sqrt{1-\eta^2}\,\tilde{d}\sin\theta\ln r}{(1-\eta^2\sin^2\theta)r} + O\left(\frac{1}{r}\right)\right)$$

and

$$\begin{aligned}&\ln\bigl(r^2\cos^2\theta + \tilde{x}_2^2\bigr) \\ &\quad = 2\ln\Bigl(r\sqrt{1-\eta^2\sin^2\theta}\Bigr) - \frac{\eta\sqrt{1-\eta^2}\tilde{d}\sin\theta}{(1-\eta^2\sin^2\theta)}\cdot\frac{\ln r}{r} + O\left(\frac{1}{r}\right).\end{aligned} \tag{6.11}$$

Substitute (6.11) into (6.8) and solve $\tilde{x}_2$ again:

$$\begin{aligned}\tilde{x}_2 = r\sin\theta\sqrt{1-\eta^2} - \eta\tilde{c} - \eta\tilde{d}\ln\Bigl(r\sqrt{1-\eta^2\sin^2\theta}\Bigr)& \\ + \frac{\eta\sqrt{1-\eta^2}\,\tilde{d}\sin\theta}{(1-\eta^2\sin^2\theta)}\cdot\frac{\ln r}{r} + O\left(\frac{1}{r}\right).&\end{aligned} \tag{6.12}$$

Substitute (6.11) into (6.6) and then substitute (6.6) and (6.12) into the third equality of (6.7), we have

$$\begin{aligned} u(r,\theta) &= \eta r\sin\theta + \sqrt{1-\eta^2}\,\tilde{c} + \sqrt{1-\eta^2}\,\tilde{d}\ln\big(r\sqrt{1-\eta^2\sin^2\theta}\big) \\ &\quad + \frac{\eta^2\tilde{d}\sin\theta}{(1-\eta^2\sin^2\theta)}\cdot\frac{\ln r}{r} + O\Big(\frac{1}{r}\Big). \end{aligned} \tag{6.13}$$

Notice that we get (6.13) with the assumption $\sin\theta \neq 0$. If $\sin\theta = 0$, then (6.8) becomes

$$\tilde{x}_2 + \eta\tilde{c} + \frac{\eta\tilde{d}}{2}\ln(r^2+\tilde{x}_2^2) + O\bigg(\frac{1}{\sqrt{r^2+\tilde{x}_2^2}}\bigg) = 0.$$

Then we have

$$\begin{aligned} \tilde{x}_2 &= -\eta\tilde{d}\ln r(1+o(1)), \\ r^2+\tilde{x}_2^2 &= r^2\Big(1+O\Big(\frac{1}{r}\Big)\Big), \\ \ln(r^2+\tilde{x}_2^2) &= 2\ln r + O\Big(\frac{1}{r}\Big), \\ \tilde{x}_2 &= -\eta\tilde{d}\ln r - \eta\tilde{c} + O\Big(\frac{1}{r}\Big), \end{aligned}$$

and hence

$$u(r,\theta) = \sqrt{1-\eta^2}\,\tilde{c} + \sqrt{1-\eta^2}\,\tilde{d}\ln r + O\Big(\frac{1}{r}\Big).$$

This means (6.13) is also true for $\sin\theta = 0$.

Let $\sqrt{1-\eta^2}\tilde{c} := c$ and $\sqrt{1-\eta^2}\tilde{d} := d$. In x-coordinates, we have

$$\begin{aligned} u(x_1,x_2) &= \eta x_2 + c + d\ln\sqrt{x_1^2+(1-\eta^2)x_2^2} \\ &\quad + \frac{\eta^2 d|x|x_2}{\sqrt{1-\eta^2}(x_1^2+(1-\eta^2)x_2^2)}\cdot\frac{\ln|x|}{|x|} + O(|x|^{-1}). \end{aligned}$$

Getting rid of the assumption $a = (0,\eta)$, it is not hard to see that

$$\begin{aligned} u(x) &= a\cdot x + c + d\ln\sqrt{|x|^2-(a\cdot x)^2} \\ &\quad + \frac{d|a||x|(a\cdot x)}{\sqrt{1-|a|^2}(|x|^2-(a\cdot x)^2)}\cdot\frac{\ln|x|}{|x|} + O(|x|^{-1}). \end{aligned}$$

By the method in step 4 of Section 6.1, we can improve $O(|x|^{-1})$ to $O_k(|x|^{-1})$. We omit the details.

The remaining task is to compute d in terms of $\operatorname{Res}[u]$ and $|a|$. For simplicity, we still assume $a = (0, \eta)$. Consider the ellipse

$$E_\rho := \{x_1^2 + (1-\eta^2)x_2^2 = \rho^2\}.$$

Use the polar coordinates, but this time we set $x_1 = r\cos\theta$, $\sqrt{1-\eta^2}x_2 = r\sin\theta$. So $E_\rho = \{(r,\theta) : r = \rho, 0 \le \theta < 2\pi\}$. On E_ρ,

$$Du(\theta) = \left(\frac{d\cos\theta}{\rho} + o(\rho^{-1}), \eta + \frac{d\sqrt{1-\eta^2}\sin\theta}{\rho} + o(\rho^{-1})\right),$$

the unit outward normal vector

$$\vec{n}(\theta) = \left(\frac{\cos\theta}{\sqrt{1-\eta^2\sin^2\theta}}, \frac{\sqrt{1-\eta^2}\sin\theta}{\sqrt{1-\eta^2\sin^2\theta}}\right),$$

and the length element

$$ds = \frac{\sqrt{1-\eta^2\sin^2\theta}}{\sqrt{1-\eta^2}}\rho\, d\theta.$$

So

$$\frac{\partial u/\partial\vec{n}}{\sqrt{1-|Du|^2}} = \frac{\eta\sin\theta}{\sqrt{1-\eta^2\sin^2\theta}} + \frac{d}{\rho\sqrt{1-\eta^2}\sqrt{1-\eta^2\sin^2\theta}} + o(\rho^{-1})$$

and hence

$$\begin{aligned}\operatorname{Res}[u] &= \frac{1}{2\pi}\int_{E_\rho}\frac{\partial u/\partial\vec{n}}{\sqrt{1-|Du|^2}}\, ds\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\eta\rho\sin\theta}{\sqrt{1-\eta^2}}\, d\theta + \frac{1}{2\pi}\int_0^{2\pi}\frac{d}{1-\eta^2}\, d\theta + o(1)\\ &= \frac{d}{1-\eta^2} + o(1).\end{aligned}$$

Letting $\rho \to +\infty$, we have

$$d = (1-\eta^2)\operatorname{Res}[u] = (1-|a|^2)\operatorname{Res}[u].$$

6.4 Case $|a| > 0, n \ge 3$

We do the same things as above. Assuming $a = \eta e_n$ with $\eta \in (0,1)$, make the Lorentz transformation $L_{-\eta}$: graph of $u \to$ graph of $\tilde{u}$, then

$$\tilde{u}(\tilde{x}) = \tilde{c} - \tilde{d}|\tilde{x}|^{2-n} + O(|\tilde{x}|^{1-n})$$

and

$$L_\eta : (\tilde{x}', \tilde{x}_n, \tilde{u}(\tilde{x}', \tilde{x}_n)) \to \left(\tilde{x}', \frac{\tilde{x}_n + \eta\tilde{u}}{\sqrt{1-\eta^2}}, \frac{\eta\tilde{x}_n + \tilde{u}}{\sqrt{1-\eta^2}}\right) = (x', x_n, u(x', x_n)).$$

Use the polar coordinates $x' = r\cos\theta\xi$, $x_n = r\sin\theta$ with $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\xi \in S^{n-2}$ the unit sphere in $\mathbb{R}^{n-1}$. Then we are going to solve $\widetilde{x}_n$ from

$$\widetilde{x}_n + \eta\widetilde{c} - \eta\widetilde{d}\big(r^2\cos^2\theta + \widetilde{x}_n^2\big)^{\frac{2-n}{2}} + O\Big(\big(r^2\cos^2\theta + \widetilde{x}_n^2\big)^{\frac{1-n}{2}}\Big) = r\sin\theta\sqrt{1-\eta^2}.$$

Suppose $\sin\theta \ne 0$. We have

$$\begin{aligned}\widetilde{x}_n &= r\sin\theta\sqrt{1-\eta^2}\left(1 + O\left(\frac{1}{r}\right)\right),\\ r^2\cos^2\theta + \widetilde{x}_n^2 &= r^2(1-\eta^2\sin^2\theta)\left(1 + O\left(\frac{1}{r}\right)\right),\\ \big(r^2\cos^2\theta + \widetilde{x}_n^2\big)^{\frac{2-n}{2}} &= r^{2-n}(1-\eta^2\sin^2\theta)^{\frac{2-n}{2}} + O(r^{1-n}),\end{aligned}$$

where $O(r^{1-n})$ is independent of small $\sin\theta$. So

$$\widetilde{u} = \widetilde{c} - \widetilde{d}r^{2-n}(1-\eta^2\sin^2\theta)^{\frac{2-n}{2}} + O(r^{1-n}),$$

and

$$\widetilde{x}_n = r\sin\theta\sqrt{1-\eta^2} - \eta\widetilde{c} + \eta\widetilde{d}r^{2-n}(1-\eta^2\sin^2\theta)^{\frac{2-n}{2}} + O(r^{1-n}).$$

Therefore, denoting $\sqrt{1-\eta^2}\widetilde{c} := c$ and $\sqrt{1-\eta^2}\widetilde{d} := d$,

$$\begin{aligned}u(x) &= \eta x_n + c - d(|x|^2 - \eta^2 x_n^2)^{\frac{2-n}{2}} + O(|x|^{1-n})\\ &= a\cdot x + c - d(|x|^2 - (a\cdot x)^2)^{\frac{2-n}{2}} + O(|x|^{1-n}).\end{aligned}$$

One can verify that the above expansion is also true in the case of $\sin\theta = 0$. Also $O(|x|^{1-n})$ can be improved to $O_k(|x|^{1-n})$. We omit the details.

Now we compute d. Assume $a = \eta e_n$ with $\eta \in (0,1)$ and

$$E_\rho := \big\{x'^2 + (1-\eta^2)x_n^2 = \rho^2\big\}.$$

Use the coordinates $x' = r\cos\theta\xi$, $\sqrt{1-\eta^2}x_n = r\sin\theta$. So

$$E_\rho = \left\{(r,\theta,\xi) : r = \rho, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \xi \in S^{n-2}\right\}.$$

On E_ρ:

$$u_i = \frac{(n-2)dx_i}{r^n} + O(r^{-n}) \quad \text{for } i = 1,\dots,n-1$$

and

$$u_n = \eta + \frac{(n-2)d(1-\eta^2)x_n}{r^n} + O(r^{-n}).$$

The unit outward normal vector is

$$\vec{n} = \left(\frac{x_1}{r\sqrt{1-\eta^2\sin^2\theta}}, \dots, \frac{x_{n-1}}{r\sqrt{1-\eta^2\sin^2\theta}}, \frac{(1-\eta^2)x_n}{r\sqrt{1-\eta^2\sin^2\theta}}\right),$$

and the surface element is

$$d\sigma = \frac{\sqrt{1-\eta^2\sin^2\theta}}{\sqrt{1-\eta^2}}\rho^{n-1}\cos^{n-2}\theta\, d\theta\, dS^{n-2}.$$

So

$$\frac{\partial u/\partial\vec{n}}{\sqrt{1-|Du|^2}} = \frac{\eta\sin\theta}{\sqrt{1-\eta^2\sin^2\theta}} + \frac{(n-2)d\rho^{1-n}}{\sqrt{1-\eta^2}\sqrt{1-\eta^2\sin^2\theta}} + O(\rho^{-n}),$$

and hence

$$\begin{aligned}&\operatorname{Res}[u]\\ &= \frac{1}{(n-2)|\partial B_1|}\int_{E_\rho}\frac{\partial u/\partial\vec{n}}{\sqrt{1-|Du|^2}}d\sigma\\ &= \frac{|S^{n-2}|}{(n-2)|\partial B_1|}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\eta\rho^{n-1}\cos^{n-2}\theta\sin\theta}{\sqrt{1-\eta^2}} + \frac{(n-2)d\cos^{n-2}\theta}{1-\eta^2} + O(\rho^{-1})d\theta\\ &= \frac{d}{1-\eta^2} + O(\rho^{-1}).\end{aligned}$$

We used the fact that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|S^{n-2}|\cos^{n-2}\theta\, d\theta = |S^{n-1}| = |\partial B_1|.$$

Letting $\rho \to +\infty$, we have

$$d = (1-\eta^2)\operatorname{Res}[u] = (1-|a|^2)\operatorname{Res}[u].$$

7 Exterior Dirichlet Problem: Proof of Theorem 1.2

Recall that w_λ is the radial solution defined by (2.2). Let $a \in B_1$; we use $w_\lambda^a(x)$ to denote the representation function of the hypersurface L_a(graph of w_λ), where the Lorentz transformation $L_a = T_a L_{|a|} T_a^{-1}$ is defined at the end of Section 2. Then the function $w_\lambda^a(x)$ has the following properties: $w_\lambda^a(0) = 0$, $w_\lambda^a(x)$ solves equation (1.1) in $\mathbb{R}^n \setminus \{0\}$, and (from the argument in the previous section or by direct calculation) for $n = 2$

$$w_\lambda^a(x) = a\cdot x + \sqrt{1-|a|^2}\,m(\lambda) + \sqrt{1-|a|^2}\,\lambda\ln\sqrt{|x|^2-(a\cdot x)^2} + o(1)$$

and for $n \geq 3$

$$w_\lambda^a(x) = a\cdot x + \sqrt{1-|a|^2}\,M(\lambda,n) - \frac{\sqrt{1-|a|^2}\lambda}{n-2}(|x|^2-(a\cdot x)^2)^{\frac{2-n}{2}} + o(|x|^{2-n})$$

as $x \to \infty$. The numbers $m(\lambda)$ and $M(\lambda,n)$ are from (2.4) and (2.3).

Now we prove Theorem 1.2. We do this in the following two subsections corresponding to the cases $n = 2$ and $n \geq 3$ respectively.

7.1 Case $n = 2$

Let A, g, a, and d be given as in Theorem 1.2 and $\sqrt{1-|a|^2}\lambda = d$. Choose constants $c^- \le 0 \le c^+$ such that $w_\lambda^a(x) + c^- \le g(x) \le w_\lambda^a(x) + c^+$ on ∂A. We claim that there exists $\breve{R}(A, g, a, d) > 0$ such that for any $R \ge \breve{R}$ there exists a solution u_R of maximal surface equation in $B_R \setminus A$ satisfying $u_R = g$ on ∂A and $u_R = w_\lambda^a$ on ∂B_R.

In fact, let ψ be a spacelike extension of g into $\mathbb{R}^2 \setminus A$. By Theorem 3.1, there exists R^* such that $|\psi(x) - \psi(y)| < |x - y|$ for any $x, y \in \partial B_{R^*}$ and $x \ne y$. Assume $|\psi| \le G$ on ∂B_{R^*}. Let $R \ge \breve{R} > R^*$ for any $x \in \partial B_{R^*}$ and $y \in \partial B_R$,

$$\left|w_\lambda^a(y) - \psi(x)\right| \le \left|w_\lambda^a(y)\right| + G < \frac{|a|+1}{2}(R - R^*) \le \frac{|a|+1}{2}|x-y|$$

provided $\breve{R}$ is chosen to be sufficiently large. So we can find a spacelike function v_R on $\partial B_R \setminus B_{R^*}$ such that $v_R = \psi$ on ∂B_{R^*} and $v_R = w_\lambda^a$ on ∂B_R. Define Ψ_R by $\Psi_R = \psi$ in $B_{R^*} \setminus A$ and $\Psi_R = v_R$ in $B_R \setminus B_{R^*}$. It is not difficult to see that Ψ_R is a spacelike function defined on $B_R \setminus A$ possessing boundary values g and w_λ^a on ∂A and ∂B_R respectively. Hence by Remark 2.6, we can get u_R by solving the Dirichlet problem. The above claim is proved.

By the comparison principle, $w_\lambda^a(x) + c^- \le u_R(x) \le w_\lambda^a(x) + c^+$ in $B_R \setminus A$. Choose any sequence of $\breve{R} < R_j \to \infty$, by compactness, there exists a subsequence of $\{u_{R_j}\}$ converging to a function u locally uniformly in $\mathbb{R}^n \setminus A$. By Lemma 2.4, u is area maximizing. If u is not maximal, then graph u contains a segment of light ray and hence the whole of the ray in $(\mathbb{R}^n \setminus A) \times \mathbb{R}$, contradicting the fact $w_\lambda^a(x) + c^- \le u(x) \le w_\lambda^a(x) + c^+$. Therefore u solves equation (1.1) in $\mathbb{R}^n \setminus A$. Moreover, $u = g$ on ∂A and

$$u(x) = a \cdot x + d \ln \sqrt{|x|^2 - (a \cdot x)^2} + O(1)$$

as $x \to \infty$.

Finally, we prove the uniqueness of u. Suppose there is another such solution v also satisfying $v = g$ on ∂A and

$$v(x) = a \cdot x + d \ln \sqrt{|x|^2 - (a \cdot x)^2} + O(1).$$

Then $w := u - v$ satisfies a divergence form elliptic equation in $\mathbb{R}^n \setminus A$, $w = 0$ on ∂A, and w is bounded. By [7, theorem 7], $w \equiv 0$ in $\mathbb{R}^n \setminus A$.

7.2 Case $n \ge 3$

Given A, g, a, and c as in Theorem 1.2, choose $\widetilde{R}$ and G such that $A \subset B_{\widetilde{R}}$ and $|g| \le G$ on ∂A. Choose $\lambda^* > 0$ such that $\sqrt{1-|a|^2}M(\lambda^*, n) \ge |c| + \widetilde{R} + G$. Denote

$$\Psi^-(x) := w_{\lambda^*}^a(x) - \sqrt{1-|a|^2}M(\lambda^*, n) + c,$$

$$\Psi^+(x) := w_{-\lambda^*}^a(x) + \sqrt{1-|a|^2}M(\lambda^*, n) + c.$$

One can verify that

$$\begin{aligned} &\Psi^-(x) \le a \cdot x + c \le \Psi^+(x) && \text{in } \mathbb{R}^n, \\ &\Psi^\pm(x) = a \cdot x + c + o(1) && \text{as } x \to \infty, \\ &\Psi^-(x) \le g \le \Psi^+(x) && \text{on } \partial A. \end{aligned}$$

For the same reason as in the two-dimensional case in the previous subsection, there exists $\breve{R}$ such that for any $R \ge \breve{R}$ there exists a solution u_R in $B_R \setminus A$ satisfying $u_R = g$ on ∂A and $u_R = a \cdot x + c$ on ∂B_R. Hence $\Psi^-(x) \le u_R \le \Psi^+(x)$ in $B_R \setminus A$. In the same way, we can construct a solution u in $\mathbb{R}^n \setminus A$ satisfying $u = g$ on ∂A and

$$u(x) = a \cdot x + c + o(1)$$

as $x \to \infty$.

The uniqueness of u follows from the comparison principle directly.

Acknowledgment. Part of this paper was completed during Hong's visit to the University of Washington (Seattle). This visit was funded by the China Scholarship Council. Yuan is partially supported by an NSF grant.

Bibliography

[1] Bers, L. Isolated singularities of minimal surfaces. *Ann. of Math.* **53** (1951), 364–386.

[2] Bartnik, R.; Simon, L. Spacelike hypersurfaces with prescribed boundary values and mean curvature. *Comm. Math. Phy.* **87** (1982), 131–152.

[3] Caffarelli, L. A.; Li, Y.-Y. An extension to a theorem of Jörgens, Calabi, and Pogorelov. *Comm. Pure Appl. Math.* **56** (2003), 549–583.

[4] Calabi, E. Examples of Bernstein problems for some nonlinear equations. *Proc. Symp. Pure Math.* **15** (1970), 223–230.

[5] Cheng, S.-Y.; Yau, S.-T. Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces. *Ann. of Math.* **104** (1976), 407–419.

[6] Ecker, K. Area maximizing hypersurfaces in Minkowski space having an isolated singularity. *Manus. Math.* **56** (1986), 375–397.

[7] Gilbarg, D.; Serrin, J. On isolated singularities of solutions of second order elliptic differential equations. *J. Analyse Math.* **4** (1955/56), 309–340.

[8] Gilbarg, D.; Trudinger, N. S. *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Springer-Verlag, Berlin, 1998.

[9] Hong, G.-H. Remarks on area maximizing hypersurfaces over $\mathbb{R}^n \setminus \{0\}$ and exterior domains. *Manus. Math.*, available online, doi:10.1007/s00229-019-01139-4.

[10] Hong, G.-H.; Zhao, Y.-Z. Infinity harmonic functions over exterior domains. *Int. Math. Res. Not. IMRN*, available online, doi:10.1093/imrn/rnz366

[11] Li, D.-S.; Li, Z.-S.; Yuan, Y. A Bernstein problem for special Lagrangian equations in exterior domains. *Adv. Math.* **361** (2020), available online, https://doi.org/10.1016/j.aim.2019.106927.

[12] Lin, F.-H.; Yang, X.-P. *Geometric measure theory—an introduction*. Advanced Mathematics (Beijing/Boston), Vol. 1. Science Press, Beijing; International Press, Boston, MA, 2002.

[13] Moser, J. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.* **14** (1961), 577–591.

[14] Simon, L. Asymptotic behaviour of minimal graphs over exterior domains. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1987), 231–242.

GUANGHAO HONG
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an, Shaanxi Province
710049
P.R. CHINA
E-mail: ghhongmath@xjtu.edu.cn

YU YUAN
Department of Mathematics
University of Washington
Seattle, WA 98195
USA
E-mail: yuan@math.washington.edu

Received March 2019.