RIGIDITY FOR GENERAL SEMICONVEX ENTIRE SOLUTIONS TO THE SIGMA-2 EQUATION

RAVI SHANKAR and YU YUAN

Abstract
We show that every general semiconvex entire solution to the sigma-2 equation is a quadratic polynomial. A decade ago, this result was shown for almost convex solutions. Two decades ago, this result was obtained in three dimensions, as a by-product of the work on special Lagrangian equations. Warren's rare saddle entire solution in 2014 confirmed the necessity of the semiconvexity assumption.

1. Introduction

In this paper, we show that every general semiconvex entire solution in $\mathbb{R}^n$ to the Hessian equation

$$\sigma_k(D^2u) = \sigma_k(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} = 1$$

with $k = 2$ must be quadratic. Here the λ_i's are the eigenvalues of the Hessian D^2u.

THEOREM 1.1
Let u be a smooth semiconvex solution to $\sigma_2(D^2u) = 1$ on $\mathbb{R}^n$ with $D^2u \ge -KI$ for a large $K > 0$. Then u is quadratic.

Recall the classical Liouville theorem for the Laplace equation $\sigma_1(D^2u) = \triangle u = 1$ or the Jörgens–Calabi–Pogorelov theorem for the Monge–Ampère equation $\sigma_n(D^2u) = \det D^2u = 1$: all convex entire solutions to those equations must be quadratic. Theorem 1.1 has been settled under an almost convexity condition $D^2u \ge (\delta - \sqrt{2/[n(n-1)]})I$ for general dimension in the joint work with Chang [4]; under the general semiconvexity condition $D^2u \ge -KI$ in three dimensions by taking advantage of the special Lagrangian form of the equation in this case (see [18]); or under the same general semiconvexity condition and an additional

DUKE MATHEMATICAL JOURNAL
Vol. 171, No. 15, © 2022 DOI 10.1215/00127094-2022-0034
Received 28 February 2021. Revision received 24 October 2021.
First published online 7 October 2022.
2020 *Mathematics Subject Classification*. Primary 35J60; Secondary 35B08.

quadratic growth assumption for general dimensions in our work [15]. Assuming a super quadratic growth condition, Bao, Chen, Guan, and Ji [1] demonstrated that all convex entire solutions to $\sigma_k(D^2u) = 1$ with $k = 1, 2, \dots, n$ are quadratic polynomials, and Chen and Xiang [5] showed that all "super quadratic" entire solutions to $\sigma_2(D^2u) = 1$ with $\sigma_1(D^2u) > 0$ and $\sigma_3(D^2u) \geq -K$ are also quadratic polynomials. Assuming only quadratic growth on entire solutions to $\sigma_2(D^2u) = 1$ in three dimensions, the same rigidity result was proved in the joint work with Warren [17]. Warren's rare saddle entire solutions for the $\sigma_2(D^2u) = 1$ case (see [16]) confirm the necessity of the semiconvexity or the quadratic growth assumption. It was "guessed" in the 2009 paper [4] that Theorem 1.1 should hold true.

The equation $\sigma_2(\kappa) = 1$ prescribes the intrinsic scalar curvature of a Euclidean hypersurface $(x, u(x))$ in $\mathbb{R}^n \times \mathbb{R}^1$ with extrinsic principal curvatures $\kappa = (\kappa_1, \dots, \kappa_n)$. The σ_2 function of the Schouten tensor arises in conformal geometry, and complex σ_2-type equations arise from the Strominger system in string theory.

Our current work, as well the previous ones [4] and [18], was inspired by Nitsche's classical paper [13], where the Legendre–Lewy transform was employed to produce an elementary proof of Jörgens' rigidity for the two-dimensional Monge–Ampère equation and, in turn, Bernstein's rigidity for the two-dimensional minimal surface equation.

The Legendre–Lewy transform of a general semiconvex solution satisfies a uniformly elliptic, saddle equation. In the almost convex case (see [4]), the new equation becomes concave; thus the Evans–Krylov–Safonov theory yields the constancy of the bounded new Hessian and, in turn, the old one. To beat the saddle case, one has to be "lucky." Recall that, in general, the Evans–Krylov–Safonov theory fails, as shown by the saddle counterexamples of Nadirashvili and Vlăduţ [12]. Our earlier trace Jacobi inequality, as an alternative log-convex vehicle, other than the maximum eigenvalue Jacobi inequality, in deriving the Hessian estimates for general semiconvex solutions in [15], could rescue the saddleness. But the trace Jacobi only holds for a large enough trace of the Hessian. It turns out that the trace added by a large enough constant satisfies the elusive Jacobi inequality (Proposition 2.1)

Equivalently, the reciprocal of the shifted trace Jacobi quantity is superharmonic, and it remains so in the new vertical coordinates under the Legendre–Lewy transformation by a transformation rule (Proposition 2.2). Then the iteration arguments developed in the joint work with Caffarelli [3] show that the "vertical" solution is close to a "harmonic" quadratic at one small scale (Proposition 3.1, two steps in the execution: the superharmonic quantity concentrates to a constant in measure by applying Krylov and Safonov's weak Harnack; a variant of the superharmonic quantity, as a quotient of symmetric Hessian functions of the new potential, is very pleasantly concave and uniformly elliptic; consequently, closeness to a "harmonic" quadratic is possible by the Evans–Krylov–Safonov theory), and the closeness improves increasingly as we rescale (this is a self-improving feature of elliptic equations; no concavity/convexity needed). Thus a Hölder estimate for the bounded Hessian is realized, and, consequently, so is the constancy of the new and then the old Hessian. See Section 3.

In closing, we remark that, in three dimensions, our proof provides a "pure" PDE way to establish the rigidity, distinct from the geometric measure theory way used in the earlier work on the rigidity for special Lagrangian equations (see [18, Theorem 1.3]).

2. Shifted trace Jacobi inequality and superharmonicity under Legendre–Lewy transform

Taking the gradient of both sides of the quadratic Hessian equation

$$F(D^2u) = \sigma_2(\lambda) = \frac{1}{2}\big[(\triangle u)^2 - |D^2u|^2\big] = 1, \tag{2.1}$$

we have

$$\triangle_F \, Du = 0, \tag{2.2}$$

where the linearized operator is given by

$$\triangle_F = \sum_{i,j=1}^{n} F_{ij}\partial_{ij} = \sum_{i,j=1}^{n} \partial_i(F_{ij}\partial_j), \tag{2.3}$$

with

$$(F_{ij}) = \triangle u I - D^2u = \sqrt{2 + |D^2u|^2}\, I - D^2u > 0. \tag{2.4}$$

Here, without loss of generality, we assume that $\triangle u > 0$ in what follows. Otherwise, the smooth Hessian D^2u would be in the $\triangle u < 0$ branch of the equation (2.1). Given the semiconvexity condition, the conclusion in Theorem 1.1 would be straightforward by the Evans–Krylov–Safonov theory.

The gradient square $|\nabla_F v|^2$ for any smooth function v with respect to the inverse "metric" (F_{ij}) is defined as

$$|\nabla_F v|^2 = \sum_{i,j=1}^{n} F_{ij}\partial_i v \partial_j v.$$

2.1. *Shifted trace Jacobi inequality*

PROPOSITION 2.1

Let u be a smooth solution to $\sigma_2(\lambda) = 1$ with $D^2u \geq -KI$. Set $b = \ln(\triangle u + J)$. Then we have

$$\triangle_F b \geq \varepsilon |\nabla_F b|^2 \tag{2.5}$$

for $J = 8nK/3$ *and* $\varepsilon = 1/3$.

Proof

Step 1. Differentiation of the trace
We derive the following formulas for function $b = \ln(\sigma_1 + J) = \ln(\triangle u + J)$:

$$|\nabla_F b|^2 = \sum_i f_i \frac{(\triangle u_i)^2}{(\sigma_1 + J)^2} \tag{2.6}$$

and

$$\begin{aligned}\triangle_F b &= \frac{1}{(\sigma_1 + J)}\Big\{6 \sum_{i>j>k} u_{ijk}^2 \\ &\quad + \Big[3 \sum_{i \neq j} u_{jji}^2 + \sum_i u_{iii}^2 - \sum_i \Big(1 + \frac{f_i}{\sigma_1 + J}\Big)(\triangle u_i)^2\Big]\Big\}\end{aligned} \tag{2.7}$$

at $x = p$, where, without loss of generality, $D^2u(p)$ is assumed to be diagonalized and $f(\lambda) = \sigma_2(\lambda)$.

Noticing (2.4), it is straightforward to have the identity (2.6) and at p,

$$\triangle_F b = \sum_{i=1}^n f_i \Big[\frac{\partial_{ii} \triangle u}{(\sigma_1 + J)} - \frac{(\partial_i \triangle u)^2}{(\sigma_1 + J)^2}\Big]. \tag{2.8}$$

Next we substitute the fourth-order derivative terms $\partial_{ii} \triangle u = \sum_{k=1}^n \partial_{ii} u_{kk}$ in the above by lower-order derivative terms. Differentiating equation (2.2) $\sum_{i,j=1}^n F_{ij} \partial_{ij} u_k = 0$ and using (2.4), we obtain at p,

$$\begin{aligned}\sum_{i=1}^n f_i \partial_{ii} \triangle u = \triangle_F u_{kk} &= \sum_{i,j=1}^n F_{ij} \partial_{ij} u_{kk} = \sum_{i,j=1}^n -\partial_k F_{ij} \partial_{ij} u_k \\ &= \sum_{i,j=1}^n -(\triangle u_k \delta_{ij} - u_{kij}) u_{kij} = \sum_{i,j=1}^n [u_{ijk}^2 - (\triangle u_k)^2].\end{aligned}$$

Plugging the above identity in (2.8), we have at p,

$$\triangle_F b = \frac{1}{(\sigma_1 + J)}\Big[\sum_{i,j,k=1}^n u_{ijk}^2 - \sum_{k=1}^n (\triangle u_k)^2 - \sum_{i=1}^n \frac{f_i (\triangle u_i)^2}{\sigma_1 + J}\Big].$$

Regrouping those terms $u_{\heartsuit\spadesuit\clubsuit}$, $u_{\spadesuit\spadesuit\heartsuit}$, $u_{\heartsuit\heartsuit\heartsuit}$, and $\triangle u_{\heartsuit}$ in the last three expressions, we obtain (2.7).

Subtracting (2.6) $*\varepsilon$ from (2.7), we have

$$\big(\triangle_F b - \varepsilon|\nabla_F b|^2\big)(\sigma_1 + J) \geq 3\sum_{i\neq j} u_{jji}^2 + \sum_i u_{iii}^2 - \sum_i \Big(1 + \delta \frac{f_i}{\sigma_1 + J}\Big)(\triangle u_i)^2$$

with $\delta = 1 + \varepsilon$.

Fix i and denote $t = (u_{11i}, \ldots, u_{nni})$ and e_i the i'th basis vector in $\mathbb{R}^n$, then the i'th term above can be written as

$$Q = 3|t|^2 - 2\langle e_i, t\rangle^2 - \Big(1 + \delta \frac{f_i}{\sigma_1 + J}\Big)\big\langle (1, \ldots, 1), t\big\rangle^2. \tag{2.9}$$

Step 2. Tangential projection

Equation (2.2) at p yields that t is tangential to the level set of the equation $\sigma_2(\lambda) = 1$, $\langle Df, t\rangle = 0$. Then by projecting e_i and $(1, \ldots, 1)$ to the tangential space,

$$E = (e_i)_T = e_i - \frac{f_i}{|Df|^2} Df \qquad \text{and}$$
$$L = (1, \ldots, 1)_T = (1, \ldots, 1) - \frac{(n-1)\sigma_1}{|Df|^2} Df.$$

The coefficients of the two negative terms in the quadratic form (2.9)

$$Q = 3|t|^2 - 2\langle E, t\rangle^2 - \Big(1 + \delta \frac{f_i}{\sigma_1 + J}\Big)\langle L, t\rangle^2$$

decrease, as simple symmetric computation shows that

$$\begin{aligned} |E|^2 &= 1 - \frac{f_i^2}{|Df|^2} < 1, \\ |L|^2 &= 1 - \frac{2(n-1)}{|Df|^2} < 1, \qquad \text{and} \\ E \cdot L &= 1 - \frac{(n-1)\sigma_1 f_i}{|Df|^2}. \end{aligned} \tag{2.10}$$

Step 3. Two anisotropic and nonorthogonal directions

We proceed to show that the quadratic form Q is positive definite. When t is perpendicular to both E and L, $Q = 3|t|^2 \geq 0$. So we only need to deal with the anisotropic case, when t is along $\{E, L\}$-space. The corresponding matrix of the quadratic form Q is

$$Q = 3I - 2E \otimes E - \eta L \otimes L$$

with $\eta = 1 + \delta \frac{f_i}{\sigma_1 + J} = 1 + (1+\varepsilon)\frac{f_i}{\sigma_1+J}$. The *real* ξ-eigenvector equation for (symmetric) Q under *nonorthogonal* basis $\{E, L\}$ is

$$\begin{pmatrix} 3-2|E|^2 & -2E\cdot L \\ -\eta L\cdot E & 3-\eta|L|^2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \xi \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

where corresponding real eigenvalues

$$\begin{aligned} \xi &= \frac{1}{2}(\mathrm{tr} \pm \sqrt{\mathrm{tr}^2 - 4\det}) \quad \text{with} \\ \mathrm{tr} &= 6 - 2|E|^2 - \eta|L|^2 \qquad \text{and} \\ \det &= 9 - 6|E|^2 - 3\eta|L|^2 + 2\eta\big[|E|^2|L|^2 - (E\cdot L)^2\big]. \end{aligned}$$

Now, by (2.10),

$$\begin{aligned} \mathrm{tr} &= 6 - 2\Big(1 - \frac{f_i^2}{|Df|^2}\Big) - \Big(1 + \delta\frac{f_i}{\sigma_1 + J}\Big)\Big(1 - \frac{2(n-1)}{|Df|^2}\Big) \\ &> 3 - \delta\frac{f_i}{\sigma_1+J} = \frac{(3-\delta)\sigma_1 + \delta\lambda_i + 3J}{\sigma_1 + J} > 0 \end{aligned} \tag{2.11}$$

for any $\delta \le 1.5$ and $J \ge 0$, given $\sigma_1 = \sqrt{|\lambda|^2 + 2} > |\lambda_i|$ in the nontrivial remaining case.

Next, again by (2.10),

$$\begin{aligned} \det &= 6\frac{f_i^2}{|Df|^2} - 3\delta\frac{f_i}{\sigma_1+J} + 3\Big(1 + \delta\frac{f_i}{\sigma_1+J}\Big)\underbrace{\frac{2(n-1)}{|Df|^2}} \\ &\quad + 2\Big(1 + \delta\frac{f_i}{\sigma_1+J}\Big)\Big[\frac{2(n-1)\sigma_1 f_i}{|Df|^2} - \frac{n f_i^2}{|Df|^2} - \underbrace{\frac{2(n-1)}{|Df|^2}}\Big] \\ &> -3\delta\frac{f_i}{\sigma_1+J} + 4\Big(1 + \delta\frac{f_i}{\sigma_1+J}\Big)\frac{(n-1)\sigma_1 f_i}{|Df|^2} \\ &\quad + \Big[6 - 2n\Big(1 + \delta\frac{f_i}{\sigma_1+J}\Big)\Big]\frac{f_i^2}{|Df|^2}. \end{aligned}$$

Then, for $\delta = 1 + \varepsilon = 4/3$, we have

$$\begin{aligned} &\det\cdot\frac{(\sigma_1+J)|Df|^2}{f_i} \\ &\quad \ge -3\delta\underbrace{\big[(n-1)\sigma_1^2 - 2\big]}_{|Df|^2} + \left\{\begin{array}{c} 4(\sigma_1 + J + \delta f_i)(n-1)\sigma_1 \\ +\big[(6-2n)(\sigma_1+J) - 2n\delta f_i\big]f_i \end{array}\right\} \end{aligned}$$

$$= \left\{ \begin{array}{c} 6\delta + 4(n-1)J\sigma_1 + 2(3-n)J \underbrace{f_i}_{\sigma_1-\lambda_i} \\ + (n-1)(4-3\delta)\sigma_1^2 + \big[2n(2\delta-1) + 6 - 4\delta\big]\sigma_1 f_i - 2n\delta f_i \underbrace{f_i}_{\sigma_1-\lambda_i} \end{array} \right\}$$

$$\overset{\delta=4/3}{=} 8 + 2(n+1)J\sigma_1 + 2(n-3)J\lambda_i + \frac{2(n+1)}{3}\sigma_1 f_i + \frac{8}{3}n\lambda_i f_i \tag{2.12}$$

$$> 2(n+1)J\sigma_1 + 2(n-3)J\lambda_i + \frac{8}{3}n\lambda_i f_i. \tag{2.13}$$

Case $\lambda_i \geq 0$: (2.13) is positive by the ellipticity $f_i > 0$ from (2.4).
Case $0 > \lambda_i \geq -K$:

$$(2.13) = 2nJ \underbrace{(\sigma_1 + \lambda_i)}_{\sqrt{2+|\lambda|^2}+\lambda_i>0} - 6J\lambda_i + 2J\sigma_1 + \frac{8}{3}n\lambda_i \underbrace{f_i}_{\sigma_1-\lambda_i<2\sigma_1} > 0$$

if $J = 8nK/3$.

Therefore, the quadratic form Q is positive definite, and we have derived the shifted Jacobi inequality (2.5) in the semiconvex case. □

Remark

In three dimensions, the Jacobi inequality (2.5) still holds for any $J \geq 0$ and $\varepsilon = 1/3$ without the semiconvexity assumption $D^2u \geq -KI$. Actually, we only need to show that, in Step 3, (2.12) with $\delta = 1 + \varepsilon = 4/3 < 1.5$ is also positive for negative λ_i. We would have the desired lower bound for (2.12),

$$\det \cdot \frac{(\sigma_1 + J)|Df|^2}{f_i} > 8f_i\Big(\frac{\sigma_1}{3} + \lambda_i\Big) > 0,$$

if we know that $\lambda_i > -\sigma_1/3$. Without loss of generality, we assume that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Because $\lambda_2 + \lambda_3 = f_1 > 0$, only the smallest eigenvalue λ_3 could be negative. In such a negative case $\lambda_3 = \frac{1-\lambda_1\lambda_2}{\lambda_1+\lambda_2}$ with $\lambda_1\lambda_2 > 1$, we do have $\lambda_i > -\sigma_1/3$ or $\frac{\sigma_1}{-\lambda_3} > 3$, because

$$\begin{aligned} \frac{\sigma_1}{-\lambda_3} &= -1 + \frac{(\lambda_1+\lambda_2)^2}{\lambda_1\lambda_2 - 1} \\ &\geq -1 + \frac{4\lambda_1\lambda_2}{\lambda_1\lambda_2 - 1} > 3. \end{aligned}$$

Note that, in three dimensions, the Jacobi inequality for the log-convex $b = \ln \triangle u = \ln \sqrt{2 + |\lambda|^2}$ (with $\varepsilon = 1/100$) was derived by Qiu [14, Lemma 3] for solutions to (2.1) along with variable right-hand side; the Jacobi inequality with $\varepsilon = 1/3$

for the log-max $b = \ln \lambda_{\max}$ (with $\varepsilon = 1/3$) was derived for solutions to (2.1) in [17, Lemma 2.2].

In general dimensions, a Jacobi inequality for sufficiently large $b = \ln u_{11}$, at points where $u_{11} = \lambda_{\max}$, was obtained for solutions having $\sigma_3(D^2u)$ lower bound to (2.1) along with variable right-hand side by Guan and Qiu [9, p. 1650]; another Jacobi inequality for sufficiently large $b = \ln \lambda_{\max}$ was derived for semiconvex solutions to (2.1) in [15, Proposition 2.1], as mentioned in the introduction.

2.2. *Superharmonicity under Legendre–Lewy transform*

Set $\tilde{u}(x) = u(x) + \bar{K}|x|^2/2$ for our K-semiconvex entire solution u and, say, $\bar{K} = J/n > K + 1$, where $J = 8nK/3$ is from Proposition 2.1. The $\bar{K}$-convexity of $\tilde{u}$ ensures that the smallest canonical angle of the "Lewy-sheared" "gradient" graph is larger than $\pi/4$. This means that we can make a well-defined Legendre reflection about the origin,

$$\big(x, D\tilde{u}(x)\big) = \big(Dw(y), y\big) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2.14}$$

where $w(y)$ is the Legendre transform of $u + \frac{\bar{K}}{2}|x|^2$ (see [10]). Note that $y(x) = Du(x) + Kx$ is a diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ and

$$0 < D^2w = (D^2u + \bar{K}I)^{-1} < I.$$

More precisely, by [4, p. 663] or [15, (2.11)], the eigenvalues $\mu_1 \le \mu_2 \le \cdots \le \mu_n$ of D^2w satisfy

$$0 < \mu_1 \le c(n) < 1 \qquad \text{and} \qquad 0 < c(n,K) \le \mu_i < 1 \quad \text{for } i \ge 2. \tag{2.15}$$

As shown in [4, p. 663] or [15, proof of Proposition 2.4], the equation solved by the vertical coordinate Lagrangian potential $w(y)$,

$$G(D^2w) = -F(D^2u) = -\sigma_2\big(-\bar{K}I + (D^2w)^{-1}\big) = -1,$$

is conformally, uniformly elliptic for K-convex solutions u, in the sense that for $H_{ij} := \sigma_n(\mu(D^2w))G_{ij}$, the linearized operator $H_{ij}\partial_{ij}$ of equation

$$\begin{aligned} 0 &= H(D^2w) \\ &= \sigma_n(D^2w)\big[G(D^2w) + 1\big] \\ &= -\sigma_{n-2}(\mu) + \underbrace{(n-1)\bar{K}}_{A_1}\sigma_{n-1}(\mu) - \underbrace{\Big[\frac{n(n-1)}{2}\bar{K}^2 - 1\Big]}_{A_2}\sigma_n(\mu) \end{aligned} \tag{2.16}$$

is uniformly elliptic:

$$c(n,K)I \le (H_{ij}) = \sigma_n(\mu)(G_{ij}) \le C(n,K)I.$$

PROPOSITION 2.2
Let $u(x)$ be a smooth solution to $\sigma_2(\lambda) = 1$ with $D^2u \geq -KI$. Set

$$a(y) = \Big(\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}\Big)^{-1/3} = \Big[\frac{\sigma_n}{\sigma_{n-1}}(\mu)\Big]^{1/3}$$

with the μ_i's being the eigenvalues of the Hessian $D^2w(y)$ of the Legendre–Lewy transform of $u(x) + \bar{K}|x|^2/2$. Then we have

$$\triangle_H a \leq 0.$$

Proof
The trace Jacobi inequality (2.5) in Proposition 2.1 with $J = n\bar{K}$ is equivalent to

$$\triangle_F (\triangle u + J)^{-1/3} = \triangle_F e^{-b/3} \leq 0.$$

Noticing that $\triangle u + n\bar{K} = \frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n}$ and applying the transformation rule from [15, Proposition 2.3], we immediately obtain the desired superharmonicity

$$\triangle_H a = \sigma_n(\mu) \triangle_G a \leq 0. \qquad \square$$

As another preparation for the arguments in next section, we extend the operator H in (2.16) to outside the box in (2.15), to a uniformly elliptic smooth operator with bounded $C^{1,1}$ norm $\|\nabla^2 H\|_{L^\infty}$, still denoted by H.

3. Hölder Hessian estimate for saddle equation and rigidity

The Hessian bound $0 < D^2w(y) \leq I$ ensures that establishing a local $C^{2,\alpha}$ estimate for such solutions to (2.16) will prove, by scaling, that $w(y)$ is a quadratic polynomial. By the iteration arguments developed in [3] for such smooth PDEs $H(D^2w) = 0$ with solutions satisfying Hessian bounds, proving $C^{2,\alpha}$ regularity at a point, say the origin, reduces to showing that $w(y)$ is close to a uniform quadratic polynomial. Namely, we have the following.

PROPOSITION 3.1
Let $u(x)$ be a smooth solution to $\sigma_2(\lambda) = 1$ with $D^2u \geq -KI$ in $\mathbb{R}^n$. Let $w(y)$ be its Legendre–Lewy transform defined in (2.14) solving (2.16) in $\mathbb{R}^n$ with $0 < D^2w \leq I$. Given any $\epsilon > 0$, there exists small $\eta = \eta(n, K, \epsilon) > 0$ and a quadratic polynomial $P(y)$ whose coefficients only depend on n, K, ϵ, such that $H(D^2P) = 0$ and

$$\Big|\frac{1}{\eta^2} w(\eta y) - P(y)\Big| \leq \epsilon$$

is valid for $|y| \leq 1$.

In the case that the level set $\{H(D^2w)=0\}$ were convex (in fact saddle from [4, p. 661]), the alternative way in [3] other than the Evans–Krylov–Safonov theory is the following. The Laplacian $\Delta w(y)$ is a sub- or supersolution of the linearized operator $\Delta_H = H_{ij}\partial^2/\partial y_i \partial y_j$ of $H(D^2w)$. The weak Harnack inequality shows that $\Delta w(y)$ concentrates in measure at a level c on a small ball $B=B_r(0)$. Solving the equation $\Delta v = c$ on B with $v=w$ on ∂B furnishes the desired smooth approximation, which is uniform by the ABP estimate. The Laplacian can be replaced with any elliptic slice of the Hessian, such that the elliptic slice is a supersolution of Δ_H, and the corresponding elliptic equation both has $C^{2,\alpha}$ interior regular solutions and allows for the ABP estimate.

However, it is not clear if the saddle level set $\{H(D^2w)=0\}$ of (2.16) is any of trace-convex (see [3]), max-min (see [2]), or twisted (see [6], [7]), so it is not clear if there are good PDEs which super-solve Δ_H. Now that the remarkable superharmonic quantity $\sigma_n(\mu)/\sigma_{n-1}(\mu)$ in Proposition 2.2 is available, the core method in [3, pp. 687–690] becomes more realistic.

There is still one more hurdle to overcome. The superharmonic, "one-step" Hessian quotient $a^3=\sigma_n(\mu)/\sigma_{n-1}(\mu)$ is well known to be concave, but it is not uniformly elliptic, because $\sigma_n(\mu)$ could be arbitrarily close to zero. This prevents applying the Evans–Krylov–Safonov theory. We resolve this by substituting the concentration of a into the "conformal" equation (2.16). This implies concentration of a better quantity. Observe that equation (2.16) can be written as

$$q(\mu) := \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}(\mu)} = \left[A_1 - A_2 \frac{\sigma_n(\mu)}{\sigma_{n-1}(\mu)}\right]^{-1}. \tag{3.1}$$

Thus, the concentration of the higher quotient $a^3=\sigma_n/\sigma_{n-1}$ implies the concentration of the lower quotient $\sigma_{n-1}/\sigma_{n-2}$, which is also a concave operator (see [11, Theorem 15.18]). The almost-convex case, $D^2u \geq (-K+\delta)I$ for $K^{-2}=n(n-1)/2$ and a $\delta>0$ considered in [4], corresponds to $A_2=0$. There, it was shown that (2.16) is uniformly elliptic for arbitrarily large K, in particular, the lower quotient $\sigma_{n-1}/\sigma_{n-2}=A_1^{-1}$ for $K^{-2}=n(n-1)/2$. For arbitrary K, using the bound for μ in (2.15) and the result in [10, Theorem 15.18], we deduce the uniform ellipticity of $q(\mu)$:

$$\partial_{\mu_i} q \in \frac{\sigma_{n-1}(\mu)}{\sigma_{n-2}^3(\mu)}\sigma_{n-2,i}^2(\mu)\big[c(n),1\big] \subset \big[c(n,K),C(n,K)\big]. \tag{3.2}$$

Proof of Proposition 3.1

Step 1. Concentration of $a(y)$ in measure
Take positive small ρ, ξ, δ and large k_0, to be chosen later. We denote $a_k = \min_{B_{1/2^k}} a$ with $B_r=B_r(0)$, and define a "bad set" $E_k=\{y\in B_{1/2^k} : a > a_k+\xi\}$. We claim that there is some $1\leq l \leq k_0$ such that $|E_\ell| \leq \delta |B_{1/2^\ell}|$.

Otherwise, for all $1 \le k \le k_0$, we have $|E_k| > \delta |B_{1/2^k}|$. For each k, applying Krylov and Safonov's weak Harnack inequality (see [8, Theorem 9.22]) for supersolution $a(y)$ to $\triangle_H a \le 0$ from Proposition 2.2, we have

$$a(y) \ge \min_{B_{1/2^k}} a(y) + c(n,K)\xi\delta^{1/p_0(n,K)} \quad \text{in } B_{2^{-1}2^{-k}}.$$

That is, $a(y)$ increases by $\theta = c(n,K)\xi\delta^{1/p_0}$ each time. After

$$k_0 = \text{the integer part of } \left[\frac{\max_{B_1} a(y) - \min_{B_1} a(y)}{\theta} + 1\right] \le \frac{n^{-1/3}}{\theta} + 1$$

steps, we obtain

$$a(y) \ge \min_{B_{1/2^{k_0}}} a(y) + \operatorname*{osc}_{B_1} a(y) + \theta > \max_{B_1} a(y) \quad \text{in } B_{2^{-k_0-1}}.$$

This is a contradiction. Thus our claim holds.

Before we move to Step 2, set $w_l(y) = 2^{2l} w(2^{-l} y)$.

Remark

In the alternative case of the claim with k_0 replaced by $k_0 - 1$, for all $1 \le k \le k_0 - 1$, $|E_k| > \delta |B_{1/2^k}|$, after $k_0 - 1$ steps, we obtain

$$\operatorname*{osc}_{B_{2^{-k_0}}} a(y) \le \theta < \xi \qquad \text{or} \qquad a_{k_0-1} \le a(y) < a_{k_0-1} + \xi \quad \text{in } B_{2^{-k_0}}.$$

Following the argument for Case 2 in [3, pp. 688–690], one can also reach the conclusion of Proposition 3.1.

Step 2. Approximation of $w_l(y)$ by quadratic polynomial

First, using uniform ellipticity (3.2), let us extend $\sigma_{n-1}/\sigma_{n-2}$ to a uniformly elliptic concave smooth operator $Q(D^2 v)$ to outside the eigenvalue rectangle (2.15), $\mu \in [0, c(n)] \times [c(n,K), 1]^{n-1}$. Let $v(y) \in C^\infty(B_1)$ solve the concave equation $Q(D^2 v) = (A_1 - A_2 a_\ell^3)^{-1}$ in B_1 with $v = w_l$ on ∂B_1. Then from the quotient representation (3.1) for the equation that w_l solves, the ABP estimate (see [8, Theorem 9.1]) yields on B_1, and the claim in Step 1, we get

$$\begin{aligned}
|w_l - v| &\le C(n,K) \left\| Q(D^2 w_l) - Q(D^2 v) \right\|_{L^n(B_1)} \\
&\le C(n,K)\delta^{1/n} + C(n,K) \left\| \frac{a^3 - a_\ell^3}{(A_1 - A_2 a^3)(A_1 - A_2 a_\ell^3)} \right\|_{L^n(E_\ell^c)} \\
&\le C(n,K)(\delta^{1/n} + \xi),
\end{aligned}$$

where, in the last inequality, we used the boundedness of $(A_1 - A_2 a)^{-1}$ via (3.1) and (2.15). By the Evans–Krylov–Safonov theory applied to the smooth equation $Q(D^2 v) = (A_1 - A_2 a_\ell^3)^{-1}$, $v(y)$ has uniform interior C^3 estimates, so v can be replaced by its quadratic part $\overline{P}$ at the origin, up to a uniform $O(|y|^3)$ term. Then

$$\left|w_l(y) - \overline{P}(y)\right| \le C(n,K)\big(\delta^{1/n} + \xi + |y|^3\big).$$

Let $y = \rho z$, $\overline{\overline{P}}(z) = \rho^{-2}\, \overline{P}(\rho z)$. For $|z| \le 1$, we have

$$\left|\rho^{-2} w_l(\rho z) - \overline{\overline{P}}(z)\right| \le C(n,K)\Big(\frac{\delta^{1/n} + \xi}{\rho^2} + \rho\Big).$$

Still $H(D^2 w_l(\rho z)) = 0$. By [3, Lemma 2], which was proved by applying the maximum principle in a contradiction argument, and noticing that $F(D^2\overline{\overline{P}} + sI) = H(D^2\overline{\overline{P}} + sI)$ is uniformly increasing as s crosses $s = 0$, we perturb $\overline{\overline{P}}(z)$ to another quadratic polynomial $P(z)$ so that $H(D^2 P) = 0$ with

$$\left|\rho^{-2} w_l(\rho z) - P(z)\right| \le C(n,K)\Big(\frac{\delta^{1/n} + \xi}{\rho^2} + \rho\Big).$$

Finally, we choose ρ, then ξ, δ, and fix k_0 successively, depending on n, K, ϵ so that

$$\left|\eta^{-2} w(\eta y) - P(y)\right| \le \epsilon,$$

where $\eta = \eta(n,K,\epsilon) = \rho/2^l$. □

Proof of Theorem 1.1

As indicated in the beginning of Section 2, we only need to handle the positive branch $\triangle u > 0$ of the quadratic equation $\sigma_2(D^2u) = 1$. This is because the only other possibility is that D^2u is on the negative branch $\triangle u < 0$ of the still elliptic and concave equation $\sigma_2(D^2u) = 1$. Then the semiconvex solutions must have bounded Hessian, and consequently, the conclusion in Theorem 1.1 is straightforward by the Evans–Krylov–Safonov theory.

Now armed with Proposition 3.1, the initial closeness of w to a "harmonic" quadratic on the unit ball, and repeating the proof of Proposition 2 in [3] with the equation there replaced by our smooth uniformly elliptic extended equation H, with bounded $C^{1,1}$ norm $\|\nabla^2 H\|_{L^\infty}$, from (2.16), in the end of Section 2, we see that the closeness to "harmonic" quadratics accelerates. As in [3, p. 692], we obtain that $D^2 w$ is Hölder at the origin. Similarly, one proves that $D^2 w$ is Hölder in the half-ball

$$[D^2 w]_{C^\alpha(B_{1/2})} \le C(n,K),$$

where $\alpha = \alpha(n,K) > 0$.

By quadratic scaling $R^2 w(y/R)$, we get

$$[D^2 w]_{C^\alpha(B_{R/2})} \leq \frac{C(n,K)}{R^\alpha} \longrightarrow 0 \quad \text{as } R \to \infty.$$

We conclude that $D^2 w$ is a constant matrix, and in turn, so is $D^2 u$. □

Acknowledgments. Both authors are partially supported by National Science Foundation grants DMS-1800495 and DMS-2054973.

References

[1] J. BAO, J. CHEN, B. GUAN, and M. JI, *Liouville property and regularity of a Hessian quotient equation*, Amer. J. Math. **125** (2003), no. 2, 301–316. MR 1963687. *(3202)*

[2] X. CABRÉ and L. A. CAFFARELLI, *Interior $C^{2,\alpha}$ regularity theory for a class of nonconvex fully nonlinear elliptic equations.* J. Math. Pures Appl. (9) **82** (2003), no. 5, 573–612. MR 1995493. DOI 10.1016/S0021-7824(03)00029-1. *(3210)*

[3] L. A. CAFFARELLI and Y. YUAN, *A priori estimates for solutions of fully nonlinear equations with convex level set*, Indiana Univ. Math. J. **49** (2000), no. 2, 681–695. MR 1793687. DOI 10.1512/iumj.2000.49.1901. *(3202, 3209, 3210, 3211, 3212)*

[4] S.-Y. A. CHANG and Y. YUAN, *A Liouville problem for the sigma-2 equation.* Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 659–664. MR 2644763. DOI 10.3934/dcds.2010.28.659. *(3201, 3202, 3208, 3210)*

[5] L. CHEN and N. XIANG, *Rigidity theorems for the entire solutions of 2-Hessian equation*, J. Differential Equations **267** (2019), no. 9, 5202–5219. MR 3991557. DOI 10.1016/j.jde.2019.05.028. *(3202)*

[6] T. C. COLLINS, *$C^{2,\alpha}$ estimates for nonlinear elliptic equations of twisted type*, Calc. Var. Partial Differential Equations **55** (2016), no. 1, art. 6, 11 pp. MR 3441283. DOI 10.1007/s00526-015-0950-y. *(3210)*

[7] T. C. COLLINS and G. SZÉKELYHIDI, *Convergence of the J-flow on toric manifolds*, J. Differential Geom. **107** (2017), no. 1, 47–81. MR 3698234. DOI 10.4310/jdg/1505268029. *(3210)*

[8] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Classics in Math., Springer, Berlin, 2001. MR 1814364. *(3211)*

[9] P. GUAN and G. QIU, *Interior C^2 regularity of convex solutions to prescribing scalar curvature equations*, Duke Math. J. **168** (2019), no. 9, 1641–1663. MR 3961212. DOI 10.1215/00127094-2019-0001. *(3208)*

[10] H. LEWY, *A priori limitations for solutions of Monge-Ampère equations, II*, Trans. Amer. Math. Soc. **41** (1937), no. 3, 365–374. MR 1501906. DOI 10.2307/1989787. *(3208, 3210)*

[11] G. M. LIEBERMAN, *Second Order Parabolic Differential Equations*, World Sci., River Edge, 1996. MR 1465184. DOI 10.1142/3302. *(3210)*

[12] N. NADIRASHVILI and S. VLĂDUŢ, *Nonclassical solutions of fully nonlinear elliptic equations*, Geom. Funct. Anal. **17** (2007), no. 4, 1283–1296. MR 2373018. DOI 10.1007/s00039-007-0626-7. *(3202)*

[13] J. C. C. NITSCHE, *Elementary proof of Bernstein's theorem on minimal surfaces*, Ann. of Math. (2) **66** (1957), 543–544. MR 0090833. DOI 10.2307/1969907. *(3202)*

[14] G. QIU, *Interior Hessian estimates for Sigma-2 equations in dimension three*, preprint, arXiv:1711.00948 [math.AP]. *(3207)*

[15] R. SHANKAR and Y. YUAN, *Hessian estimate for semiconvex solutions to the sigma-2 equation*, Calc. Var. Partial Differential Equations **59** (2020), no. 1, paper no. 30, 12 pp. MR 4054864. DOI 10.1007/s00526-019-1690-1. *(3202, 3208, 3209)*

[16] M. WARREN, *Nonpolynomial entire solutions to σ_k equations*, Comm. Partial Differential Equations **41** (2016), no. 5, 848–853. MR 3508325. DOI 10.1080/03605302.2015.1123277. *(3202)*

[17] M. WARREN and Y. YUAN, *Hessian estimates for the sigma-2 equation in dimension 3*, Comm. Pure Appl. Math. **62** (2009), no. 3, 305–321. MR 2487850. DOI 10.1002/cpa.20251. *(3202, 3208)*

[18] Y. YUAN, *A Bernstein problem for special Lagrangian equations*, Invent. Math. **150** (2002), no. 1, 117–125. MR 1930884. DOI 10.1007/s00222-002-0232-0. *(3201, 3202, 3203)*

Shankar
Department of Mathematics, University of Washington, Seattle, Washington, USA; *current:* Department of Mathematics, Princeton University, Princeton, New Jersey, USA;
rs1838@princeton.edu

Yuan
Department of Mathematics, University of Washington, Seattle, Washington, USA;
yuan@math.washington.edu