An L_p-theory for non-divergence form SPDEs driven by Lévy processes

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Abstract

In this paper we present an L_p-theory for a class of stochastic partial differential equations (SPDEs in abbreviation) driven by Lévy processes. Existence and uniqueness of solutions in Sobolev spaces are obtained. The number of derivatives of the solution can be any real number, and the coefficients of SPDEs under consideration are random functions depending on time and space variables.

Keywords: Stochastic partial differential equation, Lévy process, L_p-theory, Sobolev space, martingale.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_t, t \geq 0\}$ an increasing filtration of quasi-left continuous σ-fields on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F}_t \subset \mathcal{F}$, each contains all $(\mathcal{F}, \mathbb{P})$-null sets. We assume that on $(\Omega, \mathbb{P})$, there are a family of independent one-dimensional Lévy processes $Z^1, Z^2, \dots$, such that for each $k \geq 1$ and $0 < s < t$, Z^k_t is $\mathcal{F}_t$-measurable and $Z^k_t - Z^k_s$ is independent of $\mathcal{F}_s$. We call $Z^1, Z^2, \dots$, Lévy processes relative to $\{\mathcal{F}_t, t \geq 0\}$.

In this article we are concerned with $W^{n,p}$-theory of the following quasi-linear stochastic partial differential equation on $\mathbb{R}^d$

$$du = \left(a^{ij}u_{x^ix^j} + b^i u_{x^i} + cu + f(u)\right) dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k(u)\right) dZ^k_t \tag{1.1}$$

for $t \geq 0$ and $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, where $u_{x^i} := \frac{\partial u}{\partial x^i}$ and $u_{x^ix^j} := \frac{\partial^2 u}{\partial x^i \partial x^j}$. Here $p \in [2, \infty)$ and $n \in \mathbb{R}$. Indices i and j go from 1 to d, and k runs through $\{1, 2, ...\}$ with the summation convention on i, j, k being enforced. The coefficients a^{ij}, b^i, c, σ^{ik}, μ^k depend on (ω, t, x), and the

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functions $f(u) = f(\omega, t, x, u)$ and $g^k(u) = g^k(\omega, t, x, u)$ satisfy certain properties to be specified in Assumption 4.9.

Stochastic partial differential equations (SPDEs) of type (1.1) arise naturally in applications when the objects are subject to randomness and high variability. For instance, such equations arise in nonlinear filtering theory of partially observable diffusion processes, in relativistic quantum mechanics and population models with geographical structures (See, for instance, [16]).

In this article we establish the unique solvability for SPDEs (1.1) in Sobolev spaces $W^{n,p}(\mathbb{R}^d)$. The order of derivatives n of the solution can be any real number. In particular, it may be negative or fractional. Our L_p-theory for equation (1.1) also can be used as an instrument for treating SPDEs with measure-valued processes, for example, driven by Lévy space-time white noise (see Theorem 4.13).

If $\{Z^k, k \geq 1\}$ are independent one-dimensional Wiener processes, L_p-theory for SPDE (1.1) has been well studied. An L_p-theory of SPDEs with Wiener processes defined on $\mathbb{R}^n$ was first introduced by Krylov in [9]. Subsequently, Krylov and Lototsky [11, 12] developed an L_p-theory of such equations in half space $\mathbb{R}^n_+$ with constant coefficients. These results were later extended to SPDEs with variable coefficients defined in bounded domains of $\mathbb{R}^n$ by several authors, see, for instance, [8, 7, 13].

However as far as we know, most of previous works (except for the L_2-theory of equations of divergence form type, see (1.3) below) on SPDEs driven by Lévy processes were devoted to study the mild or pathwise solutions of SPDEs of the type

$$du = (\mathcal{A}u + f)dt + \sum_{k=1}^{n} g^k(u)dZ^k_t, \tag{1.2}$$

where $\mathcal{A}$ is the generator of certain semigroup and $g^k(u)$ satisfies certain continuity conditions. Semi-group approach has been one of main tools of such studies. For a sample of such study, we refer the reader to Peszat and Zabczyk's book [15], and some other recent works [2, 5, 14, 17] and the references therein. The main differences between equation (1.1) and equation (1.2) are that (i) the operator $\mathcal{A}$ in (1.2) is non-random and independent of t, and (ii) unlike in (1.1) the first derivatives of the solutions do not appear in the stochastic part of (1.2).

We refer to [6] and [4] for an L_2-theory of equations of divergence form type. Roughly speaking, unique solvability of the equations of the type

$$du = \left((a^{ij}u_{x^j} + f^i)_{x^i} + b^i u_{x^i} + cu + f\right) dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k\right) dZ^k_t \tag{1.3}$$

in space $L_2(\Omega \times [0,T], H^1_2)$ can be found in [6] and [4].

Our approach in this paper is different from those in [15, 2, 5, 6, 14, 17]. We adopt an analytic approach introduced by Krylov in [9]. Our results are new even for the stochastic heat equation

$$du = \Delta u \, dt + g \, dZ_t,$$

since we establish unique solvability result in $L_p(\Omega \times [0,T], H^n_p)$ for every $p \in [2, \infty)$ and $n \in \mathbb{R}$, not just in $L_2(\Omega \times [0,T], H^1_2)$ space. (The definition of Sobolev space $H^n_p = W^{n,p}$ will be given in next section.) This allows us to obtain various regularity results of solutions (see Remark 4.5(ii)).

In section 2 we introduce some stochastic Banach spaces, and in section 3 we deal with equations with constant coefficients. Our main results are given in section 4 and consist of Theorem 4.2 (L_2-theory for linear equations), Theorem 4.4 (L_p-theory for linear equations with $p > 2$) and Theorem 4.10 (L_p-theory for quasi-linear equations with $p \geq 2$).

We end the introduction with some notation. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, ..., x^d)$, and $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ denotes the Euclidean ball in $\mathbb{R}^d$ of radius r centered at x. For $i = 1, ..., d$, multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$ with $\alpha_i \in \{0, 1, 2, ...\}$ and functions $u(x)$ on $\mathbb{R}^d$, we set

$$D_i u := u_{x^i} := \partial u/\partial x^i, \qquad D^\alpha u := D_1^{\alpha_1} \cdot ... \cdot D_d^{\alpha_d} u, \qquad |\alpha| := \alpha_1 + ... + \alpha_d.$$

For $a, b \in \mathbb{R}^d$, we define $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For $p \geq 1$, we will use $\|u\|_p$ to denote the L_p-norm of u in $L_p(\mathbb{R}^d; dx)$. For scalar functions f, g on $\mathbb{R}^d$, $(f, g) := \int_{\mathbb{R}^d} f(x)g(x)dx$. We will use C and c to denote a constant whose value may change from line to line.

2 Stochastic Banach spaces

For $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, define

$$N^k(t, A) := \#\left\{0 \leq s \leq t; \, Z^k_s - Z^k_{s-} \in A\right\}, \quad \widetilde{N}^k(t, A) := N^k(t, A) - t\nu_k(A)$$

where $\nu_k(A) := \mathbb{E}[N_k(1, A)]$ is the Lévy measure of Z^k. By Lévy-Itô decomposition, there exist constants α^k, β^k and Brownian motion B^k so that

$$Z^k(t) = \alpha^k t + \beta^k B^k_t + \int_{|z|<1} z\widetilde{N}^k(t, dz) + \int_{|z|\geq 1} zN^k(t, dz). \tag{2.1}$$

Denote

$$\widehat{c}_{k,p} := \left(\int_{\mathbb{R}} |z|^p \nu_k(dz)\right)^{1/p}, \quad \widehat{c}_{k,2,p} := \widehat{c}_{k,2} \vee \widehat{c}_{k,p}.$$

Note that for $2 < q < p$, by Hölder's inequality,

$$\widehat{c}_{k,q} \leq \left(\int_{\mathbb{R}} |z|^2 \nu_k(dz)\right)^{(p-q)/(q(p-2))} \left(\int_{\mathbb{R}} |z|^p \nu_k(dz)\right)^{(q-2)/(q(p-2))} \leq \widehat{c}_{k,2,p}.$$

Assumption 2.1 *$p \in [2, \infty)$ and*

$$\widehat{c}_0 := \sup_k \widehat{c}_{k,2,p} < \infty.$$

Note that due to the assumption $\widehat{c}_{k,2} < \infty$, we have $\int_{|z|>1} |z|\nu_k(dz) < \infty$. Denoting $a^k = \alpha^k + \int_{|z|>1} z\nu_k(dz)$, we get

$$Z^k_t = a^k t + \beta^k B^k_t + \int_{\mathbb{R}^1} z\widetilde{N}^k(t, dz) =: a^k t + \beta^k B^k_t + \bar{Z}^k_t. \tag{2.2}$$

Throughout this paper, for simplicity, we may and do assume $a^k = \beta^k = 0$ (equivalently we consider equation (1.1) with $\bar{Z}_t^k$ in place of Z_t^k). See Remark 4.3 for the case $\beta^k \neq 0$. The theory of equation (1.1) for the case $a^k \neq 0$ can be reduced to the case $a^k = 0$ by moving the term $\sum_k a^k(\sigma^{ik} u_{x^i} + \mu^k u + g^k)$ to the deterministic part.

For $n = 0, 1, 2, ...,$ define Sobolev space

$$H_p^n := H_p^n(\mathbb{R}^d) = \left\{ u : u, Du, ..., D^n u \in L_p(\mathbb{R}^d) \right\},$$

where for integer $k \geq 1$, $D^k u$ denotes all the kth order derivatives of u of in the distributional sense. In literature, H_p^n is also denoted as $W^{n,p}(\mathbb{R}^d)$. In general, for $\gamma \in \mathbb{R}$ define the space $H_p^\gamma = H_p^\gamma(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2} L_p$ (called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the set of all distributions u such that $(1-\Delta)^{\gamma/2} u \in L_p$. For $u \in H_p^\gamma$, we define

$$\|u\|_{H_p^\gamma} := \|(1-\Delta)^{\gamma/2} u\|_p := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\gamma/2} \mathcal{F}(u)(\xi)]\|_p, \tag{2.3}$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ the inverse Fourier transform.

Denote by $\mathcal{P}$ the predictable σ-field generated by the filtration $\{\mathcal{F}_t, t \geq 0\}$. Let $\mathcal{P}^{dP \times dt}$ be the completion of $\mathcal{P}$ with respect to $d\mathbb{P} \times dt$. Denote by $\mathbb{H}_p^\gamma(T)$ the space of all $\mathcal{P}^{d\mathbb{P} \times dt}$-measurable processes $u : [0, T] \times \Omega \to H_p^\gamma$ so that

$$\|u\|_{\mathbb{H}_p^\gamma(T)} := \left(\mathbb{E} \left[\int_0^T \|u\|_{H_p^\gamma}^p \, dt \right] \right)^{1/p} < \infty.$$

Remark 2.2 It follows from (2.3) that for any $\mu, \gamma \in \mathbb{R}$, the operator $(1-\Delta)^{\mu/2} : H_p^\gamma \to H_p^{\gamma-\mu}$ is an isometry. Indeed,

$$\|(1-\Delta)^{\mu/2} u\|_{H_p^{\gamma-\mu}} = \|(1-\Delta)^{(\gamma-\mu)/2}(1-\Delta)^{\mu/2} u\|_p = \|(1-\Delta)^{\gamma/2} u\|_p = \|u\|_{H_p^\gamma}.$$

We will use $|g|_{\ell^2}$ to denote the ℓ^2-norm of $g \in \ell^2$. For ℓ^2-valued processes $g = (g^1, g^2, ...)$, we say $g \in \mathbb{H}_p^\gamma(T, \ell^2)$ if $g^k \in \mathbb{H}_p^\gamma(T)$ for every $k \geq 1$ and

$$\|g\|_{\mathbb{H}_p^\gamma(T,\ell^2)} := \left(\mathbb{E} \left[\int_0^T \| \, |(1-\Delta)^{\gamma/2} g|_{\ell^2} \, \|_p^p \, dt \right] \right)^{1/p} < \infty. \tag{2.4}$$

Finally, we say $u_0 \in U_p^\gamma$ if u_0 is $\mathcal{F}_0$-measurable and

$$\|u_0\|_{U_p^\gamma} := \left(\mathbb{E} \left[\|u_0\|_{H_p^{\gamma-(2/p)}}^p \right] \right)^{1/p} < \infty.$$

Remark 2.3 (i) For a square integrable martingale M, we will use $[M]$ to denotes its quadratic variation process. The dual predictable projection of $[M]$ will be denoted as $\langle M \rangle$. Under Assumption 2.1 for $p = 2$, Z^k is a square integrable martingale with $\langle Z^k \rangle_t = \widehat{c}_{k,2}^2 t$. For every $\mathcal{P}^{d\mathbb{P} \times dt}$-measurable process $H \in L_2(\Omega \times [0, T])$, $M_t := \int_0^t H_s dZ_s^k$ is a square integrable martingale with

$$\mathbb{E}[M_t^2] = \mathbb{E} \left[\int_0^t H_s^2 \, d[Z^k]_s \right] = \mathbb{E} \left[\int_0^t H_s^2 \, d\langle Z^k \rangle_s \right] = \widehat{c}_{k,2}^2 \mathbb{E} \left[\int_0^t H_s^2 ds \right]. \tag{2.5}$$

(ii) In many other articles, the equation of the type

$$du = (Au + f)dt + g(u(t-))dZ_t$$

has been studied. The expression $u(t-)$ is used so that it is predictable and the integral $\int_0^t g(u(t-))dZ_t$ becomes a martingale. Such notation is not used in this article, because by (i), stochastic integral can be defined for a process H in $L_2(\Omega\times[0,T])$ which is not predictable but has a predictable version $\tilde{H}$, and (a.s.)

$$\int_0^t H_s dZ_s = \int_0^t \tilde{H}_s dZ_s, \quad \forall t \le T.$$

(iii) Let $p \ge 2$. For any $g \in \mathbb{H}_p^\gamma(T,\ell^2)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, the series of stochastic integral $\sum_{k=1}^\infty \int_0^t (g^k,\phi)dZ_s^k$ defines a square integrable martingale on $[0,T]$, which is right continuous with left limits. Indeed,

$$\mathbb{E}\left[\Big(\int_0^t (g^k,\phi)dZ_s^k\Big)^2\right] = \mathbb{E}\left[\int_0^t (g^k,\phi)^2 d[Z^k]_s\right] = \widehat{c}_{k,2}^2\mathbb{E}\left[\int_0^t (g^k,\phi)^2 ds\right]$$

and

$$\begin{aligned}
&\sum_{k=1}^\infty \widehat{c}_k^2\,\mathbb{E}\left[\int_0^T (g^k,\phi)^2 ds\right]\\
\le\;& \widehat{c}_0^2\sum_{k=1}^\infty \mathbb{E}\left[\int_0^T ((1-\Delta)^{\gamma/2}g^k,(1-\Delta)^{-\gamma/2}\phi)^2\,ds\right]\\
\le\;& \widehat{c}_0^2\|(1-\Delta)^{-\gamma/2}\phi\|_1\,\mathbb{E}\left[\int_0^T \Big(\sum_k |(1-\Delta)^{\gamma/2}g^k|^2,\,|(1-\Delta)^{-\gamma/2}\phi|\Big)\,ds\right]\\
\le\;& \widehat{c}_0^2\|(1-\Delta)^{-\gamma/2}\phi\|_1\,\|(1-\Delta)^{-\gamma/2}\phi\|_q\,\mathbb{E}\left[\int_0^T \Big\|\sum_k |(1-\Delta)^{\gamma/2}g^k|^2\Big\|_{p/2}ds\right]\\
\le\;& \widehat{c}_0^2\|(1-\Delta)^{-\gamma/2}\phi\|_1\,\|(1-\Delta)^{-\gamma/2}\phi\|_q\,T^{1-\frac{2}{p}}\,\|g\|^2_{\mathbb{H}_p^\gamma(T,l^2)} < \infty,
\end{aligned}$$

where $q = p/(p-2)$ ($\frac{2}{0} := \infty$). Thus the series of the square integrable martingales $\sum_{k=1}^\infty \int_0^t (g^k,\phi)dZ_s^k$ converges in L_2 and hence in probability uniformly in $[0,T]$.

Definition 2.4 *Write $u \in \mathcal{H}_p^{\gamma+2}(T)$ if $u \in \mathbb{H}_p^{\gamma+2}(T)$ with $u(0) \in U_p^{\gamma+2}$, and for some $f \in \mathbb{H}_p^\gamma(T)$ and $g \in \mathbb{H}_p^{\gamma+1}(T,\ell^2)$*

$$du = fdt + g^k dZ_t^k \quad \textit{for } t \in [0,T]$$

in the distributional sense, that is, for any testing function $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$(u(t),\phi) = (u(0),\phi) + \int_0^t (f,\phi)dt + \sum_k \int_0^t (g^k,\phi)dZ_t^k \tag{2.6}$$

holds for all $t \in [0,T]$ a.s.. Denote f and g by $\mathbb{D}u$ and $\mathbb{S}u$ respectively, and define

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} := \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(T)} + \|\mathbb{S}u\|_{\mathbb{H}_p^{\gamma+1}(T,\ell^2)} + \|u(0)\|_{U_p^{\gamma+2}}.$$

We will show that $\mathcal{H}_p^{\gamma+2}(T)$ is a Banach space. For this, we need the following result.

Lemma 2.5 *Suppose $g(\omega,t) = (g^1, g^2, \cdots)$ is an ℓ^2-valued predictable process so that each g^k is bounded. Then there is a constant $C = C(p, \widehat{c}_0)$ so that*

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^t \int |g^k(s)|^2\,|z|^2 N^k(dz,ds)\right)^{p/2}\right] \\
\leq\ & C(p,\widehat{c}_0)\,\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^{\infty} |g^k(s)|^2 ds\right)^{p/2} + \int_0^t \sum_{k=1}^{\infty} |g^k(s)|^p\,ds\right].
\end{aligned} \tag{2.7}
$$

Proof. Fix a positive integer M. By monotone convergence theorem,

$$
\begin{aligned}
A :\ &= \mathbb{E}\left[\left(\sum_{k=1}^{M}\int_0^t \int |g^k(s)|^2|z|^2 N^k(ds,dz)\right)^{p/2}\right] \\
&= \lim_{N\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{M}\int_0^t \int_{|z|\leq N} |g^k(s)|^2|z|^2 N^k(ds,dz)\right)^{p/2}\right].
\end{aligned}
$$

Since $(a+b)^{p/2} \leq 2^{p/2}(|a|^{p/2}+|b|^{p/2})$ and $\tilde{N}^k(ds,dz) := N^k(ds,dz) - \nu_k(dz)ds$,

$$
\begin{aligned}
A\ &\leq\ 2^{p/2}\lim_{N\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{M}\int_0^t \int_{|z|\leq N} |g^k(s)|^2|z|^2 \widetilde{N}^k(ds,dz)\right)^{p/2}\right] \\
&\quad +2^{p/2}\,\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} \sum_{k=1}^{M} |g^k(s)|^2\,|z|^2\nu_k(dz)ds\right)^{p/2}\right] \\
&\leq\ C(p)\lim_{N\to\infty}\mathbb{E}\left[\left(\sum_{k=1}^{M}\int_0^t \int_{|z|\leq N} |g^k(s)|^4|z|^4 N^k(ds,dz)\right)^{p/4}\right] \\
&\quad +C(p)\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^{M} |\widehat{c}_2^k g^k(s)|^2\,ds\right)^{p/2}\right],
\end{aligned}
$$

where second inequality follows from the Burkholder-Davis-Gundy inequality and the fact that the stochastic process $J_t := \sum_{k=1}^{M}\int_0^t\int_{|z|\leq N} |g^k(s)|^2|z|^2\widetilde{N}^k(ds,dz)$ is a square integrable martingale with quadratic variation process

$$
[J]_t = \sum_{k=1}^{M}\int_0^t \int_{|z|\leq N} |g^k(s)|^4|z|^4 N^k(ds,dz) = \sum_{k=1}^{M}\sum_{0\leq s\leq t} |g^k(s)|^4|\Delta Z_s^k|^4 I_{|\Delta Z_s^k|\leq N}.
$$

Recall that for any $q>1$, $(\sum |a_n|^q)^{1/q} \le \sum |a_n|$. Thus if $2<p\le 4$, then

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^M \int_0^t \int_{|z|\le N} |g^k(s)|^4\, |z|^4 N^k(ds,dz)\right)^{p/4}\right] \\
\le\ & \mathbb{E}\left[\left(\sum_{k=1}^M \sum_{0\le s\le t} |g^k(s)|^4\, |\Delta Z_s^k|^4\right)^{p/4}\right] \le \mathbb{E}\left[\sum_{k=1}^M \sum_{0\le s\le t} |g^k(s)|^p\, |\Delta Z_s^k|^p\right] \\
\le\ & \widehat{c}_0^p \mathbb{E}\left[\int_0^t \sum_{k=1}^M |g^k(s)|^p\, ds\right].
\end{aligned}
$$

If $4<p\le 8$ then

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^M \int_0^t \int_{|z|\le N} |g^k(s)|^4\, |z|^4 N^k(ds,dz)\right)^{p/4}\right] \\
\le\ & \mathbb{E}\left[\left(\sum_{k=1}^M \int_0^t \int_{|z|\le N} |g^k(s)|^4\, |z|^4 \widetilde{N}^k(ds,dz) + \sum_{k=1}^M \int_0^t \int_{|z|\le N} |g(s)|^4\, |z|^4 \nu_k(dz)ds\right)^{p/4}\right] \\
\le\ & C(p)\mathbb{E}\left[\left(\sum_{k=1}^M \int_0^t \int_{|z|\le N} |g^k(s)|^8 |z|^8 N^k(ds,dz)\right)^{p/8} + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^4 ds\right)^{p/4}\right] \\
\le\ & C(p,\widehat{c}_0)\mathbb{E}\left[\int_0^t \sum_{k=1}^M |g^k(s)|^p\, ds + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^4 ds\right)^{p/4}\right].
\end{aligned}
$$

Similarly, in general, for $p\in(2^{n-1},2^n]$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^M \int_0^t \int |g^k(s)|^2\, |z|^2 N^k(dz,ds)\right)^{p/2}\right] \\
\le\ & C(p,\widehat{c}_0)\,\mathbb{E}\left[\sum_{j=1}^M \left(\int_0^t \sum_{k=1}^M |g^k(s)|^{2^j} ds\right)^{p2^{-j}}\right] + C(p,\widehat{c}_0)\,\mathbb{E}\left[\int_0^t \sum_k |g^k(s,x)|^p ds\right].
\end{aligned}
$$

Also since for each $2\le q\le p$,

$$
\left(\int_0^t \sum_{k=1}^M |g(s)|^q ds\right)^{1/q} \le C(q)\left(\left(\int_0^t \sum_{k=1}^M |g^k(s)|^2 ds\right)^{1/2} + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^p ds\right)^{1/p}\right),
$$

we get

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{k=1}^M \int_0^t \int |g^k(s)|^2\, |z|^2 N^k(dz,ds)\right)^{p/2}\right] \\
\le\ & C(p,\widehat{c}_0)\,\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^\infty |g^k(s)|^2 ds\right)^{p/2} + \int_0^t \sum_{k=1}^\infty |g^k(s)|^p\, ds\right]. \qquad (2.8)
\end{aligned}
$$

Passing $M \to \infty$ establishes the lemma. $\square$

Theorem 2.6 *For each $p \in [2, \infty)$ and $\gamma \in \mathbb{R}$, $\mathcal{H}_p^{\gamma+2}(T)$ is a Banach space with norm $\|\cdot\|_{\mathcal{H}_p^{\gamma+2}(T)}$. Moreover, there is a constant $C = C(d, p, \widehat{c}_0, T) > 0$ such that for every $u \in \mathcal{H}_p^{\gamma+2}(T)$,*

$$\mathbb{E}\left[\sup_{t \le T} \|u(t)\|_{H_p^\gamma}^p\right] \le C \left(\|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(T)}^p + \|\mathbb{S}u\|_{\mathbb{H}_p^\gamma(T,\ell^2)}^p + \mathbb{E}\left[\|u_0\|_{H_p^\gamma}^p\right]\right). \tag{2.9}$$

Consequently, for each $t > 0$,

$$\|u\|_{\mathbb{H}_p^\gamma(t)}^p \le C \int_0^t \|u\|_{\mathcal{H}_p^{\gamma+2}(s)}^p ds. \tag{2.10}$$

Proof. In view of Remark 2.2, it suffices to prove the theorem only for $\gamma = 0$. First we prove (2.9). Let $du = f dt + g^k dZ_t^k$ with $u(0) = u_0$. Assume that $g^k = 0$ for all $k > M$ and g^k is of the type

$$g^k(t, x) = \sum_{i=0}^m I_{(\tau_i^k, \tau_{i+1}^k]}(t) g^{ki}(x), \tag{2.11}$$

where τ_i^k are bounded stopping times and $g^{ki} \in C_c^\infty(\mathbb{R}^d)$. Define

$$v(t, x) = \sum_{k=1}^M \int_0^t g^k dZ_s^k.$$

Then by the Burkholder-Davis-Gundy inequality and Lemma 2.5,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s \le t} |v(s, x)|^p\right] \\
\le\;& C(p)\,\mathbb{E}\left[\left(\sum_{k=1}^\infty \int_0^t \int |g^k(s, x)|^2 |z|^2 N^k(ds, dz)\right)^{p/2}\right] \\
\le\;& C(p, \widehat{c}_0)\,\mathbb{E}\left[\left(\int_0^t \sum_{k=1}^\infty |g^k(s, x)|^2 ds\right)^{p/2} + \int_0^t \sum_k |g^k(s, x)|^p ds\right].
\end{aligned}$$

Since $\sum_n |a_n|^p \le (\sum_n |a_n|^2)^{p/2}$ and $(\int_0^t |h| ds)^p \le t^{p-1} \int_0^t |h|^p ds$, by integrating over $\mathbb{R}^d$ we get

$$\mathbb{E}\left[\sup_{s \le t} \|v\|_p^p\right] \le C(d, p, c_0, t)\, \|g\|_{\mathbb{H}_p^0(t,\ell^2)}^p := C(d, p, c_0, t)\,\mathbb{E} \int_0^t |g|_{\ell^2}^p ds. \tag{2.12}$$

Next we show that (2.12) holds for general $g \in \mathbb{H}_p^0(T, \ell^2)$. By Theorem 3.10 in [9], we can take a sequence $g_n \in \mathbb{H}_p^0(T, \ell^2)$ so that for each fixed n, $g_n^k = 0$ for all large k and each g_n^k is of the type (2.11), and $g_n \to g$ in $\mathbb{H}_p^0(T, \ell^2)$ as $n \to \infty$. Define $v_n(t, x) = \sum_{k=1}^\infty \int_0^t g_n^k dZ_t^k$, then

$$\mathbb{E}\left[\sup_{s \le t} \|v_n\|_p^p\right] \le C(d, p, c_0, t)\, \|g_n\|_{\mathbb{H}_p^0(t,\ell^2)}^p, \quad \mathbb{E}\left[\sup_{s \le t} \|v_m - v_n\|_p^p\right] \le C(d, p, c_0, t)\, \|g_m - g_n\|_{\mathbb{H}_p^0(t,\ell^2)}^p.$$

Thus (2.12) follows by taking $n \to \infty$. Now note that

$$d(u-v) = f dt \quad \text{with} \quad (u-v)(0) = u_0.$$

Thus it is easy to check that

$$\mathbb{E}\left[\sup_{s \le t} \|u-v\|_p^p\right] \le C\,\mathbb{E}\left[\|u_0\|_p^p\right] + C\,\mathbb{E}\left[\int_0^t \|f(s,\cdot)\|_p^p\,ds\right].$$

Consequently,

$$\mathbb{E}\left[\sup_{s \le t} \|u\|_p^p\right] \le C(d,p,c_0,t)\left(\|f\|^p_{\mathbb{H}^0_p(t)} + \|g\|^p_{\mathbb{H}^0_p(t,\ell^2)} + \mathbb{E}\|u_0\|^p_{L_p}\right).$$

Next we prove the completeness of the space $\mathcal{H}^2_p(T)$ using (2.9). Let $\{u_n : n = 1, 2, \cdots\}$ be a Cauchy sequence in $\mathcal{H}^2_p(T)$. Then $\{u_n\}$, $\{\mathbb{D}u_n\}$, $\{\mathbb{S}u_n\}$ and $\{u_n(0)\}$ are Cauchy sequences in $\mathbb{H}^2_p(T), \mathbb{L}_p(T), \mathbb{H}^1_p(T,\ell^2)$ and U^1_p respectively. Thus there exist $u \in \mathbb{H}^2_p(T)$, $f \in \mathbb{L}_p(T), g \in \mathbb{H}^1_p(T,\ell^2)$ and $u_0 \in U^1_p$ so that $u_n, \mathbb{D}u_n, \mathbb{S}u_n, u_n(0)$ converge to u, f, g, u_0 respectively, that is,

$$\|u_n - u\|_{\mathbb{H}^2_p(T)} + \|\mathbb{D}u_n - f\|_{\mathbb{L}_p(T)} + \|\mathbb{S}u_n - g\|_{\mathbb{H}^1_p(T,\ell^2)} + \|u_n(0) - u_0\|_{U^1_p} \to 0$$

as $n \to \infty$. Thus, by Definition 2.4, to prove $u \in \mathcal{H}^2_p(T)$ and $u_n \to u$ in $\mathcal{H}^2_p(T)$, we only need to show that for any $\phi \in C^\infty_c(\mathbb{R}^d)$, the equality

$$(u(t),\phi) = (u_0,\phi) + \int_0^t (f,\phi)dt + \sum_k \int_0^t (g^k,\phi)dZ^k_t \tag{2.13}$$

holds for all $t \le T$ (a.s.). Taking the limit from

$$(u_n(t),\phi) = (u_n(0),\phi) + \int_0^t (\mathbb{D}u_n,\phi)dt + \sum_k \int_0^t (\mathbb{S}^k u_n,\phi)dZ^k_t$$

and using the argument given in Remark 2.3(iii) one can show that (2.13) holds in $\Omega \times [0,T]$ (a.e.). Also the inequality (see (2.9))

$$\mathbb{E}\sup_{t \le T} \|u_n - u_m\|^p_{L_p} \le C\|u_n - u_m\|_{\mathcal{H}^2_p(T)}$$

show that for some subsequence n_m, $\lim_{m\to\infty} \sup_{t \le T} \|u_{n_m} - u\|_{L_p} = 0$ a.s. It follows that $(u_{n_m}(t),\phi) \to (u(t),\phi)$ uniformly in $t \in [0,T]$ a.s., and thus $(u(t),\phi)$ is right continuous with left limits. Consequently (2.13) holds for all $t \le T$ a.s. The theorem is proved. □

3 SPDEs with coefficients independent of x

In this section we study the equation

$$du = \left(a^{ij}u_{x^ix^j} + f\right)\,dt + \left(\sigma^{ik}u_{x^i} + g^k\right)\,dZ_t^k, \tag{3.1}$$

where the coefficients $a^{ij} = a^{ij}(\omega, t), \sigma^{ik} = \sigma^{ik}(\omega, t)$ are independent of x.

Assumption 3.1 (i) *The coefficients a^{ij} and σ^{ik} are predictable.*

(ii) $a^{ij} = a^{ji}$, and there exist constants $\delta, K > 0$ so that

$$|a^{ij}| + |\sigma^i|_{\ell^2} \leq K,$$

$$\delta I_{d\times d} \leq (a^{ij} - \alpha^{ij}) \leq (a^{ij}) \leq KI_{d\times d}, \tag{3.2}$$

where $\alpha^{ij} := \frac{1}{2}\sum_{k=1}^{\infty}\widehat{c}_{k,2}^{\,2}\sigma^{ik}\sigma^{jk}$ and $I_{d\times d}$ denotes the $(d\times d)$-identity matrix.

First we give an L_2-theory for SPDE (3.1).

Theorem 3.2 *Let Assumption 3.1 be satisfied. For every $f \in \mathbb{H}_2^\gamma(T)$, $g \in \mathbb{H}_2^{\gamma+1}(T, \ell^2)$ and $u_0 \in U_2^{\gamma+2}$, equation (3.1) with initial data u_0 has a unique solution $u \in \mathcal{H}_2^{\gamma+2}(T)$, and*

$$\|D^2u\|_{\mathbb{H}_2^\gamma(T)} \leq C\left(\|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}}\right) \tag{3.3}$$

$$\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} \leq C\,e^{CT}\left(\|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}}\right), \tag{3.4}$$

where $C = C(\delta_0, d, \widehat{c}_0, K) > 0$ is a constant independent of T.

Proof. Owing to Remark 2.2, we may assume $\gamma = 0$. Indeed, suppose that the theorem holds when $\gamma = 0$. Then for general $\gamma \in \mathbb{R}$, notice that $u \in \mathcal{H}_2^{\gamma+2}(T)$ is a solution of the equation if and only if $v := (1-\Delta)^{\gamma/2}u \in \mathcal{H}_2^2(T)$ is a solution of the equation with $\bar{f} := (1-\Delta)^{\gamma/2}f$, $\bar{g} := (1-\Delta)^{\gamma/2}g$ and $\bar{u}_0 := (1-\Delta)^{\gamma/2}u_0$ in place of f, g and u_0, respectively. Thus we have

$$\begin{aligned}\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} = \|v\|_{\mathbb{H}_2^2(T)} &\leq C\left(\|\bar{f}\|_{\mathbb{H}_2^0(T)} + \|\bar{g}\|_{\mathbb{H}_2^1(T,\ell^2)} + \|\bar{u}_0\|_{U_2^2}\right)\\ &= C\left(\|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}}\right).\end{aligned}$$

Let $f \in \mathbb{H}_2^0(T)$, $g \in \mathbb{H}_2^1(T, \ell^2)$ and $u_0 \in U_2^2$.

Step 1. First we prove that (3.3) and (3.4) hold for any solution $u \in \mathcal{H}_2^2(T)$ of (3.1) with initial value u_0. Fix $\ell \in \{1, 2, \cdots, d\}$ and denote $v = u_{x^\ell}$. Then

$$dv = (a^{ij}v_{x^ix^j} + f_{x^\ell})dt + (\sigma^{ik}v_{x^i} + g^k_{x^\ell})dZ_t^k. \tag{3.5}$$

Let $\bar{h}^k$ be a predictable version of $\sigma^{ik}v_{x^i}+g^k_{x^\ell}$. By Itó's formula (e.g. Theorem 4.4.10 of [1]. Also Remark 3.3 below)

$$|v(t)|^2=|(u_0)_{x^\ell}|^2+\int_0^t 2v dv+\sum_{k=1}^{\infty}\sum_{0\le s\le t}|\Delta Z^k_s|^2|\bar{h}^k(s)|^2. \tag{3.6}$$

Take expectation and integrate over $\mathbb{R}^d$ to get

$$0\le \mathbb{E}\|Du_0\|^2_{L_2}+\mathbb{E}\int_0^T\int_{\mathbb{R}^d}2v(a^{ij}v_{x^ix^j}+f_{x^\ell})dxdt+\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\widehat{c}^2_{k,2}(\sigma^{ik}v_{x^i}+g^k_{x^\ell})^2dxdt.$$

The last term is equal to

$$\begin{aligned}&\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\left(2\alpha^{ij}v_{x^i}v_{x^j}+2\sum_i v_{x^i}\sum_k \widehat{c}^2_{k,2}\sigma^{ik}g^k_{x^\ell}+|g_x|^2_{\ell_2}\right)dxds\\ &\le \mathbb{E}\int_0^T\int_{\mathbb{R}^d}2\alpha^{ij}v_{x^i}v_{x^j}\,dxds+\varepsilon\|Dv\|^2_{\mathbb{L}_2(T)}+C(\varepsilon)\|Dg\|^2_{\mathbb{L}_2(T,\ell_2)},\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d}2v(a^{ij}v_{x^ix^j}+f_{x^\ell})dx &= \int_{\mathbb{R}^d}(-2a^{ij}v_{x^i}v_{x^j}-2v_{x^\ell}f)dx\\ &\le -2\int_{\mathbb{R}^d}a^{ij}v_{x^i}v_{x^j}dx+\varepsilon\int_{\mathbb{R}^d}|Dv|^2dx+C(\varepsilon)\int_{\mathbb{R}^d}|f|^2dx.\end{aligned}$$

It follows that

$$2\mathbb{E}\int_0^T\int_{\mathbb{R}^d}(a^{ij}-\alpha^{ij})v_{x^i}v_{x^j}dxdt\le \mathbb{E}\|Du_0\|^2_{L_2}+\varepsilon\|Dv\|^2_{\mathbb{L}_2(T)}+C(\|f\|^2_{\mathbb{L}_2(T)}+\|Dg\|^2_{\mathbb{L}_2(T,\ell_2)}).$$

This and condition (3.2) prove (3.3).

Now using (3.3), the definition of $\|u\|^2_{\mathcal{H}^2_2(t)}$ and the inequalities

$$\|u\|_{H^2_2}=\|(1-\Delta)u\|_{L_2}\le\|u\|_{L_2}+\|D^2u\|_{L_2},\quad \|Du\|_{H^\gamma_2}\le\varepsilon\|D^2u\|_{H^\gamma_2}+C(\varepsilon)\|u\|_{H^\gamma_2}, \tag{3.7}$$

we get for each $t\le T$

$$\begin{aligned}\|u\|^2_{\mathcal{H}^2_2(t)} &= \left(\|u\|_{\mathbb{H}^2_2(t)}+\|a^{ij}u_{x^ix^j}+f\|_{\mathbb{H}^0_2(t)}+\|\sigma Du+g\|_{\mathbb{H}^1_2(t,\ell^2)}\right)^2\\ &\le C\left(\|u\|^2_{\mathbb{H}^0_2(t)}+\|D^2u\|^2_{\mathbb{H}^2_2(t)}+\|f\|^2_{\mathbb{H}^0_2(t)}+\|g\|^2_{\mathbb{H}^1_2(t,\ell^2)}\right)\\ &\le C\left(\|u\|^2_{\mathbb{H}^0_2(t)}+\|f\|^2_{\mathbb{H}^0_2(t)}+\|g\|^2_{\mathbb{H}^1_2(t,\ell^2)}+\|u_0\|^2_{U^2_2}\right)\\ &\le C\int_0^t\|u\|^2_{\mathcal{H}^2_2(s)}ds+C\left(\|f\|^2_{\mathbb{H}^0_2(T)}+C\|g\|^2_{\mathbb{H}^1_2(T,\ell^2)}+\|u_0\|^2_{U^2_2}\right),\end{aligned} \tag{3.8}$$

where the last inequality is from (2.10). Finally Gronwall's inequality leads to (3.4).

Step 2. Due to the method of continuity (see the proof of Theorem 5.1 of [9]), to finish the proof for $\gamma = 0$, we only need to prove that for each $f \in \mathbb{H}_2^0(T), g \in \mathbb{H}_2^1(T, \ell^2)$ and $u_0 \in U_2^2$, the equation

$$du = (\Delta u + f)dt + g^k dZ_t^k, \quad u(0) = u_0 \tag{3.9}$$

has a solution $u \in \mathcal{H}_2^2(T)$. Let $v \in \mathcal{H}_2^2(T)$ be the solution for

$$dv = (\Delta v + f)dt, \quad v(0) = u_0$$

(cf. Theorem 5.1 of [9]). Then $\widetilde{u} := u - v$ would be a solution for (3.9) with $f = u_0 = 0$. The solvability of equation (3.9) with $f = u_0 = 0$ can be easily proved based on a priori estimates (3.3) and (3.4). See the proof of Theorem 3.8 (Steps 1 and 2) below for details. This completes the proof of theorem. □

Remark 3.3 Actually, to get (3.6), we assumed that v satisfies (3.5) in the classical sense (not in the sense of distributions) and is sufficiently smooth in x. To be more precise, one has to go through the arguments in the proof of Theorem 3.7 of [9], and we leave this to the readers.

Lemma 3.4 *For $c > 0$, let $Z_t^k(c) := c^{-1} Z_{c^2 t}^k$ and ν_c^k be the Lévy measure of $Z_t^k(c)$. Then*

$$\int_{\mathbb{R}} z^2 \nu^k(dz) = \int_{\mathbb{R}} z^2 \nu_c^k(dz).$$

Proof. Denote $N_c^k(t, A) = \#\{s \le t; \Delta Z_s^k(c) \in A\}$. Then $N_c^k(t, A) = N^k(c^2 t, cA)$ and

$$\nu_c^k(A) = \mathbb{E}\left[N^k(c^2, cA)\right] = c^2 \nu^k(cA).$$

Hence the lemma follows from a change of variables. Indeed, let $f(z) = cz$ and $h(z) = z^2/c^2$. Then $\nu_c^k \circ f^{-1} = c^2 \nu^k$, and so

$$\int z^2 \nu_c^k(dz) = \int h(f(z)) \nu_c^k(dz) = \int h(z) c^2 \nu^k(dz) = \int z^2 \nu^k(dz).$$

□

Remark 3.5 Fix $c > 0$ and consider equation (3.1) with $Z_t^k(c)$ in place of Z_t^k:

$$du = \left(a^{ij} u_{x^i x^j} + f\right) dt + \left(\sigma^{ik} u_{x^i} + g^k\right) dZ_t^k(c), \tag{3.10}$$

It follows from Lemma 3.4 that Theorem 3.2 holds for equation (3.10) in place of (3.1) with the same constant $C = C(\delta_0, d, \widehat{c}_0, K)$ in (3.3).

Let T_t denote the semigroup associated with the Laplacian Δ on $\mathbb{R}^d$, that is,

$$T_t f(x) = P_t * f(x), \qquad \text{where} \quad P_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/(2t)}.$$

Lemma 3.6 *Let $p \in [2, \infty)$ and $g = (g^1, g^2, \cdots) \in \mathbb{H}^0_p(T, \ell^2) := L_p((0,T) \times \mathbb{R}^d, \ell^2)$. Then*

$$\int_{\mathbb{R}^d} \int_0^T \left(\int_0^t |DT_{t-s} g(s,x)|^2_{\ell^2} \, ds \right)^{p/2} dt dx \leq C(d,p) \int_{\mathbb{R}^d} \int_0^T |g(t,x)|^p_{\ell^2} \, dt dx. \tag{3.11}$$

Proof. See Lemma 4.1 in [9]. □

To establish an L_p-theory of SPDE, we need an estimate on $\sum_k \int_{\mathbb{R}^d} \int_0^T \int_0^t |DT_{t-s} g^k(s,x)|^p ds dt dx$.

Lemma 3.7 *Let $p \in (2, \infty)$ and $f \in L_p((0,T) \times \mathbb{R}^d)$. Then for any $\varepsilon > 1 - 2/p$,*

$$\int_{\mathbb{R}^d} \int_0^T \int_0^t |DT_{t-s} f(s,x)|^p \, ds dt \, dx \leq C \int_0^T \|f(t, \cdot)\|^p_{H^\varepsilon_p} \, dt, \tag{3.12}$$

where $C = C(d, p, \varepsilon, T)$.

Proof. Without loss of generality we may assume $\varepsilon < 1$. Let $q > p$ be chosen so that $1/p = (1-\varepsilon)/2 + \varepsilon/q$. Define an operator $\mathcal{A}$ by

$$\mathcal{A} f(t,s,x) = \begin{cases} DT_{t-s} f & \text{if } s < t, \\ 0 & \text{otherwise.} \end{cases}$$

First observe that, due to Lemma 3.6 and the inequality $\|T_{t-s} Df\|_q \leq \|Df\|_q$, the linear mappings

$$\mathcal{A} : L_2([0,T], L_2(\mathbb{R}^d)) \to L_2([0,T] \times [0,T] \times \mathbb{R}^d)$$

and

$$\mathcal{A} : L_q([0,T], H^1_q) \to L_q([0,T] \times [0,T] \times \mathbb{R}^d)$$

are bounded. Since $\varepsilon = (1-\varepsilon) \cdot 0 + \varepsilon \cdot 1$ and $1/p = (1-\varepsilon)/2 + \varepsilon/q$, it follows from the interpolation theory (see, for instance, [3, Theorem 5.1.2]) that $\mathcal{A}$ is a bounded operator from $L_p([0,T], H^\varepsilon_p)$ to $L_p([0,T] \times [0,T] \times \mathbb{R}^d)$. The lemma is proved. □

Theorem 3.8 *Let $p > 2$ and Assumption 3.1 be satisfied. Assume $\sigma^{ik} = 0$ for each i,k and $\varepsilon_0 > 1 - 2/p$. For every $f \in \mathbb{H}^\gamma_p(T)$, $g \in \mathbb{H}^{\gamma+1+\varepsilon_0}_p(T, \ell^2)$ and $u_0 \in U^{\gamma+2}_p$, equation (3.1) with initial data u_0 has a unique solution $u \in \mathcal{H}^{\gamma+2}_p(T)$, and*

$$\|u\|_{\mathcal{H}^{\gamma+2}_p(T)} \leq C \left(\|f\|_{\mathbb{H}^\gamma_p(T)} + \|g\|_{\mathbb{H}^{\gamma+1+\varepsilon_0}_p(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_p} \right), \tag{3.13}$$

where $C = C(\delta_0, d, p, \widehat{c}_0, \varepsilon_0, K, T)$.

Proof. Again by Remark 2.2, we only need to prove the theorem for $\gamma = -1$. Since we assume $\sigma^{ik} = 0$, the uniqueness of solution of equation (3.1) follows from the uniqueness result of deterministic equations. Thus we only need to show that there is a solution $u \in \mathcal{H}^1_p(T)$ and u satisfies (3.13).

Step 1. We assume first that $g^k = 0$ for $k > M$ and each $g^k (k \le M)$ is of the type

$$g^k(t,x) = \sum_{i=0}^{m(k)} I_{(\tau_i^k, \tau_{i+1}^k]}(t) g^{ki}(x), \tag{3.14}$$

where τ_i^k are bounded stopping times and $g^{ki}(x) \in C_c^\infty(\mathbb{R}^d)$. We prove the theorem for the stochastic heat equation:

$$du = \Delta u dt + \sum_{k=1}^{M} g^k dZ_t^k, \quad u(0) = 0. \tag{3.15}$$

Define

$$v(t,x) := \sum_{k=1}^{M} \int_0^t g^k(s,x) dZ_s^k = \sum_{k=1}^{M} \sum_{i=1}^{m(k)} g^{ki}(x)(Z^k_{t\wedge\tau^k_{i+1}} - Z^k_{t\wedge\tau^k_i})$$

and

$$u(t,x) := v(t,x) + \int_0^t T_{t-s} \Delta v_s \, ds.$$

Since for any sufficient smooth function h, $w = \int_0^t T_{t-s} h(s) ds$ satisfies $w_t = \Delta w + h$, we get $d(u-v) = (\Delta(u-v) + \Delta v)dt = \Delta u dt$. Therefore

$$du = \Delta u dt + dv = \Delta u dt + g^k dZ_t^k.$$

Also by stochastic Fubini theorem, almost surely,

$$\begin{aligned} u(t,x) &= v(t,x) + \sum_{k=1}^{M} \int_0^t \int_0^s T_{t-s} \Delta g^k(r,x) dZ_r^k ds \\ &= v(t,x) - \sum_{k=1}^{M} \int_0^t \int_r^t \frac{\partial}{\partial s} T_{t-s} g^k(r,x) ds dZ_r^k \\ &= \sum_{k=1}^{M} \int_0^t T_{t-r} g^k dZ_r^k. \end{aligned} \tag{3.16}$$

Similarly,

$$\frac{\partial}{\partial x^i} u(t,x) = \sum_{k=1}^{M} \int_0^t \frac{\partial}{\partial x^i} T_{t-s} g^k dZ_s^k.$$

Thus by the Burkholder-Davis-Gundy's inequality and (2.8), we have

$$\begin{aligned} \mathbb{E}\left[|u_x(t,x)|^p\right] &\le C(p)\mathbb{E}\left[\left(\sum_{k=1}^{M} \int_0^t \int |\frac{\partial}{\partial x} T_{t-s} g^k|^2 |z|^2 N^k(dz,ds)\right)^{p/2}\right] \\ &\le C(p,\widehat{c}_0)\, \mathbb{E}\left[\left(\int_0^t \sum_{k=1}^{\infty} |\frac{\partial}{\partial x} T_{t-s} g^k|^2 ds\right)^{p/2} + \int_0^t \sum_{k=1}^{\infty} |\frac{\partial}{\partial x} T_{t-s} g^k|^p ds\right]. \end{aligned} \tag{3.17}$$

By Lemma 3.6,

$$\mathbb{E}\int_{\mathbb{R}^d}\int_0^T\left(\int_0^t\sum_{k=1}^{\infty}|\frac{\partial}{\partial x}T_{t-s}g^k|^2ds\right)^{p/2}dtdx \le C\,\mathbb{E}\int_{\mathbb{R}^d}\int_0^T|g|^p_{\ell^2}dtdx = C\,\|g\|^p_{\mathbb{H}^0_p(T,\ell^2)}.$$

Also by Lemma 3.7,

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^d}\int_0^T\sum_k|\frac{\partial}{\partial x}T_{t-s}g^k|^pdsdtdx\\
\le\ & C\mathbb{E}\int_0^T\sum_k\|g^k\|^p_{H^{\varepsilon_0}_p}ds = C\,\mathbb{E}\int_0^T\int_{\mathbb{R}^d}\sum_k|(1-\Delta)^{\varepsilon_0/2}g^k|^pdxds\\
\le\ & C\,\mathbb{E}\int_0^T\int_{\mathbb{R}^d}|(1-\Delta)^{\varepsilon_0/2}g|^p_{\ell^2}dxds = C\,\|g\|^p_{\mathbb{H}^{\varepsilon_0}_p(T,\ell^2)},
\end{aligned}$$

where for the second inequality we use the inequality $\sum_k|a_k|^p \le (\sum_k|a_n|^2)^{p/2}$. Thus it follows from (3.17) that

$$\mathbb{E}\left[\int_0^T\|u_x\|^p_{L_p}\,dt\right] \le C\mathbb{E}\left[\int_0^T\|g\|^p_{H^{\varepsilon_0}_p(\ell^2)}\,dt\right]. \tag{3.18}$$

Now we prove (3.13). As before, (3.16) yields

$$\mathbb{E}\left[|u(t,x)|^p\right] \le C(p,\widehat{c}_0)\,\mathbb{E}\left[\left(\int_0^t\sum_{k=1}^M|T_{t-s}g^k|^2ds\right)^{p/2}+\int_0^t\sum_{k=1}^M|T_{t-s}g^k|^p\,ds\right]. \tag{3.19}$$

Since $(\sum_{k=1}^M|a_n|^2)^{p/2} \le M^{p/2}\sum_{k=1}^M|a_n|^p$ and $\|T_tf\|_{L_p} \le \|f\|_{L_p}$, it easily follows that $u \in \mathbb{H}^0_p(T)$, and consequently $u \in \mathcal{H}^1_p(T)$. By Theorem 2.6 and (3.18),

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\le T}\|u\|^p_{H^{-1}_p}\right] &\le C\left(\|\mathbb{D}u\|^p_{\mathbb{H}^{-1}_p(T)}+\|\mathbb{S}u\|^p_{\mathbb{L}_p(T,\ell^2)}\right)\\
&= C\left(\|\Delta u\|^p_{\mathbb{H}^{-1}_p(T)}+\|g\|^p_{\mathbb{H}^{-1}_p(T,\ell^2)}\right)\\
&\le C\left(\|u_x\|^p_{\mathbb{L}_p(T)}+\|g\|^p_{\mathbb{H}^{-1}_p(T,\ell^2)}\right) \le C\|g\|^p_{\mathbb{H}^{\varepsilon_0}_p(T,\ell^2)}.
\end{aligned}$$

Note that the constant C is independent of M. This together with (3.18) and the inequality

$$\|u\|_{H^1_p} = \|(1-\Delta)u\|_{H^{-1}_p} \le \|u\|_{H^{-1}_p}+\|\Delta u\|_{H^{-1}_p} \le \|u\|_{H^{-1}_p}+\|u_x\|_p$$

proves (3.13).

Step 2. We now prove the theorem for equation (3.15) for arbitrary $g \in \mathbb{H}^{\varepsilon_0}_p(T,\ell^2)$. By Theorem 3.10 in [9], we can take a sequence $g_n \in \mathbb{H}^{\varepsilon}_p(T,\ell^2)$ so that each g_n satisfies the extra assumptions specified in **Step 1** and $g_n \to g$ in $\mathbb{H}^{\varepsilon_0}_p(T,\ell^2)$. Then by the result of **Step 1**, equation (3.15) with g_n in place of g has a unique solution $u_n \in \mathcal{H}^1_p(T)$, and thus u_n-u_m satisfies

$$d(u_n-u_m) = \Delta(u_n-u_m)dt+(g^k_n-g^k_m)dZ^k_t.$$

Again by **Step 1**,

$$\|u_n - u_m\|_{\mathcal{H}_p^1(T)} \le C(p, d, \widehat{c}_0, T)\|g_n - g_m\|_{\mathbb{H}_p^{\varepsilon_0}(T,\ell^2)}.$$

It follows that $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}_p^1(T)$, and there exists $u \in \mathcal{H}_p^1(T)$ so that $u_n \to u$ in $\mathcal{H}_p^1(T)$. Taking the limit from

$$(u_n(t), \phi) = \int_0^t (\Delta u_n, \phi)dt + \int_0^t (g_n^k, \phi)dZ_t^k$$

one easily checks (see the last part of the proof of Theorem 2.6) that u is a solution of equation (3.15).

Step 3. Finally we consider equation (3.1). Let $v \in \mathcal{H}_p^1(T)$ be the solution of equation (3.15), where the existence of the solution is obtained in **Step 2**. Also let $\bar{u} \in \mathcal{H}_p^1(T)$ be the solution of the following equation (see [9, Theorem 4.10])

$$d\bar{u} = (a^{ij}\bar{u}_{x^ix^j} + f + a^{ij}v_{x^ix^j} - \Delta v)dt, \quad \bar{u}(0) = u_0.$$

Then by **Step 1** and [9, Theorem 4.10],

$$\|v\|_{\mathbb{H}_p^1(T)} \le c\|g\|_{\mathbb{H}_p^{\varepsilon_0}(\ell^2)},$$

$$\|\bar{u}\|_{\mathbb{H}_p^1(T)} \le C(\|D^2v\|_{\mathbb{H}_p^{-1}(T)} + \|f\|_{\mathbb{H}_p^{-1}(T)} + \|u_0\|_{U_p^1}).$$

Note that $u := \bar{u} + v$ satisfies

$$du = (a^{ij}u_{x^ix^j} + f)dt + g^k dZ_t^k \quad \text{with } u(0) = u_0,$$

and estimate (3.13) follows. Remember that uniqueness was already discussed at the beginning of the proof. The theorem is proved. □

4 Main Results

First we consider the following linear equation :

$$du = \left(a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f\right)dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k\right)dZ_t^k, \tag{4.1}$$

where all the coefficients depend on (ω, t, x). To explain conditions on the coefficients, we introduce the space of point-wise multipliers in H_p^γ. Fix $\kappa_0 > 0$. For $r \ge 0$, define $r_+ = r$ if $r = 0, 1, 2, \cdots$, and $r_+ = r + \kappa_0$ otherwise. Define

$$B^r = \begin{cases} \mathcal{B}(\mathbb{R}^d) & \text{if } r = 0, \\ C^{r-1,1}(\mathbb{R}^d) & \text{if } r = 1, 2, \cdots, \\ C^r(\mathbb{R}^d) & \text{otherwise,} \end{cases} \tag{4.2}$$

where $\mathcal{B}(\mathbb{R}^d)$ is the space of bounded Borel measurable functions on $\mathbb{R}^d$, $C^{r-1,1}(\mathbb{R}^d)$ is the space of $r-1$ times continuously differentiable functions whose $(r-1)$st order derivatives are Lipschitz continuous, and $C^r(\mathbb{R}^d)$ is the usual Hölder space. We also use the Banach space B^r for ℓ^2-valued functions. For instance, if $g = (g^1, g^2, ...)$, then $|g|_{B^0} = \sup_x |g(x)|_{\ell^2}$ and

$$|g|_{C^{n-1,1}} = \sum_{|\alpha| \le n-1} |D^\alpha g|_{B^0} + \sum_{|\alpha|=n-1} \sup_{x \neq y} \frac{|D^\alpha g(x) - D^\alpha g(y)|_{\ell^2}}{|x-y|}.$$

Assumption 4.1 (i) *The coefficients $a^{ij}, b^i, c, \sigma^{ik}, \mu^k$ are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions.*

(ii) $a^{ij} = a^{ji}$*, and the functions a^{ij} and σ^i are uniformly continuous in x. In other words, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x-y| < \delta$,*

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell^2} < \varepsilon.$$

(iii) *There exist constants $\delta, K > 0$ so that*

$$|a^{ij}| + |b^i| + |c| + |\sigma^i|_{\ell^2} + |\mu|_{\ell^2} \le K,$$

$$\delta I_{d\times d} \le (a^{ij} - \alpha^{ij}) \le (a^{ij}) \le K I_{d\times d}, \tag{4.3}$$

where $\alpha^{ij} := \frac{1}{2}\sum_{k=1}^{\infty} \widehat{c}_{k,2}^2 \sigma^{ik}\sigma^{jk}$ and $I_{d\times d}$ denotes the $(d \times d)$-identity matrix.

Below are our main results for the linear equation (4.1). We formulate them into two theorems since our assumptions are stronger when $p > 2$.

Theorem 4.2 *Let $\gamma \in \mathbb{R}, T > 0$ and Assumption 4.1 hold. Assume there is a constant $L > 0$ so that for each ω, t,*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |b^i(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} + |\sigma^i(t,\cdot)|_{B^{|\gamma+1|_+}} + |\mu(t,\cdot)|_{B^{|\gamma+1|_+}} \le L.$$

Then for any $f \in \mathbb{H}_2^\gamma(T)$, $g \in \mathbb{H}_2^{\gamma+1}(T, \ell^2)$ and $u_0 \in U_2^{\gamma+2}$ equation (4.1) with initial data u_0 has a unique solution $u \in \mathcal{H}_2^{\gamma+2}(T)$, and

$$\|u\|_{\mathcal{H}_2^{\gamma+2}(T)} \le C\left(\|f\|_{\mathbb{H}_2^\gamma(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}}\right), \tag{4.4}$$

where $C = C(\delta, d, p, \widehat{c}_0, K, L, \gamma, T)$.

Remark 4.3 Condition (4.3) appears naturally when one writes down the Itô's formula for $|u|^2$, where u is a solution of (3.1) (see the proof of Theorem 3.2). Remember we assumed $\beta^k = 0$ in (2.2). From the proof of Theorem 3.2, it is easy to check that if $\beta^k \neq 0$ then one only needs to replace the condition (4.3) by

$$\delta I_{d\times d} \le (a^{ij} - \bar{\alpha}^{ij}) \le (a^{ij}) \le K I_{d\times d},$$

where $\bar{\alpha}^{ij} := \frac{1}{2}\sum_{k=1}^{\infty}((\beta^k)^2 + \widehat{c}_{k,2}^2)\sigma^{ik}\sigma^{jk}$.

Remember that for $r \geq 0$, $r_+ := r$ if r is an integer, and $r_+ := r + \kappa_0 > r$ otherwise.

Theorem 4.4 *Let $p \in (2,\infty), \gamma \in \mathbb{R}$ and $\varepsilon_0 > 1 - 2/p$. Assume Assumption 4.1 holds, $\sigma^{ik} = 0$ and there is a constant $L > 0$ so that for each ω, t,*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |b^i(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} + |\mu(t,\cdot)|_{B^{|\gamma+1+\varepsilon_0|_+}} \leq L. \tag{4.5}$$

Then for any $f \in \mathbb{H}^\gamma_p(T)$, $g \in \mathbb{H}^{\gamma+1+\varepsilon}_p(T,\ell^2)$ and $u_0 \in U^{\gamma+2}_p$, equation (4.1) with initial date u_0 has a unique solution $u \in \mathcal{H}^{\gamma+2}_p(T)$, and

$$\|u\|_{\mathcal{H}^{\gamma+2}_p(T)} \leq C\left(\|f\|_{\mathbb{H}^\gamma_p(T)} + \|g\|_{\mathbb{H}^{\gamma+1+\varepsilon_0}_p(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_p}\right), \tag{4.6}$$

where $C = C(\delta, \widehat{c}_0, K, L, p, \gamma, T)$.

Remark 4.5 (i) Note that Theorem 4.4 requires stronger conditions than those in Theorem 4.2; σ^{ik} is assumed to be zero, and the regularity condition of μ is stronger.

(ii) However Theorem 4.4 gives better regularity results of solutions; let $\gamma + 2 - d/p > 0$ and u be the solution in the above theorems. Then from the embedding $H^{\gamma+2}_p \subset C^{\gamma+2-d/p}$, it follows that

$$\mathbb{E}\left[\int_0^T |u|^p_{C^{\gamma+2-d/p}} ds\right] \leq C\left(\|f\|_{\mathbb{H}^\gamma_p(T)} + \|g\|_{\mathbb{H}^{\gamma+1+\varepsilon_0}_p(T,\ell^2)} + \|u_0\|_{U^{\gamma+2}_p}\right).$$

To prove Theorem 4.2 and Theorem 4.4 we need the following three lemmas.

Lemma 4.6 *For $p > 1$ and $\gamma \in \mathbb{R}$, there is a constant $C = C(d,p,\gamma) > 0$ so that for every $a \in B^{|\gamma|_+}$ and $u \in H^\gamma_p$,*

$$\|au\|_{H^\gamma_p} \leq C\,|a|_{B^{|\gamma|_+}}\,\|u\|_{H^\gamma_p}.$$

The same is true for ℓ^2-valued functions a in $B^{|\gamma|_+}$.

Proof. See Lemma 5.2 in [9]. □

The following result is an extension of Theorem 3.2 for SPDEs with variable coefficients that have small oscillations.

Lemma 4.7 *Let $b^i = c = \mu^k = 0$ and suppose that there is a constant $L > 0$ so that for each ω, t,*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |\sigma^i(t,\cdot)|_{B^{|\gamma+1|_+}} \leq L.$$

Define

$$\beta := \sup_{\omega,x,y}\left(|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^{ik}(t,x) - \sigma^{ik}(t,y)|_{\ell^2}\right).$$

Then there exists $\beta_0 = \beta_0(d,\delta,K) > 0$, **independent of** *$L$, so that if $\beta \leq \beta_0$ then for any solution $u \in \mathcal{H}^{\gamma+2}_2(T)$ of the equation*

$$du = \left(a^{ij}u_{x^ix^j} + f\right)\,dt + \left(\sigma^{ik}u_{x^i} + g^k\right)\,dZ^k_t, \quad u(0) = u_0 \tag{4.7}$$

we have

$$\|u\|_{\mathbb{H}_2^{\gamma+2}(T)} \le C\left(\|f\|_{\mathbb{H}_2^{\gamma}(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}}\right), \tag{4.8}$$

where $C = C(d, p, \widehat{c}_0, \gamma, \delta, K, L, T)$.

Proof. Let $u \in \mathcal{H}_2^{\gamma+2}(T)$ be a solution to equation (4.7). Denote

$$a_0^{ij}(t) = a^{ij}(t,0), \quad \sigma_0^{ik}(t) = \sigma^{ik}(t,0),$$
$$f_0 = (a^{ij} - a_0^{ij})u_{x^ix^j} + f, \quad g_0^k = (\sigma^{ik} - \sigma_0^{ik})u_{x^i} + g^k,$$
$$C_0 = \sup_{\omega,t}\left(|a^{ij} - a_0^{ij}|_{B^{|\gamma|_+}} + |\sigma^i - \sigma_0^i|_{B^{|\gamma+1|_+}}\right).$$

Then $du = (a_0^{ij}u_{x^ix^j} + f_0)dt + (\sigma_0^{ij}u_{x^i} + g_0^k)dZ_t^k$. So by Theorem 3.2, for each $t \le T$,

$$\|D^2u\|_{\mathbb{H}_2^{\gamma}(t)} \le C(d,\delta,K)\left(\|f_0\|_{\mathbb{H}_2^{\gamma}(t)} + \|g_0\|_{\mathbb{H}_2^{\gamma+1}(t,\ell^2)} + \|u_0\|_{U_2^{\gamma+2}}\right).$$

Let C_1 denote the constant C in Lemma 4.6. Then by Lemma 4.6,

$$\|(a^{ij} - a_0^{ij})u_{x^ix^j}\|_{H_2^{\gamma}} \le C_1\,|a^{ij} - a_0^{ij}|_{B^{|\gamma|_+}}\|u_{x^ix^j}\|_{H_2^{\gamma}} \le C_1C_0\|D^2u\|_{H_2^{\gamma}},$$

and similarly (also remember $\|Du\|_{H_2^{\gamma+1}} \le \varepsilon\|D^2u\|_{H_2^{\gamma}} + c(\varepsilon)\|u\|_{H_2^{\gamma}}$)

$$\|(\sigma^i - \sigma_0^i)u_{x^i}\|_{H_2^{\gamma+1}(\ell^2)} \le C_1C_0\|Du\|_{H_2^{\gamma+1}} \le C_1C_0\|D^2u\|_{H_2^{\gamma}} + C\|u\|_{H_2^{\gamma}}.$$

Thus if $C(d,\delta,K)C_1C_0 \le 1/8$, then we get

$$\|D^2u\|_{\mathbb{H}_2^{\gamma}(t)} \le C\|u\|_{\mathbb{H}_2^{\gamma}(t)} + C\|f\|_{\mathbb{H}_2^{\gamma}(T)} + C\|g\|_{\mathbb{H}_2^{\gamma+1}(T)} + C\|u_0\|_{U_2^{\gamma+2}},$$

and this easily leads to (see (3.8) and (3.7))

$$\|u\|^2_{\mathcal{H}_2^{\gamma+2}(t)} \le C\left(\|u\|^2_{\mathbb{H}_2^{\gamma}(t)} + \|f\|_{\mathbb{H}_2^{\gamma}(T)} + \|g\|_{\mathbb{H}_2^{\gamma+1}(T)} + \|u_0\|_{U_2^{\gamma+2}}\right).$$

Therefore, (2.10) and Gronwall's lemma prove the lemma (if $C(d,\delta,K)C_1C_0 \le 1/8$).

In general, for $m \ge 1$, denote $a_m^{ij}(t,x) := a^{ij}(t/m^2, x/m)$ and $\sigma_m^{ik}(t,x) := \sigma^{ik}(t/m^2, x/m)$. Then we have

$$|a_m^{ij}(t,\cdot) - a_m^{ij}(t,0)|_{B^{|\gamma|_+}} \le \beta + m^{-(|\gamma|_+ \wedge 1)}C_0,$$

and we can drop the second term on the right if $\gamma = 0$. Also we have a similar inequality for σ_m^{ik}. Observe that $u_m(t,x) := u(t/m^2, x/m)$ satisfies

$$du_m = (a_m^{ij}(u_m)_{x^ix^j} + f_m)\,dt + (\sigma_m^{ik}(u_m)_{x^i} + g_m^k)\,dZ_t^k(m^{-1}),$$

where $f_m(t,x) := m^{-2}f(t/m^2, x/m)$ and $g_m^{ik}(t,x) := m^{-1}g^{ik}(t/m^2, x/m)$. Then it follows from the above calculations and Remark 3.5 that for β sufficiently small and m sufficiently large,

$$\|u_m\|_{\mathbb{H}_2^{\gamma+2}(mt)} \le C\left(\|f_m\|_{\mathbb{H}_2^{\gamma}(mt)} + \|g_m\|_{\mathbb{H}_2^{\gamma+1}(mt,\ell_2)} + \|u_0(\cdot/m)\|_{U_2^{\gamma+2}}\right)$$

for each $t \le T$. Since $\|\cdot\|_{H_p^{\gamma}}$ norms of $u(t/m^2, x/m)$ and $u(t,x)$ are comparable, one gets inequality (4.8).

□

Lemma 4.8 *Let $\zeta_n \in C^\infty, n = 1, 2, \cdots$. Assume that for any multi-index α,*

$$\sup_x \sum_n |D^\alpha \zeta_n(x)| \le M(\alpha), \tag{4.9}$$

where $M(\alpha)$ are some constants. Then there exists a constant $C = C(d, n, \gamma, p, M)$ such that for any $f \in H_p^\gamma$,

$$\sum_n \|\zeta_n f\|_{H_p^\gamma}^p \le C \|f\|_{H_p^\gamma}^p.$$

If, in addition,

$$\sum_n |\zeta_n(x)|^p > c > 0, \tag{4.10}$$

then

$$\|f\|_{H_p^\gamma}^p \le C(d, n, \gamma, p, M, c) \sum_n \|\zeta_n f\|_{H_p^\gamma}^p.$$

Proof. See Lemma 6.7 in [9]. □

Proof of Theorem 4.2. In view of Theorem 3.2 and the method of continuity (see the proof of [4, Theorem 2.11]), we only need to show that a priori estimate (4.4) holds for any solution $u \in \mathcal{H}_2^{\gamma+2}(T)$ of (4.1). Let β_0 be the constant in Lemma 4.7. Since a^{ij}, σ^i are uniformly continuous, we can fix $\delta_0 > 0$ so that

$$|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,y)|_{\ell^2} \le \beta_0/4 \qquad \text{for } |x-y| \le 2\delta_0.$$

Fix a smooth function $\zeta \in C_c^\infty(B_1(0))$ so that $0 \le \zeta \le 1$ and $\zeta(x) = 1$ if $|x| \le 1/2$. Take a sequence of smooth functions $\{\zeta_n : n = 1, 2, \cdots\}$ so that $\zeta_n = \zeta(2(x - x_n)/\delta_0)$ for some $x_n \in \mathbb{R}^d$, and (4.9) and (4.10) hold. Then by Lemma 4.8, for each $t \le T$,

$$\|u\|_{\mathbb{H}_2^{\gamma+2}(t)}^2 \le C \sum_n \|u\zeta_n\|_{\mathbb{H}_2^{\gamma+2}(t)}^2. \tag{4.11}$$

Let $\xi_n(x) = \zeta(\frac{x - x_n}{\delta_0})$ and

$$a_n^{ij}(t,x) = \xi_n(x) a^{ij}(t,x) + (1 - \xi_n(x)) a^{ij}(t, x_n), \quad \sigma_n^{ik}(t,x) = \xi_n(x)\sigma^{ik}(t,x) + (1 - \xi_n(x))\sigma^{ik}(t, x_n).$$

Then a_n and σ_n satisfy (4.3) with the same constants δ, K,

$$|a_n^{ij}(t,x) - a_n^{ij}(t,y)| + |\sigma_n^i(t,x) - \sigma_n^i(t,y)|_{\ell^2} \le \beta_0, \quad \forall \omega, t, x, y$$

and $u\zeta_n$ satisfies

$$d(u\zeta_n) = (a_n^{ij}(u\zeta_n)_{x^i x^j} + f_n)dt + (\sigma_n^{ik}(u\zeta_n)_{x^i} + g_n^k)dZ_t^k,$$

where

$$f_n = -2a^{ij}u_{x^i}\zeta_{nx^j} - a^{ij}u\zeta_{nx^i x^j} + b^i u_{x^i}\zeta_n + cu\zeta_n + f\zeta_n$$
$$g_n^k = -\sigma^{ik}u\zeta_{nx^i} + \mu^k u\zeta_n + g^k\zeta_n.$$

By Lemma 4.7 and Lemma 4.6

$$
\begin{aligned}
&\|u\zeta_n\|^2_{\mathbb{H}^{\gamma+2}_2(t)} \leq C\left(\|f_n\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|g_n\|^2_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\zeta_n\|^2_{U^{\gamma+2}_2}\right) \\
\leq\ & C\Big(\sum_{i,j=1}^{d} \|u_{x_i}(\zeta_n)_{x_j}\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|uD^2\zeta_n\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|uD\zeta_n\|^2_{\mathbb{H}^{\gamma+1}_2(t)} + \|f\zeta_n\|^2_{\mathbb{H}^{\gamma}_2(t)} \\
&+ \|g\zeta_n\|^2_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\zeta_n\|^2_{U^{\gamma+2}_2}\Big).
\end{aligned}
$$

Thus it follows from (4.11) and Lemma 4.8 that

$$
\|u\|^2_{\mathbb{H}^{\gamma+2}_2(t)} \leq C\left(\|u\|^2_{\mathbb{H}^{\gamma+1}_2(t)} + \|f\|^2_{\mathbb{H}^{\gamma}_2(t)} + \|g\|^2_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\|^2_{U^{\gamma+2}_2}\right). \tag{4.12}
$$

By the definition of $\|\cdot\|_{\mathcal{H}^{\gamma+2}_2(t)}$ and Lemma 4.6,

$$
\begin{aligned}
\|u\|_{\mathcal{H}^{\gamma+2}_2(t)} &:= \|u\|_{\mathbb{H}^{\gamma+2}_2(t)} + \|a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f\|_{\mathbb{H}^{\gamma}_2(t)} \\
&\quad + \|\sigma^i u_{x^i} + \mu u + g\|_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2} \\
&\leq C\|u\|_{\mathbb{H}^{\gamma+2}_2(t)} + \|f\|_{\mathbb{H}^{\gamma}_2(t)} + \|g\|_{\mathbb{H}^{\gamma+1}_2(t,\ell^2)} + \|u_0\|_{U^{\gamma+2}_2}.
\end{aligned}
$$

This together with (4.12), the embedding inequality (see, for instance, [18])

$$
\|u\|^2_{H^{\gamma+1+\beta}_2} \leq \varepsilon\|u\|^2_{H^{\gamma+2}_2} + C(\varepsilon,\beta)\|u\|^2_{H^{\gamma}_2}, \quad \forall\beta\in[0,1) \tag{4.13}
$$

and Theorem 2.6 yields that

$$
\|u\|^2_{\mathcal{H}^{\gamma+2}_2(t)} \leq C\int_0^t \|u\|^2_{\mathcal{H}^{\gamma+2}_2(s)}ds + C\left(\|f\|^2_{\mathbb{H}^{\gamma}_2(T)} + \|g\|^2_{\mathbb{H}^{\gamma+1}_2(T,\ell^2)} + \|u_0\|^2_{U^{\gamma+2}_2}\right).
$$

A priori estimate (4.4) now follows from Gronwall's inequality. The theorem is proved. □

Proof of Theorem 4.4. Again in view of Theorem 3.8 and the method of continuity, we only need to prove that a priori estimate (4.6) holds for any solution $u \in \mathcal{H}^{\gamma+2}_p(T)$ of (4.1). One can prove the theorem by modifying the proof of Theorem 4.2. But since σ^{ik} is assumed to be zero, the proof of this theorem is much easier.

Let $v \in \mathcal{H}^{\gamma+2}_p(T)$ be the solution of

$$
dv = \Delta v dt + (\mu^k u + g^k)dZ^k_t, \quad v(0) = u_0.
$$

The existence of the solution of the above equation is guaranteed by Theorem 3.8. By Theorem 3.8 and Lemma 4.6,

$$
\begin{aligned}
\|v\|_{\mathbb{H}^{\gamma+2}_p(t)} &\leq C\left(\|\mu u + g\|_{\mathbb{H}^{\gamma+1+\varepsilon_0}_p(t,\ell^2)} + \|u_0\|_{U^{\gamma+2}_p}\right) \\
&\leq C\left(\|u\|_{\mathbb{H}^{\gamma+1+\varepsilon_0}_p(t)} + \|g\|_{\mathbb{H}^{\gamma+1+\varepsilon_0}_p} + \|u_0\|_{U^{\gamma+2}_p}\right).
\end{aligned} \tag{4.14}
$$

Note that $\bar{u} := u - v \in \mathcal{H}_p^{\gamma+2}(T)$ satisfies

$$d\bar{u} = (a^{ij}\bar{u}_{x^i x^j} + b^i \bar{u}_{x^i} + c\bar{u} + \bar{f})dt, \quad \bar{u}(0) = 0, \tag{4.15}$$

where $\bar{f} = a^{ij}v_{x^i x^j} - \Delta v + b^i v_{x^i} + cv + f$. By a classical result for deterministic equations (see, for instance, Theorem 5.1 in [9])

$$\|\bar{u}\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq C\|\bar{f}\|_{\mathbb{H}_p^{\gamma}(t)} \leq C\left(\|v\|_{\mathbb{H}_p^{\gamma+2}(t)} + \|f\|_{\mathbb{H}_p^{\gamma}(t)}\right).$$

Consequently, for each $t \leq T$, by (4.14) and (4.15)

$$\begin{aligned}
\|u\|_{\mathbb{H}_p^{\gamma+2}(t)} &\leq C\left(\|\bar{u}\|_{\mathbb{H}_p^{\gamma+2}(t)} + \|v\|_{\mathbb{H}_p^{\gamma+2}(t)}\right) \\
&\leq C\left(\|u\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon_0}(t)} + \|f\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|g\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon_0}(t)} + \|u_0\|^p_{U_p^{\gamma+2}}\right).
\end{aligned}$$

This together with the embedding inequality (see (4.13))

$$\|u\|_{H_p^{\gamma+1+\varepsilon_0}} \leq \delta\|u\|_{H_p^{\gamma+2}} + C(\delta, \varepsilon_0)\|u\|_{H_p^{\gamma}},$$

yields that

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq C\left(\|u\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|f\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|g\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon_0}(t)} + \|u_0\|^p_{U_p^{\gamma+2}}\right).$$

As in the proof of Theorem 4.2, this easily leads to (4.6). The theorem is proved. □

Finally we present our L_p-theory for the following equation :

$$du = \left(a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + f(u)\right)dt + \left(\sigma^{ik}u_{x^i} + \mu^k u + g^k(u)\right)dZ_t^k, \quad u(0) = u_0, \tag{4.16}$$

where $f(u) = f(\omega, t, x, u)$ and $g^k(u) = g^k(\omega, t, x, u)$.

Assumption 4.9 *Fix ε_0 so that $\varepsilon_0 = 0$ if $p = 2$, and $\varepsilon_0 \in (1 - 2/p, 1)$ if $p > 2$.*

(i) *$\sigma^{ik} = 0$ if $p > 2$.*

(ii) *For any $u = u(x)$ and $v = v(x)$ in $H_p^{\gamma+2}$, and $\varepsilon > 0$, there exists a constant C_ε so that*

$$\|f(t,\cdot,u) - f(t,\cdot,v)\|_{H_p^{\gamma}} + \|g(t,\cdot,u) - g(t,\cdot,v)\|_{H_p^{\gamma+1+\varepsilon_0}} \leq \varepsilon\|u-v\|_{H_p^{\gamma+2}} + C_\varepsilon\|u-v\|_{H_p^{\gamma}}. \tag{4.17}$$

Here is our main result.

Theorem 4.10 *Suppose that Assumption 4.1 and Assumption 4.9 hold. Assume also that for each ω, t,*

$$|a^{ij}(t,\cdot)|_{B^{|\gamma|_+}} + |b^i(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} + |\sigma^i(t,\cdot)|_{B^{|\gamma+1+\varepsilon_0|_+}} + |\mu(t,\cdot)|_{B^{|\gamma+1+\varepsilon_0|_+}} \leq L.$$

Then for any $u_0 \in U_p^{\gamma+2}$, the quasi-linear equation (4.16) has a unique solution $u \in \mathcal{H}_p^{\gamma+2}(T)$ and for this solution we have

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} \leq C\left(\|f(0)\|_{\mathbb{H}_p^{\gamma}(T)} + \|g(0)\|_{\mathbb{H}_p^{\gamma+1+\varepsilon_0}} + \|u_0\|_{U_p^{\gamma+2}}\right). \tag{4.18}$$

Proof. For any $u \in \mathcal{H}_p^{\gamma+2}(T)$, by using Theorem 4.2 and Theorem 4.4 we can define $v \in \mathcal{H}_p^{\gamma+2}(T)$ as the unique solution of the equation

$$dv = (a^{ij}v_{x^ix^j} + b^i v_{x^i} + cv + f(u))dt + (\sigma^{ik}v_{x^i} + \mu^k v + g^k(u))dZ_t^k, \quad u(0) = u_0.$$

By denoting $v = \mathcal{R}u$ we define an operator $\mathcal{R} : \mathcal{H}_p^{\gamma+2}(T) \to \mathcal{H}_p^{\gamma+2}(T)$. By Theorem 4.2, Theorem 4.4 and (4.17) for each $t \le T$,

$$\begin{aligned} \|\mathcal{R}u - \mathcal{R}v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} &\le C\left(\|f(u) - f(v)\|^p_{\mathbb{H}_p^{\gamma}(t)} + \|g(u) - g(v)\|^p_{\mathbb{H}_p^{\gamma+1+\varepsilon_0}(t,\ell_2)}\right) \\ &\le C\left(\varepsilon^p \|u - v\|^p_{\mathcal{H}_p^{\gamma+2}(t)} + C_\varepsilon^p \int_0^t \mathbb{E}\|u(s) - v(s)\|^p_{H_p^\gamma} ds\right). \end{aligned}$$

The proof of Theorem 6.4 in [9] shows that using this inequality one can find a positive integer m depending only on d, p, δ, K so that $\mathcal{R}^m$ is a contraction in $\mathcal{H}_p^{\gamma+2}(T)$ with contraction constant 1/2, and this yields all the assertions of the theorem. The theorem is proved. □

Example 4.11 (SPDEs with non-local operators) Consider

$$f(u) = \bar{b}(\omega, t, x)\Delta^{\beta_1/2}u + f_0, \quad g^k(u) = \bar{\nu}^k(\omega, t, x)\Delta^{\beta_2/2}u + g_0^k,$$

where $\Delta^{\alpha/2}u := \mathcal{F}^{-1}(-|\xi|^\alpha \mathcal{F}(u)(\xi))$. Assume $\beta_1 < 2$, $\beta_2 + \varepsilon_0 < 1$, and $f_0 \in \mathbb{H}_p^\gamma(T)$ and $g_0 \in \mathbb{H}_p^{\gamma+1+\varepsilon_0}(T, \ell_2)$. Assume for each ω, t,

$$|\bar{b}|_{B^{|\gamma|_+}} + |\bar{\nu}|_{B^{|\gamma+1+\varepsilon_0|_+}} \le K.$$

Then by Lemma 4.6,

$$\begin{aligned} &\|f(u) - f(v)\|_{H_p^\gamma} + \|g(u) - g(v)\|_{H_p^{\gamma+1+\varepsilon_0}} \\ \le\ & C\left(\|\Delta^{\beta_1/2}(u - v)\|_{H_p^\gamma} + \|\Delta^{\beta_2/2}(u - v)\|_{H_p^{\gamma+1+\varepsilon_0}}\right) \\ \le\ & C\left(\|u - v\|_{H_p^{\gamma+\beta_1}} + \|u - v\|_{H_p^{\gamma+1+\varepsilon_0+\beta_2}}\right). \end{aligned}$$

Since for any $\alpha < 2$ and $\varepsilon > 0$, $\|u\|_{H_p^{\gamma+\alpha}} \le \varepsilon\|u\|_{H_p^{\gamma+2}} + C(\varepsilon, \alpha)\|u\|_{H_p^\gamma}$, one easily gets (4.17). Thus Theorem 4.10 holds with this choice of $f(u)$ and $g^k(u)$.

Example 4.12 (SPDEs driven by Lévy space-time white noise) Let $d = 1$ and consider the equation

$$du = (a(t, x)u''(t, x) + b(t, x)u'(t, x) + f(t, x, u(t, x)))dt + h(t, x, u(t, x))d\mathcal{Z}_t \tag{4.19}$$

where $\mathcal{Z}_t$ is a cylindrical Lévy process on $L_2(\mathbb{R})$, that is $\mathcal{Z}_t$ has an expansion of the form

$$\mathcal{Z}_t = \sum_{k=1}^\infty \eta^k(x) Z_t^k$$

where $\{\eta^k : k = 1, 2, \cdot\}$ is an orthonormal basis in L_2, Z_t^k are identically distributed independent one-dimensional $\mathcal{F}_t$-adapted Lévy processes with Lévy measure ν that has finite second moment. Using the expansion we can rewrite (4.19) as follows :

$$du = (a(t,x)u''(t,x) + b(t,x)u'(t,x) + f(t,x,u(t,x)))dt + \sum_{k=1}^{\infty} g^k(u)dZ_t^k, \tag{4.20}$$

where $g^k(u) := h(t,x,u(t,x))\eta^k$. An L_p-theory of this equation is well studied in section 8.3 of [9] when Z_t^k are independent one-dimensional Wiener processes. By repeating the approach in [9] almost word for word, one can easily establish the estimate (4.17). Consequently, one gets from Theorem 4.10 a corresponding L_p-theory when Z_t^k are independent one-dimensional Lévy processes. Below we only give an L_2-theory for the simplicity of the presentation. Assume

$$\widehat{c}_0 =: \left(\int_{\mathbb{R}} |z|^2 \nu(dz) \right)^{1/2} < \infty, \quad |a|_{B^2} + |b|_{B^2} \le K,$$

$$|f(t,x,u) - f(t,x,v)| \le K|u-v|, \quad |h(t,x,u) - h(t,x,v)| \le \xi(t,x)|u-v|, \quad \sup|\xi| < \infty.$$

Theorem 4.13 *Let $\kappa \in (0, 1/2)$ and $u_0 \in U_2^{1/2-\kappa}$. Assume*

$$I(T) := \left(\mathbb{E} \int_0^T \left(\|f(t,\cdot,0)\|^2_{H_2^{-3/2-\kappa}} + \|\bar{h}(t,\cdot,0)\|^2_{L_2} \right) ds \right)^{1/2} < \infty,$$

where

$$\bar{h}(t,x,0) := \left(\int_{\mathbb{R}} R^2(x-y) h^2(t,y,0) dy \right)^{1/2}, \quad R(x) := |x|^{(2\kappa-1)/2} \int_0^\infty t^{-(5-2\kappa)/4} e^{-tx^2 - 1/(4t)} dt.$$

Then equation (4.19) with initial data u_0 has a unique solution $u \in \mathcal{H}_2^{1/2-\kappa}(T)$ and for this solution,

$$\|u\|_{\mathcal{H}_2^{1/2-\kappa}(T)} \le C \left(I(T) + \|u_0\|_{U_2^{1/2-\kappa}} \right).$$

Proof. Let $\gamma = -\kappa - 3/2$. Then $f(u)$ and $g(u)$ satisfy condition (4.17). We refer to the proof of Theorem 8.5 in [9] for the details. Then the claim follows from Theorem 4.10. □

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