

# ARCHIMEDES' PRINCIPLE FOR BROWNIAN LIQUID

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ABSTRACT. We consider a family of hard core objects moving as independent Brownian motions confined to a vessel by reflection. These are subject to gravitational forces modeled by drifts. The stationary distribution for the process has many interesting implications, including an illustration of the Archimedes' principle. The analysis rests on constructing reflecting Brownian motion with drift in a general open connected domain and studying its stationary distribution. In dimension two we utilize known results about sphere packing.

## 1. INTRODUCTION

We consider a model involving “hard core objects” (typically, spheres) moving as independent Brownian motions, reflecting from each other, and subjected to a constant “force”, that is, having a constant drift. The objects are confined to a “vessel” by reflection, that is, they cannot leave a subset of Euclidean space. Our “toy model” illustrates several well known physical phenomena for liquids, under some technical (mathematical) assumptions. We prove some theorems for moving objects of any size and shape, but the most interesting examples involve a large number of spheres, most of them small.

The first of the three phenomena that our model generates is tight packing of the objects under large pressure, and the formation of the surface of the liquid, that is, a hyperplane such that most spheres are below the surface, and there is little room to pack any more spheres below the surface.

The second phenomenon is the “centrifuge” effect. Centrifuges are used to separate materials consisting of small particles (molecules) with different mass. In this example, we consider spheres of the same size but subject to different “forces”, that is, drifts. The “heavy” spheres tend to be closer to the bottom than light spheres.

The third phenomenon is Archimedes' principle, which says that an object immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced by the object. We will illustrate this principle by a family of two-dimensional discs. One disc is large and it is submerged in a “liquid” consisting of a very large number of much smaller discs. The large disc either “floats” or “sinks” depending on the ratio of its drift and the drift of small

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discs. This model is limited to the two-dimensional case because the classical sphere packing problem is completely understood only in this case. A similar probabilistic theorem can be stated and proved in higher dimensions if the relevant information on sphere packing is available.

Finally, we will give an example involving objects with “inertia”, in which high inertia will make the large object (disc) sink more easily, even if the drift of this object would not be sufficient to make it sink without inertia. The inertia is modeled by oblique reflection and low diffusion constant. Two spheres are said reflect in an oblique way if the amount of push (a multiple of the local time) experienced by the spheres during reflection is not identical for the two spheres.

Although Section 2 contains technical material needed for our main results, it may have some independent interest. We construct there reflected Brownian motion with drift in an arbitrary Euclidean domain and find its stationary distribution. The reason for this great level of generality is that we apply these results to the configuration space of hard core objects. Even if the objects are spheres, the configuration space does not have to be smooth. Some regularity properties of the configuration space for non-overlapping balls were proved in Proposition 4.1 in [10].

The present article has its roots in an analogous one-dimensional model studied briefly in Section 2 of [4]. A construction of an infinite system of reflecting Brownian hard core spheres was given in [22]. See the introduction and references in that paper for the history and ramifications of the problem.

We are grateful to Charles Radin for the following remarks and references (but we take the responsibility for any inaccuracies). A physical system that could reasonably be called a Brownian fluid is a colloid, like milk. Since we are using “reflecting hard spheres”, this specializes to noncohesive, hard-particle colloids. One of the classic material properties which are demonstrated in colloids is the fluid/solid phase transition known by simulation in the hard sphere model. Although this was first demonstrated earlier by others, the definitive paper seems to be [25]. A short expository introduction to related problems can be found online ([23]). A recent preprint concerned with the motion of globules is [15].

The rest of the paper is organized as follows. Section 2 is devoted to the analysis of reflected Brownian motion with drift in an arbitrary Euclidean domain. Section 3 contains some lemmas about the geometry of the configuration space of objects in a vessel. We give a sufficient condition for the existence of a stationary distribution for a family of reflecting objects in Section 4. Finally, Section 5 contains our main results, informally discussed above. Our mathematical model is formally introduced at the beginning of Sections 3 and 4.

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## 2. STATIONARY DISTRIBUTION FOR REFLECTED BROWNIAN MOTION WITH DRIFT

Constructing reflected Brownian motion with a constant drift  $\mathbf{v}$  in a *general* open connected set  $D \subset \mathbb{R}^n$  is quite delicate. Even its definition needs a careful formulation. This will be done in this section. We will then give a formula for the stationary distribution of this process. See [11, 12, 19] for some related results. A noteworthy aspect of Theorem 2.3 below is that we do not require any regularity assumptions on the boundary of  $D$ .

When  $D$  is a  $C^3$ -smooth domain in  $\mathbb{R}^n$ , a (normally) reflecting Brownian motion  $X$  with constant drift  $\mathbf{v} \in \mathbb{R}^n$  can be described by the following SDE:

$$dX_t = dB_t + \mathbf{v}dt + \mathbf{n}(X_t)dL_t, \quad t \geq 0, \quad (2.1)$$

with the constraint that  $X_t \in \bar{D}$  and  $L$  is a continuous non-decreasing process that increases only when  $X$  is on the boundary  $\partial D$ . Here in (2.1),  $B$  is Brownian motion on  $\mathbb{R}^n$  and  $\mathbf{n}$  is the unit inward normal vector field of  $D$  on  $\partial D$ . When  $D$  is  $C^3$  smooth, the strong existence and pathwise uniqueness of solution to (2.1) is guaranteed by [20]. When  $D$  is a Lipschitz domain and  $\mathbf{v} = 0$ , (2.1) has a unique weak solution by [2, Thm. 1.1 (i)]. Using a Girsanov transform, we conclude that weak existence and weak uniqueness hold for (2.1) with non-zero constant drift  $\mathbf{v}$ . When  $\mathbf{v} = 0$ , this equation has a unique strong solution when the domain  $D$  is  $C^\gamma$  with  $\gamma > 3/2$ , by [1, Thm. 1.1]. To motivate the definition of reflecting Brownian motion with constant drift  $\mathbf{v}$  in a general, possibly non-smooth, domain, observe that when  $D$  is  $C^3$  and  $\mathbb{P}_x$  denotes the law of the solution  $X$  to (2.1) with  $X_0 = x$ , then  $\{X, \mathbb{P}_x, x \in \bar{D}\}$  forms a time-homogeneous strong Markov process with state space  $\bar{D}$ . Let  $\{P_t, t \geq 0\}$  denote its transition semigroup and  $\rho(x) := e^{\mathbf{v} \cdot x}$ . It is easy to check that for every  $t > 0$ ,  $P_t$  is a symmetric contraction operator on  $L^2(\bar{D}, m)$ , where  $m(dx) := 1_D(x)\rho(x)^2 dx$ . Let  $(\mathcal{E}, \mathcal{F})$  denote the Dirichlet form of  $X$  on  $L^2(\bar{D}, m)$ ; that is,

$$\mathcal{F} = \left\{ u \in L^2(D; m) : \lim_{t \rightarrow 0} \frac{1}{t} (u - P_t u, u)_{L^2(D; m)} < \infty \right\}, \quad (2.2)$$

$$\mathcal{E}(u, v) = \lim_{t \rightarrow 0} \frac{1}{t} (u - P_t u, v)_{L^2(D; m)} \quad \text{for } u, v \in \mathcal{F}. \quad (2.3)$$

Then it is easy to check (see also the proof of Theorem 2.3(i) below) that

$$\mathcal{F} = \{ u \in L^2(D; m) : \nabla u \in L^2(D; m) \}, \quad (2.4)$$

$$\mathcal{E}(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) m(dx) \quad \text{for } u, v \in \mathcal{F}, \quad (2.5)$$

and that  $(\mathcal{E}, \mathcal{F})$  is a regular Dirichlet form on  $L^2(\overline{D}; m)$  in the sense that  $C_c(\overline{D}) \cap \mathcal{F}$  is dense both in  $C_c(\overline{D})$  with respect to the uniform norm and in  $\mathcal{F}$  with respect to the Hilbert norm  $\sqrt{\mathcal{E}_1(u, u)} := \sqrt{\mathcal{E}(u, u) + (u, u)_{L^2(D; m)}}$ . Here  $C_c(\overline{D})$  is the space of continuous functions on  $\overline{D}$  with compact support. This motivates the following definition.

**Definition 2.1.** Let  $D \subset \mathbb{R}^n$  be an open connected set. A continuous Markov process  $\{X_t, \mathbb{P}_x, x \in D\}$  taking values in  $\overline{D}$  will be called a reflecting Brownian motion with constant drift  $\mathbf{v}$  if it is symmetric with respect to the measure  $m(dx) := 1_D(x)e^{2\mathbf{v} \cdot x} dx$ , and  $\mathbb{P}_x(X_t \in \partial D) = 0$  for every  $x \in D$  and  $t > 0$ , and whose associated Dirichlet form  $(\mathcal{E}, \mathcal{F})$  in the sense of (2.2)-(2.3) is given by (2.4)-(2.5).

**Remark 2.2.** (i) Note that reflecting Brownian motion  $X$  with constant drift  $\mathbf{v}$  does not have to be a strong Markov process on  $\overline{D}$ . Nevertheless, its associated transition semigroup  $\{P_t, t \geq 0\}$  is well defined by the formula

$$\int_D f(x) P_t g(x) m(dx) = \mathbb{E}_m [f(X_0)g(X_t)] \quad \text{for every } f, g \in C_c(D).$$

Thus defined  $\{P_t, t \geq 0\}$  is a strongly continuous symmetric semigroup in  $L^2(D; m)$  and so the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  is well defined (see [17, Table 1 on p.18 and Theorems 1.3.1 and 1.4.1]).

(ii) When  $\mathbf{v} = 0$ , the Dirichlet form  $(\mathcal{E}, \mathcal{F})$  of (2.4)-(2.5) will be denoted as  $(\mathcal{E}^0, W^{1,2}(D))$ . Note that  $W^{1,2}(D)$  is the classical Sobolev space on  $D$  of order  $(1, 2)$ .

**Theorem 2.3.** *Suppose that  $\mathbf{v} \in \mathbb{R}^n$  and  $D \subset \mathbb{R}^n$  is open and connected.*

(i) *There exists a unique in law reflected Brownian motion  $X_t$  in  $D$  with constant drift  $\mathbf{v}$ .*

(ii) *Suppose that  $D_k, k \geq 1$ , is a sequence of open connected sets with smooth boundaries, such that  $D_k \subset D_{k+1}$  for  $k \geq 1$  and  $\bigcup_{k \geq 1} D_k = D$ . Let  $X_t^k$  be the reflected Brownian motion in  $D_k$  with a constant drift  $\mathbf{v}$ . Assume that  $X_0^k \rightarrow x_0 \in D$  in distribution. Then  $\{X_t^k, t \geq 0\}$  converge weakly to  $\{X_t, t \geq 0\}$ ,  $X_0 = x_0$ , in  $C([0, \infty); \overline{D})$ , as  $k \rightarrow \infty$ .*

(iii) *The function  $f(x) = \exp(2x \cdot \mathbf{v})$ ,  $x \in D$ , is the density of an invariant measure for  $X$ . If this density is integrable over  $D$ , then  $\mu(dx) := f(x)dx / \int_D f(x)dx$  is the unique stationary distribution for  $X$ .*

(iv) *Suppose that  $D$  and vector  $\mathbf{v}_1 \neq 0$  are fixed, and let  $\mathbf{v}_b = b\mathbf{v}_1$  for  $b > 0$ . Assume that  $\int_D \exp(2x \cdot \mathbf{v}_1)dx < \infty$ . Then there exists a stationary distribution  $\mu_b(dx)$  for reflected Brownian motion in  $D$  with drift  $\mathbf{v}_b$ , for every  $b \geq 1$ . Suppose that  $c_0 := \sup_{x \in D} (2x \cdot \mathbf{v}_1) < \infty$  and let  $D(\varepsilon) = \{x \in D : 2x \cdot \mathbf{v}_1 > c_0 - \varepsilon\}$  for  $\varepsilon > 0$ . Then for every  $\varepsilon > 0$  we have  $\lim_{b \rightarrow \infty} \mu_b(D(\varepsilon)) = 1$ .*

*Proof.* (i) The following facts have been established in [5, 6]. Let  $(\mathcal{E}^0, \mathcal{F})$  be the Dirichlet form defined by (2.4)-(2.5) with constant function 1 in place of  $\rho$ . There is a continuous

strong Markov process  $\{Y^*, \mathbb{P}_x^*, x \in D^*\}$  on the Martin-Kuramochi compactification  $D^*$  of  $D$ , whose associated Dirichlet space is  $(\mathcal{E}^0, W^{1,2}(D))$ . For  $1 \leq i \leq m$ , let  $f_i(x) := x_i$ . Then  $f_i$  admits a quasi-continuous extension to  $D^*$ . Define  $Y_t = (f_1(Y_t^*), \dots, f_m(Y_t^*))$ , which we call symmetric reflecting Brownian motion on  $\bar{D}$ . Process  $Y$  admits the following decomposition:

$$Y_t = Y_0 + B_t + N_t, \quad t \geq 0,$$

where  $B$  is Brownian motion on  $\mathbb{R}^n$  and  $N_t$  is an  $\mathbb{R}^n$ -valued process locally of zero quadratic variation.

We now construct reflecting Brownian motion  $X$  on  $\bar{D}$  with constant drift  $\mathbf{v}$  through Girsanov transform. Let  $\{\mathcal{F}_t, t \geq 0\}$  be the minimal augmented filtration generated by  $Y^*$ . (This notation is similar to the one used for the domain  $\mathcal{F}$  of the Dirichlet form but there will be little opportunity for confusion.) For  $x \in D^*$ , define the measure  $\mathbb{P}_x$  by

$$\frac{d\mathbb{P}_x}{d\mathbb{P}_x^*} = M_t := \exp\left(\int_0^t \mathbf{v} \cdot dB_s - \frac{1}{2} \int_0^t |\mathbf{v}|^2 ds\right) \quad \text{on each } \mathcal{F}_t.$$

Since the right hand side forms a martingale under  $\mathbb{P}_x^*$ ,  $Y^*$  under  $\mathbb{P}_x$  has infinite lifetime. By the same (but simpler) argument as that for [7, Lemma 2.4] (with  $\rho(x) := e^{\mathbf{v} \cdot \mathbf{x}}$  there), one can show that  $(Y^*, \mathbb{P}_x, x \in D^*)$  is a symmetric Markov process with respect to the measure  $m(dx) = 1_D(x)\rho(x)^2 dx$ . On the other hand, as a special case of [8, Theorem 3.1], the asymmetric Dirichlet form  $(\mathcal{E}^*, \mathcal{F}^*)$  associated with  $\{Y^*, \mathbb{P}_x\}$  in  $L^2(D^*; dx)$  is  $(\mathcal{E}^*, W^{1,2}(D))$ , where

$$\mathcal{E}^*(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx - \int_D (\mathbf{v} \cdot \nabla u(x)) v(x) dx, \quad u, v \in W^{1,2}(D).$$

Denote by  $\{G_\alpha, \alpha > 0\}$  the resolvent of  $\{Y^*, \mathbb{P}_x\}$ . The above means that (cf. [21, Theorem I.2.13])

$$\left\{ u \in L^2(D; dx) : \sup_{\beta > 0} (u - \beta G_\beta u, u)_{L^2(D; dx)} < \infty \right\} = W^{1,2}(D)$$

and for  $u, v \in W^{1,2}(D)$ ,

$$\lim_{\beta \rightarrow \infty} (u - \beta G_\beta u, v)_{L^2(D; dx)} = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) dx - \int_D (\mathbf{v} \cdot \nabla u(x)) v(x) dx. \quad (2.6)$$

Let  $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$  be the symmetric Dirichlet form of  $\{Y^*, \mathbb{P}_x\}$  in  $L^2(D^*; m)$ . Denote by  $bW_c^{1,2}(D)$  and  $b\mathcal{F}_c$  the families of bounded functions in  $W^{1,2}(D)$  and in  $\mathcal{F}$  with compact support, respectively. Note that  $bW_c^{1,2}(D) = b\mathcal{F}_c$ , and it is a dense linear subspace in both  $(W^{1,2}(D), (\mathcal{E}_1^0)^{1/2})$

and  $(\mathcal{F}, \mathcal{E}_1^{1/2})$ . By (2.6), for  $u \in bW_c^{1,2}(D) = b\mathcal{F}_c$ ,

$$\begin{aligned} \lim_{\beta \rightarrow \infty} (u - \beta G_\beta u, u)_{L^2(D; \rho^2 dx)} &= \lim_{\beta \rightarrow \infty} (u - \beta G_\beta u, \rho^2 u)_{L^2(D; dx)} \\ &= \frac{1}{2} \int_D \nabla u(x) \cdot \nabla(\rho^2 u)(x) dx - \int_D (\mathbf{v} \cdot \nabla u(x))(\rho^2 u)(x) dx \\ &= \frac{1}{2} \int_D |\nabla u(x)|^2 m(dx). \end{aligned}$$

This implies (cf. [17, Lemma 1.3.4(ii)]) that  $b\mathcal{F}_c \subset \overline{\mathcal{F}}$  and for  $u, v \in b\mathcal{F}_c$ ,

$$\overline{\mathcal{E}}(u, v) = \frac{1}{2} \int_D \nabla u(x) \cdot \nabla v(x) m(dx) = \mathcal{E}(u, v).$$

It follows that  $\mathcal{F} \subset \overline{\mathcal{F}}$  and  $\overline{\mathcal{E}} = \mathcal{E}$  on  $\mathcal{F}$ . Conversely, for  $u \in b\overline{\mathcal{F}}_c$ , we have  $\rho^{-2}u \in b\overline{\mathcal{F}}_c$ . Hence

$$\begin{aligned} \lim_{\beta > 0} (u - \beta G_\beta u, u)_{L^2(D; dx)} &= \lim_{\beta > 0} (u - \beta G_\beta u, \rho^{-2}u)_{L^2(D; m)} \\ &= \overline{\mathcal{E}}(u, u)^{1/2} \overline{\mathcal{E}}(\rho^{-2}u, \rho^{-2}u)^{1/2} < \infty. \end{aligned}$$

This implies that  $u \in bW_c^{1,2}(D) \subset \mathcal{F}$ . In other words,  $b\overline{\mathcal{F}}_c \subset \mathcal{F}$ . We conclude that  $(\overline{\mathcal{E}}, \overline{\mathcal{F}}) = (\mathcal{E}, \mathcal{F})$ . This completes the proof that  $(Y, \mathbb{P}_x)$  is a reflecting Brownian motion with constant drift on  $\overline{D}$ . To emphasize that we have constructed reflecting Brownian motion with drift as in Definition 2.1, we switch to our original notation used in that definition, that is, processes  $Y^*$  and  $Y$  under measure  $\mathbb{P}_x$  will be denoted  $X^*$  and  $X$ , respectively.

Next we establish uniqueness. Suppose that  $\tilde{X}$  is another reflecting Brownian motion with constant drift  $\mathbf{v}$  on  $D$ . By [17, Lemma 1.3.2 and Theorem 1.3.1] the transition semigroup of  $\tilde{X}$  should be the same as the transition semigroup of  $X$ . So as continuous processes,  $\tilde{X}$  and  $X$  share the same law under the initial distribution  $m$ . Since the subprocesses of  $\tilde{X}$  and  $X$  killed upon leaving  $D$  are Brownian motions in  $D$  with constant drift  $\mathbf{v}$ , it follows that  $\tilde{X}$  and  $X$  have the same distribution for every starting point  $x \in D$ .

(ii) Since  $D_k$  has smooth boundary, the Martin-Kuramochi compactification of  $D_k$  coincides with the Euclidean closure of  $D_k$ . So reflecting Brownian motion  $\{Y^k, \mathbb{P}_x^{*,k}\}$  on  $\overline{D}_k$  is a strong conservative Markov process with continuous sample paths. Each  $Y^k$  admits a Skorokhod decomposition (cf. [6])

$$Y^k = Y_0^k + B_t^k + \int_0^t \mathbf{n}_k(Y_s^k) dL_s^k, \quad t \geq 0,$$

where  $B^k$  is Brownian motion on  $\mathbb{R}^n$ ,  $\mathbf{n}_k(x)$  is the unit inward normal vector at  $x \in \partial D_k$ , and  $L^k$  is the boundary local time for reflecting Brownian motion  $Y^k$ . As we saw in (i) above, each  $X^k$  can be generated from reflecting Brownian motion  $Y^k$  on  $\overline{D}_k$  by the Girsanov transform

$$\frac{d\mathbb{P}_x^k}{d\mathbb{P}_x^*} = M_t^k := \exp \left( \int_0^t \mathbf{v} dB_s^k - \frac{1}{2} \int_0^t |\mathbf{v}|^2 ds \right) \quad \text{on each } \mathcal{F}_t^k.$$

Since  $Y_0^k = X_0^k$  is assumed to converge to  $x_0 \in D$  in distribution, it is established in [3] that  $(Y^k, B^k)$  converges weakly to  $(Y, B, \mathbb{P}_x^*)$  in the space  $C([0, \infty), \mathbb{R}^n \times \mathbb{R}^n)$  equipped with local uniform topology. By the Skorokhod representation theorem (see Theorem 3.1.8 in [13]), we can construct  $(Y^k, B^k)$  and  $(Y, B)$  on the same probability space  $(\Omega, \mathcal{F}, \bar{\mathbb{P}})$  so that  $(Y^k, B^k)$  converges to  $(Y, B)$ ,  $\bar{\mathbb{P}}$ -a.s., on the time interval  $[0, \infty)$  locally uniformly. Consequently,  $M^k$  converges to  $M$ ,  $\bar{\mathbb{P}}$ -a.s., on the time interval  $[0, \infty)$  locally uniformly, where

$$M_t = \exp \left( \int_0^t \mathbf{v} dB_s - \frac{1}{2} \int_0^t |\mathbf{v}|^2 ds \right).$$

Let  $\mathbb{P}$  be defined by  $d\mathbb{P}/d\bar{\mathbb{P}} = M_t$  on  $\mathcal{F}_t$ . Fix  $T > 0$ . It suffices to show that  $X^k$  converges weakly to  $X$  in the space  $C([0, T], \mathbb{R}^n)$ . Let  $\Phi$  be a continuous function on  $C([0, T], \mathbb{R}^n)$  with  $0 \leq \Phi \leq 1$ . Since  $\Phi(Y^k) \rightarrow \Phi(Y)$   $\bar{\mathbb{P}}$ -a.s., and  $M_T^k \rightarrow M_T$ ,  $\bar{\mathbb{P}}$ -a.s., by Fatou's lemma,

$$\mathbb{E}_{\bar{\mathbb{P}}}[\Phi(Y)M_T] \leq \liminf_{k \rightarrow \infty} \mathbb{E}_{\bar{\mathbb{P}}}[\Phi(Y^k)M_T^k] \leq \limsup_{k \rightarrow \infty} \mathbb{E}_{\bar{\mathbb{P}}}[\Phi(Y^k)M_T^k] \quad (2.7)$$

and

$$\mathbb{E}_{\bar{\mathbb{P}}}[(1 - \Phi)(Y)M_T] \leq \liminf_{k \rightarrow \infty} \mathbb{E}_{\bar{\mathbb{P}}}[(1 - \Phi)(Y^k)M_T^k]. \quad (2.8)$$

Summing (2.7) and (2.8) we obtain  $\mathbb{E}_{\bar{\mathbb{P}}}[M_T] \leq \limsup_{k \rightarrow \infty} \mathbb{E}_{\bar{\mathbb{P}}}[M_T^k]$ . Note that all  $M^k$ 's and  $M$  are continuous non-negative  $\bar{\mathbb{P}}$ -martingales. Hence,  $\mathbb{E}_{\bar{\mathbb{P}}}[M_T^k] = 1 = \mathbb{E}_{\bar{\mathbb{P}}}[M_T]$  and, therefore the inequalities in (2.7) and (2.8) are in fact equalities. It follows that

$$\lim_{k \rightarrow \infty} \mathbb{P}^k[\Phi(X^k)] = \lim_{k \rightarrow \infty} \mathbb{E}_{\bar{\mathbb{P}}}[\Phi(Y^k)M_T^k] = \mathbb{E}_{\bar{\mathbb{P}}}[\Phi(Y)M_T] = \mathbb{E}_{\mathbb{P}}[\Phi(X)].$$

This proves the weak convergence of  $X^k$  under  $\mathbb{P}^k$  to  $X$  under  $\mathbb{P}$ .

(iii) By definition,  $m(dx) = 1_D(x)e^{2\mathbf{v} \cdot x}$  is a symmetrizing measure for reflecting Brownian motion  $X$  with constant drift  $\mathbf{v}$  on  $D$ . If  $m(D) < \infty$ , then  $\mu := m/m(D)$  is the unique stationary distribution of  $X$  on  $D$ . By Theorem 2(ii) of [16], for every bounded  $f \in \mathcal{F}$ ,

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[f(X_t)] = \lim_{t \rightarrow \infty} \mathbb{E}_x[f(X_t^*)] = h(x) \quad \text{for q.e. } x \in D,$$

where  $h$  is a quasi-continuous function with  $P_t h = h$  q.e. for every  $t > 0$ . Here  $\{P_t, t \geq 0\}$  is the transition semigroup of  $X^*$ . Since  $D$  is connected, the reflecting Brownian motion  $Y^*$  on  $D$  is irreducible and so is  $X^*$ . Since  $m(D) < \infty$ , constant  $1 \in \mathcal{F}$  with  $\mathcal{E}(1, 1) = 0$ . Therefore  $X^*$  is recurrent. It follows that  $h$  is constant and equals  $\int_D f(x)\mu(dx)$ . Note that reflecting Brownian motion  $Y^*$  can be defined to start from every point in  $x \in D$  and has a transition density function with respect to the Lebesgue measure in  $D$ . As  $X^*$  can be obtained from  $Y^*$  through Girsanov transform, the same holds for  $X^*$ . It follows that for every  $x \in D$ ,

$$\lim_{t \rightarrow \infty} \mathbb{E}_x[f(X_t)] = \lim_{s \rightarrow \infty} \mathbb{E}_x[P_1 f(X_s)] = \int_D P_1 f(x)\mu(dx) = \int_D f(x)\mu(dx).$$

Since  $\mathcal{F}$  is dense in the space of bounded continuous functions on  $D$ , the last formula shows that  $\mu$  is the unique stationary distribution for  $X^*$  and  $X$ .

(iv) Since

$$\int_D \exp(2x \cdot \mathbf{v}_b) dx \leq \frac{\exp(c_0 b)}{\exp(c_0)} \int_D \exp(2x \cdot \mathbf{v}_1) dx < \infty,$$

it follows from (iii) that  $\mu_b(dx) = \exp(2x \cdot \mathbf{v}_b) dx / \int_D \exp(2x \cdot \mathbf{v}_b) dx$  is the stationary probability distribution for reflected Brownian motion in  $D$  with drift  $\mathbf{v}_b$ .

Note that for  $x \in D(\varepsilon)$ , we have  $\lim_{b \rightarrow \infty} \exp(-(c_0 - \varepsilon)b) \exp(2x \cdot \mathbf{v}_b) = \infty$ , so by the monotone convergence theorem

$$\lim_{b \rightarrow \infty} \int_{D(\varepsilon)} \exp(-(c_0 - \varepsilon)b) \exp(2x \cdot \mathbf{v}_b) dx = \infty,$$

and, for similar reasons,

$$\lim_{b \rightarrow \infty} \int_{D \setminus D(\varepsilon)} \exp(-(c_0 - \varepsilon)b) \exp(2x \cdot \mathbf{v}_b) dx = 0.$$

It follows that

$$\begin{aligned} \lim_{b \rightarrow \infty} \mu_b(D \setminus D(\varepsilon)) &\leq \lim_{b \rightarrow \infty} \frac{\mu_b(D \setminus D(\varepsilon))}{\mu_b(D(\varepsilon))} \\ &= \lim_{b \rightarrow \infty} \frac{\int_{D \setminus D(\varepsilon)} \exp(-(c_0 - \varepsilon)b) \exp(2x \cdot \mathbf{v}_b) dx}{\int_{D(\varepsilon)} \exp(-(c_0 - \varepsilon)b) \exp(2x \cdot \mathbf{v}_b) dx} = 0. \end{aligned}$$

Consequently,  $\lim_{b \rightarrow \infty} \mu(D(\varepsilon)) = 1 - \lim_{b \rightarrow \infty} \mu(D \setminus D(\varepsilon)) = 1$ .  $\square$

### 3. CONFIGURATION SPACE FOR HARD CORE OBJECTS

In this section, we will start the formal presentation of our model and we will prove two lemmas about the configuration space.

Suppose that  $d \geq 2$  and  $D \subset \mathbb{R}^d$  is open and connected. The set  $D$  represents the space where hard core objects may be located. Note that we did not impose any smoothness assumptions on  $\partial D$ .

Consider open non-empty bounded sets  $S_k \subset \mathbb{R}^d$ ,  $k = 1, \dots, N$ ,  $N \geq 1$ . The sets  $S_k$ 's represent hard core objects. We will think about  $S_k$ 's as moving or randomly placed objects, so we will use the notation  $S_k(y) = S_k + y$ . The diameter of  $S_k$  will be denoted by  $\rho_k$ .

Let  $\mathbf{D}' \subset \mathbb{R}^{Nd}$  be the set of all  $\mathbf{x} = (x^1, \dots, x^N)$ ,  $x^k \in \mathbb{R}^d$ , such that  $S_k(x^k) \subset D$  for all  $k = 1, \dots, N$ , and  $S_j(x^j) \cap S_k(x^k) = \emptyset$ , for  $j, k = 1, \dots, N$ ,  $j \neq k$ . Let  $\mathbf{D} \subset \mathbb{R}^{Nd}$  be the interior of  $\mathbf{D}'$ . We will call  $\mathbf{D}$  the configuration space.

We will prove that the configuration space  $\mathbf{D}$  is connected for two examples of  $D$  and  $S_k$ 's. We do not aim at a great generality because, first, the problem of characterizing  $D$  and  $S_k$ 's such that  $\mathbf{D}$  is connected seems to be very hard, and, second, our main examples in Section 5 are concerned with models where connectivity of  $\mathbf{D}$  is rather easy to see.

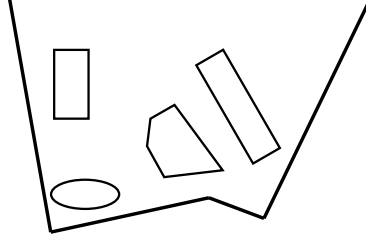


FIGURE 1. Convex objects above the graph of a function.

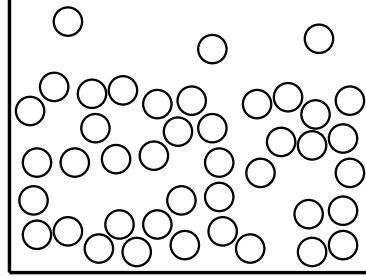


FIGURE 2. Spherical objects in a cylindrical vessel.

**Example 3.1.** (i) Suppose that there exists an upper semi-continuous function  $g : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  such that  $D = \{(x_1, \dots, x_d) : x_1 > g(x_2, \dots, x_d)\}$  and all  $S_k$ 's are convex (see Fig. 1). (ii) Suppose that  $D = \{(x_1, \dots, x_d) : x_1 > 0, x_2^2 + \dots + x_d^2 < 1\}$  is a one-sided open cylinder,  $S_k$ 's are open balls and  $\rho_k < 1$  for  $k = 1, \dots, N$  (see Fig. 2).

**Lemma 3.2.** *If  $D$  and  $S_k$ 's are such as in Example 3.1 (i) or (ii) then  $\mathbf{D}$  is pathwise connected.*

*Proof.* Suppose that  $\mathbf{x}, \mathbf{y} \in \mathbf{D}$  and  $\mathbf{x} \neq \mathbf{y}$ . We will describe a continuous motion of objects  $S_k$  inside  $D$ , such that the initial configuration is represented by  $\mathbf{x}$  and the terminal configuration is  $\mathbf{y}$ .

(i) Consider  $\mathbf{D}$  defined relative to  $D$  given in Example 3.1 (i). Let  $\rho_* = \max_{k=1, \dots, N} \rho_k$ .

Our argument will involve constants  $c_1, c_2 > 0$  whose values will be chosen later. First, we move continuously and simultaneously all objects  $S_k$  by  $c_1$  units in the direction  $(1, 0, \dots, 0)$ . Let  $\mathbf{z} = (z^1, \dots, z^N)$  denote the new configuration. Next we dilate the configuration by  $c_2$  units, that is, we fix  $S_1$  and we move continuously every object  $S_k$ ,  $k \neq 1$ , along the line segment  $[z^k, z^k + (z^k - z^1)c_2]$  away from  $S_1$ , at the speed  $c_2|z^2 - z^1|$ . We move all  $S_k$ ,  $k \neq 1$ , simultaneously. Note that the objects  $S_k$  will not intersect at any time because they are convex. Let  $\mathbf{z}_1$  denote the configuration of the objects at the end of the dilation.

Now we choose  $c_1$  and  $c_2$  so large that all objects  $S_k$  are always inside  $D$ , and the distance between any two objects is greater than  $\rho_*$ , when they are in the configuration  $\mathbf{z}_1$ .

Next, we start with the configuration  $\mathbf{y}$  and we use a similar method to move objects  $S_k$  continuously from configuration  $\mathbf{y}$  to a configuration  $\mathbf{u}$ , such that the objects do not intersect in the process of moving, they always stay inside  $D$ , and the distance between any two objects is greater than  $\rho^*$  when they are in the configuration  $\mathbf{u}$ . We make  $c_1$  larger, if necessary, so that the distance from any object in the configuration  $\mathbf{z}_1$  to any object in the configuration  $\mathbf{u}$  is greater than  $\rho_*$ .

At this point, we can move all objects continuously, one by one, from their location in configuration  $\mathbf{z}_1$  to their place in configuration  $\mathbf{u}$ . We combine motions from  $\mathbf{x}$  to  $\mathbf{z}_1$ , then to  $\mathbf{u}$ , and, by reversing an earlier motion, from  $\mathbf{u}$  to  $\mathbf{y}$ .

Consider the set  $\Gamma$  of all points  $\mathbf{z}_2$  representing the locations of objects  $S_k$  at all times during the motions. The set  $\Gamma$  is connected because the motions of the objects were continuous,  $\Gamma \subset \mathbf{D}$  because the objects always stayed in  $D$  and never intersected each other, and clearly  $\mathbf{x}, \mathbf{y} \in \Gamma$ . We have proved that for any  $\mathbf{x}, \mathbf{y} \in \mathbf{D}$ , there exists a connected subset of  $\mathbf{D}$  containing both points—this proves that  $\mathbf{D}$  is pathwise connected.

(ii) Now consider  $\mathbf{D}$  defined relative to  $D$  given in Example 3.1 (ii). Recall that  $\mathbf{x} = (x^1, \dots, x^N)$ , and let  $x^k = (x_1^k, \dots, x_d^k)$  represent the center of the  $k$ -th ball. Find a permutation  $(\pi(1), \dots, \pi(N))$  of  $(1, \dots, N)$  such that  $x_1^{\pi(1)} \geq x_1^{\pi(2)} \geq \dots \geq x_1^{\pi(N)}$ . Similarly, let  $\mathbf{y} = (y^1, \dots, y^N)$  and  $y^k = (y_1^k, \dots, y_d^k)$ . Let  $(\sigma(1), \dots, \sigma(N))$  be a permutation of  $(1, \dots, N)$  such that  $y_1^{\sigma(1)} \geq y_1^{\sigma(2)} \geq \dots \geq y_1^{\sigma(N)}$ . Let  $b = \max(x_1^{\pi(1)}, y_1^{\sigma(1)}) + 1$ . Move the  $\pi(1)$ -th ball in a continuous way to a location inside  $D$  such that the first coordinate of its center is equal to  $b + 1$  and the ball does not intersect the axis of  $D$ . Moreover, we move the ball in such a way that it does not intersect any other ball or  $\partial D$  at any time. Next, move the  $\pi(2)$ -th ball in a continuous way to a location inside  $D$  such that the first coordinate of its center is equal to  $b + 2$  and the ball does not intersect the axis of  $D$ . We move the ball in such a way that it does not intersect any other ball or  $\partial D$  at any time. Continue in this way, until we move the  $\pi(N)$ -th ball to a location inside  $D$  such that the first coordinate of its center is equal to  $b + N$  and the ball does not intersect the axis of  $D$ . We move the last ball in such a way that it does not intersect any other ball or  $\partial D$  at any time. Such continuous motions are possible because we always take the “top” ball from among those remaining in the original position, and the diameter of any ball is smaller than the radius of the cylinder  $D$ .

We now move the balls to the configuration  $\mathbf{y}$  by reversing the steps. First, we move the  $\sigma(N)$ -th ball to the location where its center is  $y^{\sigma(N)}$ , in a continuous way, such that the ball does not intersect any other ball or  $\partial D$  at any time. Next, we move the  $\sigma(N - 1)$ -st ball to the location where its center is  $y^{\sigma(N-1)}$ , in a continuous way, such that the ball does not

intersect any other ball or  $\partial D$  at any time. We continue in this way until all balls form the configuration represented by  $\mathbf{y}$ .

Consider the set  $\Gamma$  of all points  $\mathbf{z} = (z^1, \dots, z^N)$  representing the centers of all balls at all times during the motions. The set  $\Gamma$  is connected because the motions of the balls were continuous,  $\Gamma \subset \mathbf{D}$  because the balls always stayed in  $D$  and never intersected each other, and we also have  $\mathbf{x}, \mathbf{y} \in \Gamma$ . We have proved that for any  $\mathbf{x}, \mathbf{y} \in \mathbf{D}$ , there exists a connected subset of  $\mathbf{D}$  containing both points—this proves that  $\mathbf{D}$  is pathwise connected.  $\square$

**Remark 3.3.** The arguments given in the proof of Lemma 3.2 can be also used to show that  $\mathbf{D}'$  is the Euclidean closure  $\overline{\mathbf{D}}$  of  $\mathbf{D}$ . We will apply this observation to tight packings, based on the hexagonal tight packing, in proofs given later in this paper.

#### 4. EXISTENCE OF STATIONARY DISTRIBUTION

Informally speaking, we will assume that all objects  $S_k$  move as independent reflecting Brownian motions, with drifts  $(-a_k, 0, \dots, 0)$ , with  $a_k > 0$ . Formally, the evolving system of objects is represented by a stochastic process  $\mathbf{X}_t = (X_t^1, \dots, X_t^N)$  with values in  $\overline{\mathbf{D}}$ . In other words, the  $k$ -th object is represented at time  $t$  by  $S_k(X_t^k)$ . We assume that  $\mathbf{X}_t$  is  $(Nd)$ -dimensional reflected Brownian motion in  $\mathbf{D}$  with drift

$$\mathbf{v} = ((-a_1, 0, \dots, 0), \dots, (-a_N, 0, \dots, 0)), \quad (4.1)$$

where  $a_k > 0$  for  $k = 1, \dots, N$ .

The  $n$ -dimensional volume (Lebesgue measure) of a set  $A \subset \mathbb{R}^n$  will be denoted  $m_n(A)$ .

**Lemma 4.1.** *Assume that  $\mathbf{D}$  is connected. Let  $D_b = \{x = (x_1, \dots, x_d) \in D : x_1 = b\}$  and  $a_* = \min(a_1, \dots, a_N)$ . If there is  $b_0 \in \mathbb{R}$  such that  $D_b = \emptyset$  for all  $b < b_0$  and if there is some  $a < a_*$  so that*

$$\lim_{b \rightarrow \infty} m_{d-1}(D_b) \exp(-2ab) = 0, \quad (4.2)$$

*then  $\mathbf{X}$  has a unique stationary distribution.*

*Proof.* The uniqueness of the stationary distribution follows from Theorem 2.3 and the fact that  $\mathbf{D}$  is connected.

In view of Theorem 2.3 (iii), it is enough to show that  $\exp(2\mathbf{x} \cdot \mathbf{v})$  is integrable over  $\mathbf{D}$ . It follows from (4.2) that for some  $c$  and all  $b \geq b_0$ ,  $m_{d-1}(D_b) \leq c \exp(2ab)$ . This implies that

$$\begin{aligned} \int_{\mathbf{D}} \exp(2\mathbf{x} \cdot \mathbf{v}) d\mathbf{x} &\leq \int_{D^N} \exp(2\mathbf{x} \cdot \mathbf{v}) d\mathbf{x} = \prod_{k=1}^N \int_D \exp(-2a_k x_1) dx \\ &= \prod_{k=1}^N \int_{b_0}^{\infty} m_{d-1}(D_{x_1}) \exp(-2a_k x_1) dx_1 \leq \prod_{k=1}^N \int_{b_0}^{\infty} c \exp(2a_k x_1) \exp(-2a_k x_1) dx_1 < \infty. \end{aligned}$$

□

## 5. EXAMPLES OF MACROSCOPIC EFFECTS

In this section, we will be concerned with the distribution of the process  $\mathbf{X}_t$  under the stationary distribution  $\mu$ , so we will suppress the time variable  $t$ , and we will write  $\mathbf{X}$  for  $\mathbf{X}_0$ . We will also use the following notation,  $\mathbf{X} = (X^1, \dots, X^N)$  and  $X^k = (X_1^k, \dots, X_d^k)$ , for  $k = 1, \dots, N$ .

### 5.1. Surface of a liquid.

**Theorem 5.1.** *Suppose that  $D$  and  $S_k$ 's are as in Example 3.1 (i) or (ii), and  $D$  satisfies (4.2) for some  $a > 0$ . Fix some  $\alpha_k > 0$  for  $k = 1, \dots, N$  and let  $c_1 = \inf_{\mathbf{x} \in \mathbf{D}} \sum_{j=1}^N \alpha_j x_1^j$ . Let  $\lambda > 0$  and  $a_k = \lambda \alpha_k$  for  $k = 1, \dots, N$ . Let  $\lambda_* < \infty$  be so large that  $\min(a_1, \dots, a_N) > a$  for  $\lambda \geq \lambda_*$ . Assume that  $\lambda \geq \lambda_*$  and  $\mathbf{X}$  has the stationary distribution which we denote  $\mu_\lambda$ . (Note that  $(\mathbf{X}, \mu_\lambda)$  depends on  $\lambda$  through drift  $\mathbf{v}$  in (4.1).)*

(i) *For any  $p, \delta > 0$ , there exists  $\lambda_0 < \infty$  such that for  $\lambda > \lambda_0$ ,*

$$\mu_\lambda(\alpha_1 X_1^1 + \dots + \alpha_N X_1^N < c_1 + \delta) > 1 - p.$$

(ii) *For any  $p, \delta > 0$ , there exists  $\lambda_0 < \infty$  such that for  $\lambda > \lambda_0$ , with probability greater than  $1 - p$ , for every  $k = 1, \dots, N$ , for every  $z \in \mathbb{R}^d$  with  $z_1 < -\delta$ , we have,*

$$(S_k(X^k) + z) \cap \left( D^c \cup \bigcup_{j \neq k} S_j(X^j) \right) \neq \emptyset. \quad (5.1)$$

Part (i) of the above theorem says that if the drift of every process  $X^k$  is sufficiently large then the “weighted center of mass” for a typical configuration of  $S_k$ 's is within an arbitrarily small number of the infimum of weighted centers of mass over all permissible configurations, with arbitrarily large probability.

Part (ii) of the theorem says that for an arbitrarily small  $\delta > 0$ , if the drift of every process  $X^k$  is sufficiently large then with arbitrarily large probability, there is no room in the configuration to move any object  $S_k$  to a new location that would be more than  $\delta$  units in the negative  $x_1$ -direction below the current location of  $S_k$ . This means, in particular, that

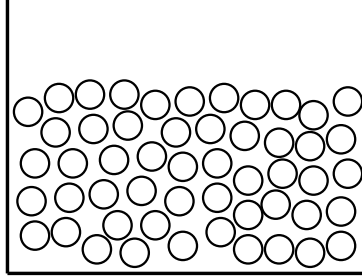


FIGURE 3. There are no “air bubbles” below the surface of this configuration.

there are no spherical holes between  $S_k$ 's with diameter  $\max_k \rho_k$  or greater,  $\delta$  units below the “surface” of  $S_k$ 's, that is, the hyperplane  $\{(x_1, \dots, x_d) \in D : x_1 = \max_k \sup_{y \in S_k(X^k)} y_1\}$  (see Fig. 3).

*Proof.* (i) Recall the notation from Theorem 2.3 (iv). We can identify  $\mathbf{v}$  in (4.1) corresponding to  $\lambda = 1$  with  $\mathbf{v}_1$  in Theorem 2.3 (iv). Then

$$\{\mathbf{x} \in \mathbf{D} : \alpha_1 x_1^1 + \dots + \alpha_N x_1^N < c_1 + \delta\} = \{\mathbf{x} \in \mathbf{D} : 2\mathbf{x} \cdot \mathbf{v}_1 > c_0 - 2\delta\}, \quad (5.2)$$

where  $c_0 := \sup_{\mathbf{x} \in \mathbf{D}} 2\mathbf{x} \cdot \mathbf{v}_1$ . It is now easy to see that part (i) of the theorem follows from Theorem 2.3 (iv).

(ii) Let  $\alpha_0 = \inf_{k=1, \dots, N} \alpha_k$ . Suppose that for some configuration  $\mathbf{x} \in \mathbf{D}$ , there exist  $k$  and  $z \in \mathbb{R}^d$  with  $z_1 < -\delta$ , such that (5.1) is not satisfied. Let  $\mathbf{y}$  represent the configuration which is obtained from  $\mathbf{x}$  by moving  $S_k$  from  $S_k(x^k)$  to  $S_k(x^k) + z$ . Note that  $\mathbf{y} \in \mathbf{D}$ . Since  $\mathbf{y} \in \mathbf{D}$  and  $z_1 < -\delta$ , we must have  $\alpha_1 y_1^1 + \dots + \alpha_N y_1^N \geq c_1 + \alpha_0 \delta$ . We now apply part (i) of the theorem to see that the family of configurations  $\mathbf{x}$  such that (5.1) is not satisfied has  $\mu_\lambda$ -probability less than  $p$ , if  $\lambda$  is sufficiently large.  $\square$

**5.2. Centrifuge effect.** In this example, all objects have the same shape but they are subject to different forces (drifts  $a_k$ ).

**Theorem 5.2.** *Suppose that  $D$  and  $S_k$ 's are as in Example 3.1 (i) or (ii),  $S_j = S_1$  for  $j = 2, \dots, N$ , and  $D$  satisfies (4.2) for some  $a > 0$ . Let the vector  $\mathbf{v}$  representing drifts be as in (4.1). Fix some  $\alpha_k > 0$  for  $k = 1, \dots, N$  and let  $c_1 = \inf_{\mathbf{x} \in \mathbf{D}} \sum_{j=1}^N \alpha_j x_1^j$ . Let  $\lambda > 0$  and  $a_k = \lambda \alpha_k$  for  $k = 1, \dots, N$ . Let  $\lambda_* < \infty$  be so large that  $\min(a_1, \dots, a_N) > a$  for  $\lambda \geq \lambda_*$ . Assume that  $\lambda \geq \lambda_*$  and  $\mathbf{X}$  has the stationary distribution  $\mu_\lambda$ . For any  $p, \delta > 0$ , there exists  $\lambda_0 < \infty$  such that for  $\lambda > \lambda_0$ , with probability greater than  $1 - p$ , for every pair  $j, k \in \{1, \dots, N\}$  with  $\alpha_j > \alpha_k$ , we have  $X_1^j < X_1^k + \delta$ .*

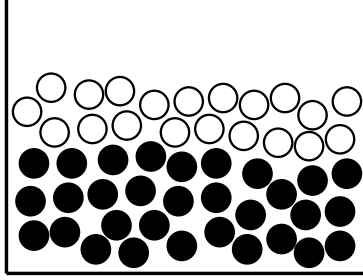


FIGURE 4. White balls have small downward drift while black balls have large downward drift. Every black ball is “almost” below every white ball.

The theorem says that in the stationary regime, with arbitrarily large probability, if the drift is very strong then the identical objects  $S_k$  are arranged in an almost monotone order, according to the strength of the drift (see Fig. 4).

*Proof.* The idea of the proof is very similar to that of the proof of Theorem 5.1(ii). Let  $\alpha_0 = \min\{\alpha_j - \alpha_k : \alpha_j > \alpha_k\}$  and note that  $\alpha_0 > 0$ . Suppose that for some configuration  $\mathbf{x} \in \mathbf{D}$ , there exist  $j$  and  $k$  such that  $\alpha_j > \alpha_k$  and  $x_1^j \geq x_1^k + \delta$ . Let  $\mathbf{y} \in \mathbf{D}$  represent the configuration which is obtained from  $\mathbf{x}$  by interchanging the positions of  $S_j(x^j)$  and  $S_k(x^k)$ . Since  $\mathbf{y} \in \mathbf{D}$ ,  $\alpha_j \geq \alpha_k + \alpha_0$  and  $x_1^j \geq x_1^k + \delta$ , we must have

$$\sum_{i=1}^N \alpha_i x_1^i = \sum_{i=1}^N \alpha_i y_1^i + (\alpha_j - \alpha_k)(x_1^j - x_1^k) \geq c_1 + \alpha_0 \delta.$$

We now apply part (i) of Theorem 5.1 to see that the family of configurations  $\mathbf{x}$  such that  $\alpha_j > \alpha_k$  and  $x_1^j \geq x_1^k + \delta$  for some  $j$  and  $k$ , has  $\mu_\lambda$ -probability less than  $p$ , if  $\lambda$  is sufficiently large.  $\square$

**5.3. Archimedes’ principle.** We will discuss the phenomena of floating and sinking assuming that  $d = 2$  and  $S_k$ ’s are discs. The reason for the limited generality of this example is that our argument is based on the classical sphere packing problem. This problem was completely solved in two dimensions long time ago (see [14]) and it also has been settled in three dimensions more recently (see [18]). The situation is more complicated in higher dimensions; see [9, 14, 18] and references therein for details. We will further limit our discussion to the cylindrical domain defined in Example 3.1(ii), because this example captures the essence of our claims. In the following, for  $x \in \mathbb{R}^d$  and  $r > 0$ ,  $B(x, r)$  denotes the open ball with radius  $r$  centered at  $x$ .

**Theorem 5.3.** *Suppose that  $d = 2$ ,  $D = (0, \infty) \times (-1, 1)$ ,  $S_1 = B(0, 1/2)$ , and  $S_k = B(0, \rho)$  for  $k = 2, \dots, N$ , where  $\rho < 1/2$ . Assume that  $a_2 = a_3 = \dots = a_N$  and  $\mathbf{X}$  has distribution  $\mu$ .*

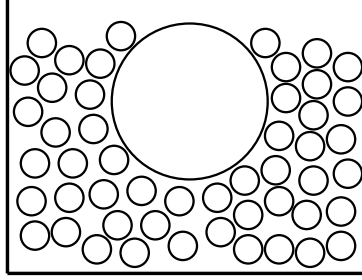


FIGURE 5. The large disc is floating because its drift is less than the sum of drifts of displaced small discs (properly scaled).

(i) For any  $p, \delta, \gamma > 0$ , there exists  $\rho_0 > 0$  and  $N_0 < \infty$  such that for  $\rho \in (0, \rho_0)$  and  $N \geq N_0$  which satisfy the condition  $\rho^2 N \sqrt{12} > 2 - \pi/4$  there exists  $a_0 > 0$  such that if  $a_2 \geq a_0$  and  $a_1/a_2 := \gamma_1 \leq (\pi/(4\sqrt{12}) - \gamma)/\rho^2$  then

$$\mu \left( X_1^1 < \max_{k=2, \dots, N} X_1^k - 1/2 - \delta \right) < p.$$

(ii) For any  $p, \delta, \gamma > 0$ , there exists  $\rho_0 > 0$  and  $N_0 < \infty$  such that for  $\rho \in (0, \rho_0)$  and  $N \geq N_0$  which satisfy the condition  $\rho^2 N \sqrt{12} > 2 - \pi/4$  there exists  $a_0 > 0$  such that if  $a_2 \geq a_0$  and  $a_1/a_2 := \gamma_2 \geq (\pi/(4\sqrt{12}) + \gamma)/\rho^2$  then

$$\mu \left( X_1^1 > 1/2 + \delta \right) < p.$$

The theorem is a form of Archimedes' principle. The first part of our result says that if the drift of the large ball is smaller than the sum of the drifts of displaced small balls then the large ball will “float”, that is, its uppermost point will be at least very close to the “surface” of the “liquid” (or above the surface). The second part says that if the drift of the large ball is greater than the sum of the drifts of displaced small balls then the large ball will sink to the bottom. Both results assume that the system is in the stationary distribution and all drifts are large (see Fig. 5).

The condition  $\rho^2 N \sqrt{12} > 2 - \pi/4$  is needed to make sure that there is an ample supply of small discs to make the large disc float.

*Proof.* (i) We will use some results about disc packing in the plane from [9, 14, 18]. The usual honeycomb lattice packing of disjoint discs has density  $\pi/\sqrt{12}$  and this is the highest possible disc packing density.

Consider the unique honeycomb packing  $\mathcal{S}$  of open discs with radii  $\rho$  in the whole plane in which some adjacent discs have their centers on the line parallel to the first axis, and one disc is centered at 0. For a set  $A$ , we will say that  $S_1, \dots, S_k$  is a *honeycomb disc packing* in  $A$  if it contains all discs in  $\mathcal{S}$  that are contained in  $A$ .

By abuse of notation, we will use  $|\cdot|$  to denote the area of a planar set and also the cardinality of a finite set.

Consider arbitrary  $p, \gamma, \delta > 0$ . We fix some  $\beta > 2 - \pi/4$  and assume that  $2 - \pi/4 < \rho^2 N \sqrt{12} < \beta$ . Note that it is sufficient to prove the theorem for every fixed  $\beta$ .

Suppose that for some configuration  $\mathbf{x} \in \mathbf{D}$ , we have

$$x_1^1 < \max_{k=2, \dots, N} x_1^k - 1/2 - \delta. \quad (5.3)$$

Let  $\alpha_1 = \gamma_1$  and  $\alpha_k = 1$  for  $2 \leq k \leq N$ . Let  $\lambda = a_2$ . Then  $\alpha_1 x_1^1 + \dots + \alpha_N x_1^N = \gamma_1 x_1^1 + x_1^2 + \dots + x_1^N$ . In view of Theorem 5.1 (i), it will suffice to show that for small  $\rho > 0$  there exists  $\delta_1 = \delta_1(\rho, N) > 0$  such that for some  $\mathbf{y} \in \mathbf{D}$ ,

$$\gamma_1 x_1^1 + x_1^2 + \dots + x_1^N > \gamma_1 y_1^1 + y_1^2 + \dots + y_1^N + \delta_1. \quad (5.4)$$

In view of Remark 3.3, it will suffice to show that (5.4) is satisfied for some  $\mathbf{y} \in \mathbf{D}' = \overline{\mathbf{D}}$ .

We divide the next part of the proof into cases and ‘‘subcases’’.

- *Case 1.* If there exist  $z = (z_1, z_2)$  and  $2 \leq k \leq N$  such that  $z_1 < -\rho$  and

$$(S_k(x^k) + z) \cap \left( D^c \cup \bigcup_{j \neq k} S_j(x^j) \right) = \emptyset, \quad (5.5)$$

then we choose the smallest  $k$  with this property and let

$$\mathbf{y} = (y^1, \dots, y^N) = (x^1, x^2, \dots, x^{k-1}, x^k + z, x^{k+1}, \dots, x^N).$$

Then (5.4) holds with  $\delta_1 = \rho$ .

- *Case 2.* In this case, we suppose that there are no  $z$  and  $k$  satisfying (5.5). Informally speaking, this implies that the disc configuration represented by  $\mathbf{x}$  has a density bounded below by some absolute constant  $c_1 > 0$ . We will now make this assertion precise. Consider a square

$$Q = Q(r_1, r_2, \rho) = \{(z_1, z_2) \in \mathbb{R}^2 : r_1 - 3\rho < z_1 < r_1 + 3\rho, r_2 - 3\rho < z_2 < r_2 + 3\rho\}$$

and assume that  $Q \subset D$  and  $r_1 + 3\rho \leq \max_{k=2, \dots, N} x_1^k$ . We have assumed that there are no  $z$  and  $k$  satisfying (5.5), so  $B((r_1, r_2), \rho)$  must intersect at least one disc  $S_k(x^k)$  with  $2 \leq k \leq N$ . It follows that  $S_k(x^k) \subset Q$  and, therefore, the area of  $(\bigcup_{2 \leq k \leq N} S_k(x^k)) \cap Q$  is greater than or equal to  $c_1 := \pi/36$  times the area of  $Q$ .

We have assumed that  $\rho^2 N \sqrt{12} < \beta$  so  $\max_{k=2, \dots, N} x_1^k < \beta_1$  for some  $\beta_1 = \beta_1(\beta) < \infty$ .

Let  $D_1 = (0, 2\beta_1) \times (-1, 1)$  and  $D_2 = D_1 \setminus S_1(x^1)$ . For  $0 < b < 2\beta_1$ , define  $D_2^b := \{(x_1, x_2) \in D_2 : x_1 < b\}$ . An upper estimate of the length of  $\partial D_2^b$  is  $c_2 := \pi + 4 + 4\beta_1$ . Let  $D_3^b = D_3^b(\rho) = \{x \in D_2^b : \text{dist}(x, \partial D_2^b) < 2\rho\}$  and  $D_4^b = D_4^b(\rho) = \{x \in (D_2^b)^c : \text{dist}(x, \partial D_2^b) < 2\rho\}$ . We have  $|D_3^b| \leq 4\rho c_2$  and  $|D_4^b| \leq 4\rho c_2$  for small  $\rho$ . Hence, the number  $N_1 = N_1(b)$  of

discs of radius  $\rho$  in the honeycomb packing in  $D_2^b$  satisfies

$$(|D_2^b| - 4\rho c_2)/(\rho^2\sqrt{12}) \leq N_1 \leq (|D_2^b| + 4\rho c_2)/(\rho^2\sqrt{12}).$$

We cannot pack  $N_1$  discs in  $D_2^{b_1}$  if  $|D_2^{b_1}| \leq |D_2^b| - 9\rho c_2$ . If  $\rho$  is small then this condition follows from  $b - b_1 \geq 9\rho c_2$ . So in any configuration of  $N_1$  discs in  $D_2$ , the  $N_1$ -th disc from the bottom will be at most  $9\rho c_2$  units below the position of the  $N_1$ -th disc from the bottom in the honeycomb packing of  $D_2$ .

If  $b_2 = b + \rho\sqrt{3}$  then the honeycomb packings of  $D_2^b$  and  $D_2^{b_2}$  differ by one row of discs, or a part of one row. The centers of the discs in the top row of  $D_2^{b_2}$  are  $\rho\sqrt{3}$  units above the centers of discs in the top row of  $D_2^b$ . There are no more than  $1/\rho$  discs in the top row of  $D_2^{b_2}$ . Consider any  $N_3$  such that  $N_1(b) \leq N_3 \leq N_1(b_2)$ . For any configuration of  $N_3$  discs in  $D_2$ , the  $N_3$ -th particle from the bottom will be at most  $9\rho c_2 + \rho\sqrt{3}$  units below the position of the  $N_3$ -th particle from the bottom in the honeycomb packing of  $D_2^{b_2}$ .

Let  $S_2(z^2), \dots, S_N(z^N)$  be the disc configuration in  $D_2$  obtained by taking  $N - 1$  discs in the honeycomb packing of  $D_2$  with the lowest first coordinates. We label  $z^k$ 's so that  $z_1^j \leq z_1^k$  if  $j < k$ . Then

$$x_1^k \geq z_1^k - (9c_2 + \sqrt{3})\rho \quad \text{for } 2 \leq k \leq N. \quad (5.6)$$

Therefore for any  $2 \leq N_2 \leq N$ ,

$$x_1^2 + \dots + x_1^{N_2} \geq z_1^2 + \dots + z_1^{N_2} - (N_2 - 1)(9c_2 + \sqrt{3})\rho. \quad (5.7)$$

Recall that we assume that there do not exist  $z$  and  $k$  satisfying (5.5).

◇ *Case 2(a)*. Suppose that  $\max_{k=2, \dots, N} x_1^k > \max_{k=2, \dots, N} z_1^k + \delta/2$ . Let  $\mathcal{K}_1 = \{2 \leq k \leq N : x_1^k > \max_{j=2, \dots, N} z_1^j + \delta/4\}$ . Note that  $|\mathcal{K}_1| \geq c_1\delta/(8\pi\rho^2)$  for small  $\rho$  because  $x^k$ 's represent a configuration with density bounded below by  $c_1$ . We have for small  $\rho$ ,

$$\sum_{k \in \mathcal{K}_1} x_1^k - \sum_{k \in \mathcal{K}_1} z_1^k \geq |\mathcal{K}_1| \delta/4 \geq c_1\delta^2/(32\pi\rho^2).$$

Let  $N_2 = N - |\mathcal{K}_1|$ . We apply (5.7) to the discs corresponding to  $1 \leq k \leq N$ ,  $k \notin \mathcal{K}_1$ , to obtain

$$\sum_{2 \leq k \leq N, k \notin \mathcal{K}_1} x_1^k - \sum_{2 \leq k \leq N, k \notin \mathcal{K}_1} z_1^k \geq -(N_2 - 1)(9c_2 + \sqrt{3})\rho.$$

Combining both estimates, we see that, for small  $\rho$ ,

$$\begin{aligned} \sum_{2 \leq k \leq N} x_1^k - \sum_{2 \leq k \leq N} z_1^k &\geq c_1\delta^2/(32\pi\rho^2) - (N_2 - 1)(9c_2 + \sqrt{3})\rho \\ &\geq c_1\delta^2/(32\pi\rho^2) - (\beta/(\rho^2\sqrt{12}) - 1)(9c_2 + \sqrt{3})\rho \\ &\geq c_1\delta^2/(64\pi\rho^2). \end{aligned}$$

Hence, for small  $\rho > 0$ , (5.4) holds with  $y^1 = x^1$ ,  $y^k = z^k$  for  $2 \leq k \leq N$ , and  $\delta_1 = c_1 \delta^2 / (64\pi\rho^2)$ . Note that  $\mathbf{y} \in \overline{\mathbf{D}}$  because  $S_2(z^2), \dots, S_N(z^N)$  is a part of the honeycomb packing of  $D_2$ , so these discs are disjoint and they are disjoint with  $S_1(x^1)$ .

◇ *Case 2(b)*. Next suppose that

$$\max_{k=2, \dots, N} x_1^k \leq \max_{k=2, \dots, N} z_1^k + \delta/2. \quad (5.8)$$

Recall  $\gamma$  from the statement of the theorem. We can assume without loss of generality that  $\gamma \in (0, 1)$ . If  $\rho$  is small then we can find  $y_1^1 \in (x_1^1 + \delta(1 - \gamma)/4, x_1^1 + \delta/4)$  such that the line  $M := \{(u_1, u_2) : u_1 = (x_1^1 + y_1^1)/2\}$  is a line of symmetry for the honeycomb packing  $\mathcal{S}$ . Let  $y^1 = (y_1^1, x_2^1)$ . Note that  $S_1(y^1) \setminus S_1(x^1)$  is “filled” with the discs from the family  $S_2(z^2), \dots, S_N(z^N)$  when  $\rho$  is small because, in view of (5.3) and (5.8),

$$\max_{k=2, \dots, N} z_1^k \geq \max_{k=2, \dots, N} x_1^k - \delta/2 > x_1^1 + 1/2 + \delta - \delta/2 = (y_1^1 + \delta/4 + 1/2) + \delta/4.$$

Let  $\mathcal{M} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the symmetry with respect to  $M$ . Let  $\mathcal{K}_2 = \{2 \leq k \leq N : S_k(z^k) \cap S_1(y^1) \neq \emptyset\}$ . For  $k \in \mathcal{K}_2$ , let  $y^k = \mathcal{M}(z^k)$ . For all other  $k$ , let  $y^k = z^k$ . Since  $\{S_2(z^2), \dots, S_N(z^N)\}$  is a part of the honeycomb packing of  $D_2 = D_1 \setminus S_1(x^1)$ ,  $\{S_2(y^2), \dots, S_N(y^N)\}$  is a part of the honeycomb packing of  $\mathbb{R}^2 \setminus S_1(y^1)$ . Since any disc in  $\mathcal{S}$  that intersects  $S_1(x^1)$  has to be in the positive half space  $\{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 > 0\}$ ,  $\{S_2(y^2), \dots, S_N(y^N)\}$  is in fact a part of the honeycomb packing of  $D_1 \setminus S_1(y^1)$ . Therefore,  $\mathbf{y} \in \overline{\mathbf{D}}$ .

The next part of our argument is best explained using physical intuition. Suppose that all discs  $S_k$  are made of material with mass density 1 and they are in gravitational field with constant acceleration 1 in the negative direction along the first axis. When we move disc  $S_1(x^1)$  to the new position at  $S_1(y^1)$  then we do  $(y_1^1 - x_1^1)\pi/4$  units of work, which is at least  $(\delta(1 - \gamma)/4)\pi/4$ . We can imagine that the mass in  $S_1(x^1) \cap S_1(y^1)$  does not move, and we only move the mass in  $S_1(y^1) \setminus S_1(x^1)$  to its symmetric image  $S_1(x^1) \setminus S_1(y^1)$  under  $\mathcal{M}$ . When  $\rho$  is very small, the discs  $S_k(z^k)$ ,  $k \in \mathcal{K}_2$ , have total mass arbitrarily close to  $(\pi/\sqrt{12})|S_1(y^1) \setminus S_1(x^1)|$ , uniformly spread over  $S_1(y^1) \setminus S_1(x^1)$ . Hence, the amount of work needed to move all discs  $S_k(z^k)$  to  $S_k(y^k)$  for  $k \in \mathcal{K}_2$  is negative and smaller than  $-(\pi/\sqrt{12})(\delta(1 - 2\gamma)/4)\pi/4$ , for small  $\rho$ . In other words,

$$\sum_{k \in \mathcal{K}_2} (y_1^k - z_1^k)\pi\rho^2 \leq -(\pi/\sqrt{12})(\delta(1 - 2\gamma)/4)\pi/4.$$

Recall the assumption that  $\gamma_1 \leq (\pi/(4\sqrt{12}) - \gamma)/\rho^2$ . We have  $y_1^1 \leq x_1^1 + \delta/4$  so

$$\gamma_1(y_1^1 - x_1^1) \leq \gamma_1\delta/4 \leq ((\pi/(4\sqrt{12}) - \gamma)/\rho^2)\delta/4.$$

Combining the last two estimates we obtain

$$\begin{aligned}
 & \gamma_1(y_1^1 - x_1^1) + \sum_{k \in \mathcal{K}_2} (y_1^k - z_1^k) \\
 & \leq -(\pi/\sqrt{12})(\delta(1 - 2\gamma)/4)(\pi/4)/(\pi\rho^2) + ((\pi/(4\sqrt{12}) - \gamma)/\rho^2)\delta/4 \\
 & \leq -\gamma\delta/(12\rho^2).
 \end{aligned} \tag{5.9}$$

Recall that  $N \leq \beta/(\rho^2\sqrt{12})$  and let  $\delta_1 = \gamma\delta/(48\rho^2)$ . We apply (5.7) and (5.9) to see that for small  $\rho > 0$ ,

$$\begin{aligned}
 \gamma_1 x_1^1 + x_1^2 + \cdots + x_1^N & \geq \gamma_1 x_1^1 + z_1^2 + \cdots + z_1^N - (N - 1)(9c_2 + \sqrt{3})\rho \\
 & \geq \gamma_1 x_1^1 + z_1^2 + \cdots + z_1^N - (9c_2 + \sqrt{3})\rho\beta/(\rho^2\sqrt{12}) \\
 & = \gamma_1 x_1^1 + \sum_{k \in \mathcal{K}_2} z_1^k + \sum_{2 \leq k \leq N, k \notin \mathcal{K}_2} z_1^k - \beta(9c_2 + \sqrt{3})/(\rho\sqrt{12}) \\
 & = \gamma_1 x_1^1 + \sum_{k \in \mathcal{K}_2} z_1^k + \sum_{2 \leq k \leq N, k \notin \mathcal{K}_2} y_1^k - \beta(9c_2 + \sqrt{3})/(\rho\sqrt{12}) \\
 & > \gamma_1 y_1^1 + \sum_{k \in \mathcal{K}_2} y_1^k + 2\delta_1 + \sum_{2 \leq k \leq N, k \notin \mathcal{K}_2} y_1^k - \beta(9c_2 + \sqrt{3})/(\rho\sqrt{12}) \\
 & = \gamma_1 y_1^1 + y_1^2 + \cdots + y_1^N + \gamma\delta/(24\rho^2) - \beta(9c_2 + \sqrt{3})/(\rho\sqrt{12}) \\
 & > \gamma_1 y_1^1 + y_1^2 + \cdots + y_1^N + \delta_1.
 \end{aligned}$$

Hence, condition (5.4) is satisfied. This completes the proof of part (i) of the theorem.

(ii) The second part of the theorem can be proved just like the first part. The proof is identical up to (5.8). In the part following (5.8), all we have to do is to take  $y_1^1 \in (x_1^1 - \delta/4, x_1^1 - \delta(1 - \gamma)/4)$  instead of  $y_1^1 \in (x_1^1 + \delta(1 - \gamma)/4, x_1^1 + \delta/4)$ , because in this part we want to move  $S_1(x^1)$  down, not up. We leave the details to the reader.  $\square$

**5.4. Inert objects.** We will model inertia of objects  $S_k$  by changing the rules of reflection. When two different objects  $S_j$  and  $S_k$  reflect from each other, they will no longer receive the same amount of push to keep them apart. One way to formalize this idea is to say that when the process  $\mathbf{X}$  hits the boundary of  $\mathbf{D}$  at the time when  $S_j$  hits  $S_k$ , then  $\mathbf{X}$  is not reflected normally but it is subject to oblique reflection. The drift  $a_k$  will not be assumed to be related to the value of inertia for  $S_k$ . In other words,  $a_k$ 's may model forces which have strength dependent on factors other than the inertial mass.

We will assume that the standard deviation for oscillations of  $S_k$  is inversely proportional to the inertia of  $S_k$ . The reason for this assumption is purely technical. The assumption allows us to transform the problem to the model covered by Theorem 2.3. In general, it is not easy to find an explicit formula for the stationary distribution of reflected Brownian motion with oblique reflection (even if the process has no drift).

Next, we formalize the above ideas. Let  $m_k > 0$  be the parameter representing the inertia for  $S_k$ . Let

$$\begin{aligned}\mathcal{T}(\mathbf{x}) &= (m_1 x^1, \dots, m_N x^N), & \mathbf{x} \in \mathbb{R}^{Nd}, \\ \tilde{\mathbf{D}} &= \mathcal{T}(\mathbf{D}), & \tilde{\mathbf{X}}_t = \mathcal{T}(\mathbf{X})_t.\end{aligned}$$

We assume that  $\tilde{\mathbf{X}}$  is reflected Brownian motion in  $\tilde{\mathbf{D}}$  (with the normal reflection), with drift

$$\tilde{\mathbf{v}} = ((-a_1 m_1, 0, \dots, 0), \dots, (-a_N m_N, 0, \dots, 0)).$$

Let  $\tilde{\mu}$  denote the stationary distribution for  $\tilde{\mathbf{X}}$  and let  $\mu'$  be the corresponding stationary distribution for  $\mathbf{X}$ . Note that under  $\mu'$ , the quadratic variation process for  $X^k$  is  $t/m_k^2$ .

In the present example, if two objects  $S_j$  and  $S_k$  reflect from each other then the infinitesimal displacement of  $S_j$  is  $m_k/m_j$  times the infinitesimal displacement of  $S_k$ .

We will illustrate the effect of inertia using the same model as in the Section 5.3.

**Theorem 5.4.** *Suppose that  $d = 2$ ,  $D$  is as in Example 3.1(ii),  $S_1 = B(0, 1/2)$ , and  $S_k = B(0, \rho)$  for  $k = 2, \dots, N$ , where  $\rho < 1/2$ . Assume that  $a_2 = a_3 = \dots = a_N$ ,  $m_1 > 1$ ,  $m_2 = \dots = m_N = 1$ , and  $\mathbf{X}$  has distribution  $\mu'$ .*

(i) *For any  $p, \delta, \gamma > 0$ , there exists  $\rho_0 > 0$  and  $N_0 < \infty$  such that for  $\rho \in (0, \rho_0)$  and  $N \geq N_0$  which satisfy the condition  $\rho^2 N \sqrt{12} > 2 - \pi/4$  there exists  $a_0 > 0$  such that if  $a_2 \geq a_0$  and  $a_1/a_2 := \gamma_1 \leq (\pi/(4\sqrt{12}) - \gamma)/(\rho^2 m_1)$  then*

$$\mu' \left( X_1^1 < \max_{k=2, \dots, N} X_1^k - 1/2 - \delta \right) < p.$$

(ii) *For any  $p, \delta, \gamma > 0$ , there exists  $\rho_0 > 0$  and  $N_0 < \infty$  such that for  $\rho \in (0, \rho_0)$  and  $N \geq N_0$  which satisfy the condition  $\rho^2 N \sqrt{12} > 2 - \pi/4$  there exists  $a_0 > 0$  such that if  $a_2 \geq a_0$  and  $a_1/a_2 := \gamma_2 \geq (\pi/(4\sqrt{12}) + \gamma)/(\rho^2 m_1)$  then*

$$\mu' (X_1^1 > 1/2 + \delta) < p.$$

The theorem says that the higher is inertia  $m_1$ , the lower is the critical drift  $a_1$  that makes the disc  $S_1$  sink. We note parenthetically that behavior of real particulate matter can be paradoxical, unlike in our example. Large and heavy particles may move to the top of a mixture of small and large particles under some circumstances; see [24].

*Proof.* We can use the same reasoning as in the proof of Theorem 5.3 but with a twist. Theorem 5.1 (i) must be applied to the process  $\tilde{\mathbf{X}}$  under  $\tilde{\mu}$ , so we have to analyze  $m_1 a_1 x_1^1 + a_2 x_1^2 + \dots + a_N x_1^N$  rather than  $a_1 x_1^1 + a_2 x_1^2 + \dots + a_N x_1^N$ . Hence,  $\gamma_1$  in (5.4) must have an extra factor of  $1/m_1$ . The constant  $\gamma_1$  in the present theorem has that extra factor, as compared to the constant  $\gamma_1$  in Theorem 5.3. With this change, the proof of Theorem 5.3 applies in the present context.  $\square$

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