# Dirichlet Heat Kernel Estimates for $\Delta^{\alpha/2} + \Delta^{\beta/2}$

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**Abstract**

For $d \geq 1$ and $0 < \beta < \alpha < 2$, consider a family of pseudo differential operators $\{\Delta^\alpha + a^\beta \Delta^{\beta/2};\ a \in [0,1]\}$ on $\mathbb{R}^d$ that evolves continuously from $\Delta^{\alpha/2}$ to $\Delta^{\alpha/2} + \Delta^{\beta/2}$. It gives arise to a family of Lévy processes $\{X^a, a \in [0,1]\}$ on $\mathbb{R}^d$, where each $X^a$ is the independent sum of a symmetric $\alpha$-stable process and a symmetric $\beta$-stable process with weight $a$. For any $C^{1,1}$ open set $D \subset \mathbb{R}^d$, we establish explicit sharp two-sided estimates (uniform in $a \in [0,1]$) for the transition density function of the subprocess $X^{a,D}$ of $X^a$ killed upon leaving the open set $D$. The infinitesimal generator of $X^{a,D}$ is the non-local operator $\Delta^\alpha + a^\beta \Delta^{\beta/2}$ with zero exterior condition on $D^c$. As consequences of these sharp heat kernel estimates, we obtain uniform sharp Green function estimates for $X^{a,D}$ and uniform boundary Harnack principle for $X^a$ in $D$ with explicit decay rate.

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## 1 Introduction

It is well-known that, for a second order elliptic differential operator $\mathcal{L}$ on $\mathbb{R}^d$ satisfying some natural conditions, there is a diffusion process $X$ on $\mathbb{R}^d$ with $\mathcal{L}$ as its infinitesimal generator. The fundamental solution $p(t,x,y)$ of $\partial_t u = \mathcal{L}u$ (also called the heat kernel of $\mathcal{L}$) is the transition density function of $X$. Thus obtaining sharp two-sided estimates for $p(t,x,y)$ is a fundamental problem in both analysis and probability theory. Such relationship is also true for a large class of Markov processes with discontinuous sample paths, which constitute an important family of stochastic processes in probability theory. They have been widely used in various applications.

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One of the most important and most widely used family of Markov processes is the family of (rotationally) symmetric $\alpha$-stable processes on $\mathbb{R}^d$, $0<\alpha\le 2$. A symmetric $\alpha$-stable process $X=\{X_t, t\ge 0, \mathbb{P}_x, x\in\mathbb{R}^d\}$ on $\mathbb{R}^d$ is a Lévy process such that

$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t-X_0)}\right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x\in\mathbb{R}^d \text{ and } \xi\in\mathbb{R}^d.$$

When $\alpha=2$, $X$ is a Brownian motion on $\mathbb{R}^d$ whose infinitesimal generator is the Laplacian $\Delta$. When $0<\alpha<2$, the infinitesimal generator of a symmetric $\alpha$-stable process $X$ in $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2}u(x) = \mathcal{A}(d,-\alpha)\lim_{\varepsilon\downarrow 0}\int_{\{y\in\mathbb{R}^d:\,|y-x|>\varepsilon\}}(u(y)-u(x))\frac{dy}{|x-y|^{d+\alpha}} \tag{1.1}$$

for some constant $\mathcal{A}(d,-\alpha):=\alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Here and in the sequel, we use $:=$ as a way of definition. Here $\Gamma$ is the Gamma function defined by $\Gamma(\lambda):=\int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda>0$.

Two-sided heat kernel estimates for diffusions in $\mathbb{R}^d$ have a long history and many beautiful results have been established. See [11, 13] and the references therein. But, due to the complication near the boundary, two-sided estimates for the transition density functions of killed diffusions in a domain $D$ (equivalently, the Dirichlet heat kernels) have been established only recently. See [12, 13, 14] for upper bound estimates and [29] for the lower bound estimates of the Dirichlet heat kernels in bounded $C^{1,1}$ domains. In a recent paper [3], we succeeded in establishing sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ with zero exterior condition on $D^c$ (or equivalently, the transition density function of the killed $\alpha$-stable process) in any $C^{1,1}$ open set.

The approach developed in [3] can be adapted to establish heat kernel estimates of other jump processes in open subsets of $\mathbb{R}^d$. In [4], the ideas of [3] were adapted to establish two-sided heat kernel estimates of censored stable processes in $C^{1,1}$ open subsets of $\mathbb{R}^d$. One of the main tools used in [4] is the boundary Harnack principle established in [2] and [17].

In [5] the ideas of [3] were adapted to establish two-sided heat kernel estimates of relativistic stable processes in $C^{1,1}$ open subsets of $\mathbb{R}^d$. One of main facts we used in [5] is that relativistic stable processes can be regarded as perturbations of symmetric stable processes in bounded open sets and therefore the Green functions of killed relativistic stable processes in bounded open sets are comparable to the Green functions of killed stable processes in the same open sets.

The goal of this paper is to prove sharp two-sided estimates for the independent sum of an $\alpha$-stable process and a $\beta$-stable process, $0<\beta<\alpha<2$, in $C^{1,1}$ open subsets of $\mathbb{R}^d$. Note that these processes can not be obtained from symmetric stable processes through a combination of Girsanov transform and Feynman-Kac transform. So the method of [5] can not be used to establish the comparability of the Green functions of these processes and the Green functions of symmetric stable processes in bounded open sets. Since the differences of the Lévy measures of these processes and those of symmetric stable processes have infinite total mass, the method of [22] and [16] also could not be used to establish the comparability of the Green functions of these processes and the Green functions of symmetric stable processes in bounded open sets. The approach of this paper will be described in the second paragraph below after the statement of Corollary 1.2.

Let us first recall some basic facts about the independent sum of stable processes and state our main result.

Throughout the remainder of this paper, we assume that $d \geq 1$ and $0 < \beta < \alpha < 2$. The Euclidean distance between $x$ and $y$ will be denoted as $|x-y|$. We will use $B(x,r)$ to denote the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$

Suppose $X$ is a symmetric $\alpha$-stable process and $Y$ is a symmetric $\beta$-stable process on $\mathbb{R}^d$ and that $X$ and $Y$ are independent. For any $a \geq 0$, we define $X^a$ by $X_t^a := X_t + aY_t$. We will call the process $X^a$ the independent sum of the symmetric $\alpha$-stable process $X$ and the symmetric $\beta$-stable process $Y$ with weight $a$. The infinitesimal generator of $X^a$ is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Let $p^a(t,x,y)$ denote the transition density of $X^a$ (or equivalently the heat kernel of $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$) with respect to the Lebesgue measure on $\mathbb{R}^d$. We will use $p(t,x,y) = p^0(t,x,y)$ to denote the transition density of $X = X^0$ . Recently it is proven in [8] that

$$p^1(t,x,y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}}\right) \quad \text{on } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{1.2}$$

Here and in the sequel, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$; for any two positive functions $f$ and $g$, $f \asymp g$ means that there is a positive constant $c \geq 1$ so that $c^{-1} g \leq f \leq c\, g$ on their common domain of definition.

For every open subset $D \subset \mathbb{R}^d$, we denote by $X^{a,D}$ the subprocess of $X^a$ killed upon leaving $D$. The infinitesimal generator of $X^{a,D}$ is $(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2})|_D$, the sum of two fractional Laplacians in $D$ with zero exterior condition. It is known (see [8]) that $X^{a,D}$ has a Hölder continuous transition density $p_D^a(t,x,y)$ with respect to the Lebesgue measure.

Unlike the case of the symmetric $\alpha$-stable process $X := X^0$, $X^a$ does not have the stable scaling for $a > 0$. Instead, the following approximate scaling property is true and will be used several times in the rest of this paper: If $\{X_t^{a,D}, t \geq 0\}$ is the subprocess of $X^a$ killed upon leaving $D$, then $\{\lambda^{-1} X_{\lambda^\alpha t}^{a,D}, t \geq 0\}$ is the subprocess of $\{X_t^{a\lambda^{(\alpha-\beta)/\beta}}, t \geq 0\}$ killed upon leaving $\lambda^{-1}D$. So for any $\lambda > 0$, we have

$$p_{\lambda^{-1}D}^{a\lambda^{(\alpha-\beta)/\beta}}(t,x,y) = \lambda^d p_D^a(\lambda^\alpha t, \lambda x, \lambda y) \qquad \text{for } t > 0 \text{ and } x, y \in \lambda^{-1}D. \tag{1.3}$$

In particular, letting $a = 1$, $\lambda = a^{\beta/(\alpha-\beta)}$ and $D = \mathbb{R}^d$, we get

$$p^a(t,x,y) = a^{\frac{\beta d}{\alpha-\beta}} p^1(a^{\frac{\alpha\beta}{\alpha-\beta}} t, a^{\frac{\beta}{\alpha-\beta}} x, a^{\frac{\beta}{\alpha-\beta}} y) \qquad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d.$$

So we deduce from (1.2) that for any $M > 0$ there exists a constants $C > 1$ depending only on $d, \alpha, \beta$ and $M$ such that for any $a \in (0, M]$ and $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$C^{-1} f^a(t,x,y) \leq p^a(t,x,y) \leq C f^a(t,x,y), \tag{1.4}$$

where

$$f^a(t,x,y) := \left((a^\beta t)^{-d/\beta} \wedge t^{-d/\alpha}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a^\beta t}{|x-y|^{d+\beta}}\right).$$

The purpose of this paper is to establish the following two-sided sharp estimates on $p_D^a(t,x,y)$ in Theorem 1.1 for every $t > 0$. To state this theorem, we first recall that an open set $D$ in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a (uniform) $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$

and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \le \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda_0|x - z|$, and an orthonormal coordinate system $CS_z$ with its origin at $z$ such that

$$B(z, R_0) \cap D = \{y = (\widetilde{y}, y_d) \text{ in } CS_z : |y| < R_0, y_d > \phi(\widetilde{y})\}.$$

The pair $(R_0, \Lambda_0)$ is called the characteristics of the $C^{1,1}$ open set $D$. Note that a $C^{1,1}$ open set $D$ with characteristics $(R_0, \Lambda_0)$ can be unbounded and disconnected; the distance between two distinct components of $D$ is at least $R_0$. It is well known that any $C^{1,1}$ open set $D$ satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 < R_0$ such that for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \overline{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \overline{D}$ for $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is a positive constant $R_0$.

**Theorem 1.1** *Suppose $M > 0$. Let $D$ be a $C^{1,1}$ open subset of $\mathbb{R}^d$ and $\delta_D(x)$ the Euclidean distance between $x$ and $D^c$.*

(i) *For every $T > 0$, there is a positive constant $C_1 = C_1(D, M, \alpha, \beta, T) \ge 1$ such that for every $a \in (0, M]$,*

$$C_1^{-1}\, f_D^a(t, x, y) \le p_D^a(t, x, y) \le C_1\, f_D^a(t, x, y),$$

*where*

$$f_D^a(t, x, y) := \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a^\beta t}{|x-y|^{d+\beta}}\right)\right).$$

(ii) *Suppose in addition that $D$ is bounded. For every $T > 0$, there is a constant $C_2 = C_2(D, M, \alpha, \beta, T) \ge 1$ so that for every $a \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,*

$$C_2^{-1}\, e^{-\lambda_1 t}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2} \;\le\; p_D^a(t, x, y) \;\le\; C_2\, e^{-\lambda_1 t}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2},$$

*where $\lambda_1 > 0$ is the smallest eigenvalue of $-(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2})|_D$.*

Letting $a \to 0$, Theorem 1.1 recovers the heat kernel estimates for symmetric $\alpha$-stable processes obtained in [3]. By integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to $t$, we obtain the following estimates on the Green function $G_D^a(x, y) := \int_0^\infty p_D^a(t, x, y)dt$, which mean that, for bounded $C^{1,1}$ domains $D$, $G_D^a$ and $G_D^0$ are comparable, see [9] and [20]. To the best of our knowledge, the Green function estimates in the corollary below are new.

**Corollary 1.2** *Suppose $M > 0$. For any bounded $C^{1,1}$-open set $D$ in $\mathbb{R}^d$, there is a constant $C_3 = C_3(D, M, \alpha, \beta) \ge 1$ so that for every $a \in (0, M]$,*

$$C_3^{-1}\, g_D(x, y) \le G_D^a(x, y) \le C_3\, g_D(X, y) \qquad \text{for } x, y \in D,$$

*where*

$$g_D(x,y) := \begin{cases} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) |x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log\left(1 + \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d = 1 = \alpha, \\ \left(\delta_D(x)\delta_D(y)\right)^{(\alpha-1)/2} \wedge \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha. \end{cases} \tag{1.5}$$

Theorem 1.1(i) will be established through Theorems 2.8 and 3.5, which give the upper bound and lower bound estimates, respectively. Theorem 1.1(ii) is a consequence of the intrinsic ultracontractivity of $X^a$ in a bounded open set and the continuity of eigenvalues proved in [10]. In fact, the upper bound estimates in both Theorem 1.1 and Corollary 1.2 hold for any open set $D$ with a weak version of the *uniform exterior ball condition* in place of the $C^{1,1}$ condition, while the lower bound estimates in both Theorem 1.1 and Corollary 1.2 hold for any open set $D$ with the *uniform interior ball condition* in place of the $C^{1,1}$ condition (see Theorems 2.8 and 3.5, and the proofs for Theorem 1.1(ii) and Corollary 1.2).

Although we follow the main ideas we developed in [3], there are several new difficulties in obtaining two-sided Dirichlet heat kernel estimates for $X^a$: Even though the boundary Harnack principle has been extended in [19] to a large class of pure jump Lévy processes including $X^a$, the explicit decay rate of harmonic functions of $X^a$ near the boundary of $D$ was unknown. Instead, following the approach in [6], we establish necessary estimates using suitably chosen subharmonic and superharmonic functions of the process $X^a$. As in [6], we need to use finite range (or truncated) symmetric $\beta$-stable process $\widehat{Y}^\lambda$ obtained from $Y$ by suppressing all its jumps of size larger than $\lambda$. The infinitesimal generator of $\widehat{Y}^\lambda$ is

$$\widehat{\Delta}_\lambda^{\beta/2} u(x) := \mathcal{A}(d,-\alpha) \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d :\, \varepsilon < |y-x| \le \lambda\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\beta}}. \tag{1.6}$$

When $\lambda = 1$, we will simply denote $\widehat{\Delta}_\lambda^{\beta/2}$ by $\widehat{\Delta}^{\beta/2}$. We first establish the desired estimates for the Lévy process $\widehat{X}^a := X + a\widehat{Y}^{1/a}$. The infinitesimal generator of $\widehat{X}^a$ is $\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}$. The desired estimates for $X^a = X + aY$ can then be obtained by adding back those jumps of $Y$ of size larger than $1/a$. To obtain the lower bound of $p^a(t,x,y)$, we use the Dirichlet heat kernel estimate for the fractional Laplacian in [3] and a comparison of the killed subordinate stable process with the subordinate killed stable process where we will use some of the results obtained in [26].

We like to point out that unlike [3] the boundary Harnack principle for $X^a$ is not used in this paper, which indicates that it might be possible to obtain sharp heat kernel estimate for processes for which the boundary Harnack principle fails.

As a consequence of Corollary 1.2, we have the following uniform boundary Harnack principle with explicit decay rate.

**Theorem 1.3** *Suppose that $M > 0$. For any $C^{1,1}$ open set $D$ in $\mathbb{R}^d$ with the characteristics $(R_0, \Lambda)$, there exists a positive constant $C_4 = C_4(\alpha, \beta, d, \Lambda, R_0, M) \ge 1$ such that for $a \in [0, M]$, $r \in (0, R_0]$, $Q \in \partial D$ and any nonnegative function $u$ in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q,r)$ with respect to $X^a$ and vanishes continuously on $D^c \cap B(Q,r)$, we have*

$$\frac{u(x)}{u(y)} \le C_4 \frac{\delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(y)} \qquad \text{for every } x, y \in D \cap B(Q, r/2). \tag{1.7}$$

Throughout this paper, we will use capital letters $C_1, C_2, \dots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \dots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the constants on dimension $d$ may not be mentioned explicitly. For every function $f$, let $f^+ := f \vee 0$. We will use $\partial$ to denote a cemetery point and for every function $f$, we extend its definition to $\partial$ by setting $f(\partial) = 0$. We will use $dx$ to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

## 2 Upper bound estimate

Throughout this section we assume that $D$ is an open set satisfying the uniform exterior ball condition with radius $r_0 > 0$ in the following sense: for every $z \in \partial D$ and $r \in (0, r_0)$, there is a ball $B^z$ of radius $r$ such that $B^z \subset \mathbb{R}^d \setminus \overline{D}$ and $\partial B^z \cap \partial D = \{z\}$. The goal of this section is to establish the upper bound for the transition density (heat kernel) $p^a_D(t, x, y)$. One of the main difficulties of getting the upper bound for $p^a_D(t, x, y)$ is to obtaining the correct boundary decay rate.

Recall that $\Delta^{\alpha/2}$ and $\widehat{\Delta}^{\beta/2}_\lambda$ are defined by (1.1) and (1.6). The next two lemmas can be proved by direct computation, whose proofs can be found in [17] and [6], respectively.

For $p > 0$, let $w_p(x) := (x_1^+)^p$.

**Lemma 2.1** *For any $x \in (0, \infty) \times \mathbb{R}^{d-1}$, we have*

$$\Delta^{\alpha/2} w_{\alpha/2}(x) \ = 0. \tag{2.1}$$

*Moreover, for every $p \in (\alpha/2, \alpha)$, there is a positive constant $C_5 = C_5(d, \alpha, p)$ such that for every $x \in (0, \infty) \times \mathbb{R}^{d-1}$*

$$\Delta^{\alpha/2} w_p(x) = C_5 \, x_1^{p-\alpha}. \tag{2.2}$$

**Lemma 2.2** *There are constants $R_* \in (0, 1)$, $C_6 > C_7 > 0$ depending on $p$, $d$ and $\alpha$ only such that for every $x \in (0, R_*] \times \mathbb{R}^{d-1}$*

$$C_7 x_1^{p-\alpha} \le \widehat{\Delta}^{\alpha/2} w_p(x) \le C_6 x_1^{p-\alpha} \qquad \text{for } \alpha/2 < p < \alpha, \tag{2.3}$$

$$|\widehat{\Delta}^{\alpha/2} w_p(x)| \le C_6 \, |\log x_1| \qquad \text{for } p = \alpha \tag{2.4}$$

*and*

$$|\widehat{\Delta}^{\alpha/2} w_p(x)| \le C_6 \qquad \text{for } p > \alpha. \tag{2.5}$$

In the remainder of this paper, $R_*$ will always stand for the constant in Lemma 2.2. The following result and its proof are similar to Lemma 3.2 of [6] and the proof there. For reader's convenience, we spell out the details of the proof here.

**Lemma 2.3** *Assume that $r_1 \in (0, 1/2]$ and $p \ge \frac{\alpha}{2}$. Let $\delta_1 := R_* \wedge (r_1/4)$, $U := \left\{ z \in \mathbb{R}^d : \, r_1 < |z| < 3r_1/2 \right\}$ and*

$$h_p(y) := \left( y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right)^p \mathbf{1}_{U \cap \{z_d > 0, |\tilde{z}| < r_1/2\}}(y).$$

*Then there exist $C_i = C_i(\alpha, p, r_1) > 0$, $i = 8, \cdots, 12$, such that*

(i) *when* $p \in (\alpha/2, \alpha)$, *we have for all* $y \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\}$,

$$C_8 \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right)^{p-\alpha} \le \widehat{\Delta}^{\alpha/2} h_p(y) \le C_9 \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right)^{p-\alpha} \tag{2.6}$$

*and*

$$C_8 \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right)^{p-\alpha} \le \Delta^{\alpha/2} h_p(y) \le C_9 \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right)^{p-\alpha}; \tag{2.7}$$

(ii) *when* $p > \alpha$, *we have*

$$|\widehat{\Delta}^{\alpha/2} h_p(y)| \le C_{10} \quad \text{for all } y \in \left\{ z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4 \right\}; \tag{2.8}$$

(iii) *when* $p = \alpha/2$, *we have*

$$|\Delta^{\alpha/2} h_{\alpha/2}(y)| \le C_{11} \quad \text{for all } y \in \left\{ z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta, |\widetilde{z}| < r_1/4 \right\}; \tag{2.9}$$

(iv) *when* $p = \alpha$, *we have for every* $y \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta, |\widetilde{z}| < r_1/4\}$,

$$|\widehat{\Delta}^{\alpha/2} h_{\alpha/2}(y)| \le C_{12} \left| \log \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right) \right|. \tag{2.10}$$

**Proof.** Let

$$\Gamma(\tilde{y}) := \sqrt{r_1^2 - |\tilde{y}|^2} \quad \text{and} \quad \underline{h}(y) := y_d - \Gamma(\tilde{y}), \quad y \in U.$$

Fix $x \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + R_* \wedge (r_1/8), |\widetilde{z}| < r_1/4\}$ and choose a point $x_0 \in \partial B_+(0, r_1) := \{z_d > 0, |z| = r_1\}$ satisfying $\widetilde{x} = \widetilde{x_0}$. Denote by $\overrightarrow{n}(x_0)$ the inward unit normal vector at $x_0$ for the exterior ball $B(0, r_1)^c$ and set $\Phi(y) = \langle y - x_0, \overrightarrow{n}(x_0)\rangle$ for $y \in \mathbb{R}^d$. $\Pi = \{y : \Phi(y) = 0\}$ is the plane tangent to $\partial B_+(0, r_1)$ at the point $x_0$. Let $\Gamma^* : \widetilde{x} \in \mathbb{R}^{d-1} \to \mathbb{R}$ be the function describing the hyperplane $\Pi$, that is, $\langle (\widetilde{x}, \Gamma^*(\widetilde{x})) - x_0, \overrightarrow{n}(x_0)\rangle = 0$. We also let

$$\begin{aligned} E &:= \{y = (\widetilde{y}, y_d) : y \in U, \ \ |y - x| < r_1/4\}, \\ A &:= \{y : \Gamma^*(\widetilde{y}) > y_d > \Gamma(\widetilde{y}),\ |y - x| < r_1/4\} \end{aligned}$$

and $\overline{h}(y) := (y_d - \Gamma^*(\widetilde{y}))\mathbf{1}_{\{y_d > \Gamma^*(\widetilde{y})\}}(y)$ for $y \in \mathbb{R}^d$. Since $\nabla\Gamma(\widetilde{x}) - \nabla\Gamma^*(\widetilde{x}) = 0$, by the mean value theorem

$$|\overline{h}(y) - \underline{h}(y)| \le |\Gamma(\widetilde{y}) - \Gamma^*(\widetilde{y})| \le \Lambda |\widetilde{y} - \widetilde{x}|^2, \qquad y \in E. \tag{2.11}$$

Let $\delta_\Pi(y) = \mathrm{dist}(y, \Pi)$ for $y \in \mathbb{R}^d$ and $U_{\Gamma^*} = \{y \in \mathbb{R}^d : y_d > \Gamma^*(\widetilde{y})\}$. Let $b_x := \sqrt{1 + |\nabla\Gamma(\widetilde{x})|^2}$ and

$$h_{x,p}(y) := (\overline{h}(y))^p.$$

Note that $h_{x,p}(x) = h_p(x)$ and $B(x, r_1/4) \cap U \subset E$. Since $\overline{h}(y) = b_x \delta_\Pi(y)$ on $D_{\Gamma^*}$, by Lemma 2.1,

$$\Delta^{\alpha/2} h_{x,\alpha/2}(x) \ = \ 0 \tag{2.12}$$

and, if $\alpha/2 < p < \alpha$,

$$\Delta^{\alpha/2} h_{x,p}(x) \;=\; c_1\, b_x^p \delta_{\Pi}^{p-\alpha}(x) \;=\; c_1\, b_x^{\alpha} (\underline{h}(x))^{p-\alpha} \tag{2.13}$$

for some $c_1 > 0$. By Lemma 2.2, there are constants $c_i > 0$, $i = 2\dots 6$, such that for $y \in D_{\Gamma^*}$ and $\delta_{\Pi}(y) < R_*$, when $\alpha/2 < p < \alpha$,

$$c_2(\underline{h}(x))^{p-\alpha} \le c_3\, b_x^p \delta_{\Pi}^{p-\alpha}(x) \le \widehat{\Delta}^{\alpha/2} h_{x,p}(x) \;=\; b_x^p \widehat{\Delta}^{\alpha/2}(\delta_{\Pi}(x))^p \;\le\; c_4\, b_x^p \delta_{\Pi}^{p-\alpha}(x) \le c_5(\underline{h}(x))^{p-\alpha}, \tag{2.14}$$

when $p > \alpha$,

$$|\widehat{\Delta}^{\alpha/2} h_{x,p}(x)| \;=\; b_x^p |\widehat{\Delta}^{\alpha/2}(\delta_{\Pi}(x))^p| \;\le\; c_6 \tag{2.15}$$

and when $p = \alpha$,

$$|\widehat{\Delta}^{\alpha/2} h_{x,p}(x)| \le c_6\, |\log(\underline{h}(x))|. \tag{2.16}$$

Note that

$$\begin{aligned} |\widehat{\Delta}^{\alpha/2}(h_p - h_{x,p})(x)| &= \mathcal{A}(d,-\alpha)\left| \lim_{\varepsilon\downarrow 0} \int_{\{1\ge|y-x|>\varepsilon\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}}\, dy \right| \\ &\le\; \mathcal{A}(d,-\alpha)\left| \int_{\{1\ge|y-x|>r_1/4\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}}\, dy \right| \qquad (2.17)\\ &\quad + \mathcal{A}(d,-\alpha) \lim_{\varepsilon\downarrow 0} \int_{\{r_1/4\ge|y-x|>\varepsilon\}} \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}}\, dy \\ &\le\; c_7 + \mathcal{A}(d,-\alpha) \int_A \frac{h_p(y) + h_{p,x}(y)}{|x-y|^{d+\alpha}}\, dy + \mathcal{A}(d,-\alpha) \int_E \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} \\ &=:\; c_7 + I_1 + I_2 \end{aligned} \tag{2.18}$$

and, similarly,

$$\begin{aligned} &|\Delta^{\alpha/2}(h_p - h_{x,p})(x)| \\ &\le \mathcal{A}(d,-\alpha)\left| \int_{\{|y-x|>r_1/4\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}}\, dy \right| \\ &\quad + \mathcal{A}(d,-\alpha) \int_A \frac{h_p(y) + h_{p,x}(y)}{|x-y|^{d+\alpha}}\, dy + \mathcal{A}(d,-\alpha) \int_E \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} =: I_3 + I_1 + I_2. \end{aligned} \tag{2.19}$$

Since for $y \in B(x, r_1/4)^c$,

$$|h_{x,p}(y) - h_{x,p}(x)| \le c_8 |y-x|^p \quad \text{and} \quad |h_p(y)| \le c_8$$

and $h_p(y) = 0$ for $|\widetilde{y}| > r_1/2$, for $\alpha/2 \le p < \alpha$ we get

$$\begin{aligned} I_3 \le& \mathcal{A}(d,-\alpha) \int_{B(x,r_1/4)^c} \frac{|h_{x,p}(y) - h_{x,p}(x)|}{|x-y|^{d+\alpha}}\, dy + \mathcal{A}(d,-\alpha) \int_{B(x,r_1/4)^c \cap \{|\widetilde{y}|\le r_1/2\}} \frac{|h_p(y) - h_p(x)|}{|x-y|^{d+\alpha}} dy \\ &+ \mathcal{A}(d,-\alpha)\left| \int_{B(x,r_1/4)^c \cap \{|\widetilde{y}|>r_1/2\}} \frac{h_p(x)}{|x-y|^{d+\alpha}}\, dy \right| \\ \le& c_9 \int_{B(x,r_1/4)^c} \frac{1}{|x-y|^{d+\alpha-p}} dy + c_9 \int_{B(x,r_1/4)^c} \frac{1}{|x-y|^{d+\alpha}} dy \le c_{10} < \infty. \end{aligned} \tag{2.20}$$

We claim that, if $p \ge \alpha/2$,

$$I_1 \,+\, I_2 \,\le\, c_{11} \,<\, \infty. \tag{2.21}$$

Note that for $y \in A$

$$\begin{aligned} |h_{x,p}(y)| + |h_p(y)| &\le |y_d - \Gamma^*(\widetilde{y})|^p + |y_d - \Gamma(\widetilde{y})|^p \le 2|\Gamma(\widetilde{y}) - \Gamma^*(\widetilde{y})|^p \\ &\le 2|\Gamma(\widetilde{y}) - \Gamma(\widetilde{x}) - \nabla\Gamma(\widetilde{x}) \cdot (\widetilde{y} - \widetilde{x})|^p \le 2c_{12}^p |\widetilde{y} - \widetilde{x}|^{2p}. \end{aligned} \tag{2.22}$$

Furthermore, since $|\Gamma(\widetilde{y}) - \Gamma^*(\widetilde{y})| \le c_{13}|\widetilde{y} - \widetilde{x}|^2 \le c_{12}r^2$ on $|y - x| = r$, this together with (2.22) yields that

$$\begin{aligned} I_1 &\le c_{14} \int_0^{r_1/4} r^{2p-\alpha-d} \int_{|y-x|=r} \mathbf{1}_A(y)\; m_{d-1}(dy) dr \\ &= c_{14} \int_0^{r_1/4} r^{2p-\alpha-d} m_{d-1}\big(\{y:\, |y-x| = r,\; \Gamma^*(\widetilde{y}) > y_d > \Gamma(\widetilde{y})\}\big) dr \\ &\le c_{15} \int_0^{r_1/4} r^{2p-\alpha} dr \,<\, \infty. \end{aligned}$$

Note that for $y \in E$

$$|h_p(y) - h_{x,p}(y)| \le c_{16} |(\overline{h}(y))^p - (\underline{h}(y))^p| \le c_{17} (\overline{h}(y))^{(p-1)_-} |\overline{h}(y) - \underline{h}(y)|, \tag{2.23}$$

where $(p-1)_- := (p-1) \wedge 0$. In the last inequality above, we have used the inequalities

$$|b^p - a^p| \le b^{p-1}|b - a| \qquad \text{for } a, b > 0,\; 0 < p \le 1$$

and

$$|b^p - a^p| \le (p+1)|b - a| \qquad \text{for } a, b \in (0,1),\; p > 1.$$

For $y = (\widetilde{y}, y_d) \in \mathbb{R}^d$, we use an affine coordinate system $z = (\widetilde{z}, z_d)$ to represent it so that $z_d = y_d - \Gamma^*(\widetilde{y})$ and $\widetilde{z}$ is the coordinates in an orthogonal coordinate system centered at $x_0$ for the $(d-1)$-dimensional hyperplane $\Pi$ for the point $(\widetilde{y}, \Gamma^*(\widetilde{y}))$. Denote such an affine transformation $y \mapsto z$ by $z = \Psi(y)$. It is clear that there is a constant $c_{18} > 1$ so that for every $y \in \mathbb{R}^d$,

$$c_{18}^{-1}|\widetilde{y} - \widetilde{x}| \le |\widetilde{z}| \le c_{18}|\widetilde{y} - \widetilde{x}|, \qquad c_{18}^{-1}|y - x| \le |\Psi(y) - \Psi(x)| \le c_{18}|y - x|$$

and that

$$\Psi(E) \subset \{z = (\widetilde{z}, z_d) \in \mathbb{R}^d :\; |\widetilde{z}| < c_{18} r_1 \text{ and } 0 < z_d \le c_{18} r_1\}.$$

Denote $x_d - \Gamma^*(\widetilde{x})$ by $w$; that is, $\Psi(x) = (\widetilde{0}, w)$. Hence by (2.11) and (2.23) and applying the transform $\Psi$, we have by using polar coordinates for $\widetilde{z}$ on the hyperplane $\Pi$,

$$\begin{aligned} I_2 &\le c_{19} \int_E \frac{\overline{h}(y)^{(p-1)_-}\, |\widetilde{y} - \widetilde{x}|^2}{|y - x|^{d+\alpha}}\, dy \le c_{19} \int_{\Psi(E)} \frac{z_d^{(p-1)_-}\, |\widetilde{z}|^2}{|z - (\widetilde{0}, w)|^{d+\alpha}}\, dz \\ &\le c_{20} \int_0^{c_{18}r_1} z_d^{(p-1)_-} \left( \int_0^{c_{18}r_1} \frac{r^{d-2}}{(r + |z_d - w|)^{d+\alpha-2}} dr \right) dz_d \\ &\le c_{20} \int_0^{c_{18}r_1} z_d^{(p-1)_-} \left( \int_0^{c_{18}r_1} \frac{1}{(r + |z_d - w|)^{\alpha}} dr \right) dz_d \\ &\le c_{21} \int_0^{c_{18}r_1} z_d^{(p-1)_-} \left( \frac{1}{|z_d - w|^{\alpha-1}} - \frac{1}{(c_{18}r_1 + |z_d - w|)^{\alpha-1}} \right) dz_d \\ &< c_{22} \int_0^{c_{18}r_1} \frac{1}{z_d^{(1-p)^+}\, |z_d - w|^{\alpha-1}} dz_d \le c_{23} < \infty, \end{aligned}$$

where all the constants depend on $\alpha$, $p$ and $r_1$. The last inequality is due to the fact that since $p>0$, $0<\alpha<2$ and $(1-p)^+ + \alpha - 1 = \max\{\alpha - p, \alpha - 1\} < 1$, by the dominated convergence theorem, $\phi(w) := \int_0^{c_{18}r_1} \frac{1}{z_d^{(1-p)^+}\,|z_d-w|^{\alpha-1}} dz_d$ is a strictly positive continuous function in $x_d \in [0, c_{18}r_1]$ and hence is bounded.

Thus we have proved the claim (2.21). The desired estimates (2.6)-(2.10) now follow from (2.12)-(2.21). □

It is well-known that $X^1$ has Lévy intensity

$$J^1(x,y) = j^1(|x-y|) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} + \frac{\mathcal{A}(d,-\beta)}{|x-y|^{d+\beta}}.$$

A scaling argument yields that

$$J^a(x,y) = j^a(|x-y|) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} + a^\beta \frac{\mathcal{A}(d,-\beta)}{|x-y|^{d+\beta}}.$$

Put

$$\psi^a(r) := 1 + a^\beta \frac{\mathcal{A}(d,-\beta)}{\mathcal{A}(d,-\alpha)} r^{\alpha-\beta}, \quad r \geq 0. \tag{2.24}$$

Clearly for $a \in (0, M]$ and $r > 0$, $1 \leq \psi^a(r) \leq 1 + cM^\beta r^{\alpha-\beta}$ and

$$J^a(x,y) = j^a(|x-y|) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}} \psi^a(|x-y|).$$

The Lévy intensity gives rise to a Lévy system for $X^a$, which describes the jumps of the process $X^a$: for any non-negative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time $T$ (with respect to the filtration of $X^a$),

$$\mathbb{E}_x\Big[\sum_{s\leq T} f(s, X^a_{s-}, X_s)\Big] = \mathbb{E}_x\Big[\int_0^T \Big(\int_{\mathbb{R}^d} f(s, X^a_s, y) J^a(X^a_s, y) dy\Big) ds\Big]. \tag{2.25}$$

(See, for example, [7, Proof of Lemma 4.7] and [8, Appendix A]).

For any open set $D \subset \mathbb{R}^d$, let $\tau^a_D = \tau^a(D) := \inf\{t > 0 : X^a_t \notin D\}$ denote the first exit time from $D$ by $X^a$.

The next lemma follows immediately from a special case of [19, Proposition 2.10 and Lemma 3.6] and the scaling property of $X^a$.

**Lemma 2.4** *For any $b, M \in (0,\infty)$, there exists $C_{13} = C_{13}(M, b, \alpha, \beta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $a \in [0, M]$ and $r \in (0, b]$,*

$$\mathbb{E}_x\left[\tau^a_{B(x_0,r)}\right] \leq C_{13}\, r^{\alpha/2}(r - |x-x_0|)^{\alpha/2}, \qquad \text{for } x \in B(x_0, r). \tag{2.26}$$

For $\lambda > 0$, $\widehat{Y}^\lambda = (\widehat{Y}^\lambda_t, \mathbb{P}_x)$ is a Lévy process in $\mathbb{R}^d$ such that

$$\mathbb{E}_x\left[e^{i\xi\cdot(\widehat{Y}^\lambda_t - \widehat{Y}^\lambda_0)}\right] = e^{-t\psi(\xi)} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

with

$$\psi(\xi) = \mathcal{A}(d, -\beta) \int_{\{|y| \le \lambda\}} \frac{1 - \cos(\xi \cdot y)}{|y|^{d+\beta}} dy.$$

In other words, $\widehat{Y}^\lambda$ is a pure jump symmetric Lévy process in $\mathbb{R}^d$ with a Lévy density given by $\mathcal{A}(d, -\beta)|x|^{-d-\beta} 1_{\{|x| \le \lambda\}}$. For $a > 0$, suppose $\widehat{Y}^{1/a}$ is independent of the symmetric $\alpha$-stable process $X$ on $\mathbb{R}^d$. Define

$$\widehat{X}_t^a := X_t + a\widehat{Y}_t^{1/a}, \qquad t \ge 0.$$

We will call the process $\widehat{X}^a$ the independent sum of the symmetric $\alpha$-stable process $X$ and the truncated symmetric $\beta$-stable process $\widehat{Y}^{1/a}$ with weight $a > 0$. The infinitesimal generator of $\widehat{X}^a$ is $\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}$.

For any open set $U \subset \mathbb{R}^d$, let $\widehat{\tau}_U^a = \inf\{t > 0 : \widehat{X}_t^a \notin U\}$ be the first exit time from $U$ by $\widehat{X}^a$. The truncated process $\widehat{X}^a$ will be used in the proof of next lemma.

**Lemma 2.5** *Assume* $r_1 \in (0, \frac{1}{4}]$ *and* $M > 0$. *Let* $U := \left\{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\right\}$. *There are constants* $C_{14} = C_{14}(r_1, \alpha) > 0$ *and* $C_{15} = C_{15}(r_1, M, \alpha, \beta) > 0$ *such that for every* $a \in [0, M]$

$$\mathbb{E}_x[\tau_U^a] \le C_{14} \mathbb{P}_x\left(|X_{\tau_U^a}^a| \ge 3r_1/2\right) \le C_{15} \delta_U(x)^{\alpha/2}, \qquad \textit{for } r_1 < |x| < 5r_1/4. \tag{2.27}$$

**Proof.** The first inequality in (2.27) is easy. In fact, by the Lévy system (2.25) with

$$f(s, x, y) = \mathbf{1}_U(x) \mathbf{1}_{\{5r_1 < |y| < 10r_1\}}(y)$$

and $T = \tau_U^a$, we have that for $x \in U$

$$\begin{aligned}
&\mathbb{P}_x\left(|X_{\tau_U^a}^a| \ge 3r_1/2\right) \ge \mathbb{P}_x\left(10r_1 > |X_{\tau_U^a}^a| > 5r_1\right) \\
&= \mathbb{E}_x\left[\int_0^{\tau_U^a} \int_{\{5r_1 < |y| < 10r_1\}} J^a(X_s^a, y) dy ds\right] \\
&\ge \mathbb{E}_x\left[\int_0^{\tau_U^a} \int_{\{5r_1 < |y| < 10r_1\}} \frac{\mathcal{A}(d, -\alpha)}{|X_s^a - y|^{d+\alpha}} dy ds\right] \ge c_1 \mathbb{E}_x[\tau_U^a],
\end{aligned}$$

where $c_1 = c(r_1, \alpha) > 0$.

It is enough to prove the second inequality in (2.27) for $r_1 < |x| < r_1 + \delta$ for some small $\delta > 0$. Without loss of generality, we assume $\widetilde{x} = \widetilde{0}$ and $x_d > 0$. Let $p > 0$ be such that $p \ne \beta$ and

$$\alpha - (\beta/2) < p < (\alpha - (\beta/2) + (\alpha - \beta)/3) \wedge \alpha.$$

Note that $\alpha/2 < p < 3\alpha/2 - \beta$. Define

$$h(y) := \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{\alpha/2} \mathbf{1}_{U \cap \{z_d > 0, |\widetilde{z}| < r_1/2\}}(y),$$

$$g_p(y) := \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^p \mathbf{1}_{U \cap \{z_d > 0, |\widetilde{z}| < r_1/2\}}(y),$$

and let $\phi$ be a smooth function on $\mathbb{R}^d$ with bounded first and second partial derivatives such that $\phi(y) = 2^{4+p}|\widetilde{y}|^2/r_1^2$ for $y \in \{z_d > 0, r_1 < |y| < 4r_1/5, |\widetilde{z}| < r_1/4\}$ and $2^p \le \phi(y) \le 4^p$ if $|\widetilde{y}| \ge r_1/2$ or $|y| \ge 3r_1/2$.

Since $r_1 \le 1/4$, it is easy to see that $\|g_p\|_\infty < 1$. Now we define

$$u(y) := h(y) + \phi(y) - g_p(y).$$

By Taylor's expansion with the remainder of order 2, we get that for any $a \in (0, M]$ and $y \in \mathbb{R}^d$,

$$\left|(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})\phi(y)\right| \le \left\|\Delta^{\alpha/2}\phi\right\|_\infty + M^\beta \left\|\widehat{\Delta}^{\beta/2}\phi\right\|_\infty \le c_2(\alpha, \beta, M) < \infty. \tag{2.28}$$

Moreover, by (2.6)–(2.8), there exist $c_3 = c_3(\alpha, \beta) > 0$ and $\delta_1 = \delta_1(\alpha, \beta) \in (0, r_1/8)$ such that

$$\Delta^{\alpha/2} g_p(y) \ge c_3 \delta_D(y)^{p-\alpha} \quad \text{for } y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\right\}$$

and

$$\widehat{\Delta}^{\beta/2} g_p(y) \ge -c_3 \delta_D(y)^{(p-\beta)\wedge 0} \quad \text{for } y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\right\}.$$

Thus there exist $c_4 = c_4(\alpha, \beta, M) > 0$ and $\delta_2 = \delta_2(\alpha, \beta, M) \in (0, \delta_1)$ such that for all $a \in (0, M]$ and $y \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_2, |\widetilde{z}| < r_1/4\}$,

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}) g_p(y) \ge c_3 \delta_D(y)^{p-\alpha} - c_3 M^\beta \delta_D(y)^{(p-\beta)\wedge 0} \ge c_4 \delta_D(y)^{p-\alpha}. \tag{2.29}$$

Furthermore by (2.6) and (2.8)–(2.10), there exist $c_5 = c_5(\alpha, \beta, M) > 0$ and $\delta_3 = \delta_3(\alpha, \beta) \in (0, \delta_1)$ such that for all $a \in (0, M]$ and for every $y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_3, |\widetilde{z}| < r_1/4\right\}$,

$$\begin{aligned}\left|(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}) h(y)\right| &\le \left|\Delta^{\alpha/2} h(y)\right| + M^\beta \left|\widehat{\Delta}^{\beta/2} h(y)\right| \\ &\le \begin{cases} c_5 + c_5 \delta_D(y)^{(\alpha/2-\beta)\wedge 0} & \text{if } \beta \ne \alpha/2, \\ c_5 + c_5 |\log \delta_D(y)| & \text{if } \beta = \alpha/2. \end{cases}\end{aligned} \tag{2.30}$$

Since $p - \alpha < \alpha/2 - \beta$, by (2.28)–(2.29), there exists $\delta_4 = \delta_4(\alpha, \beta, M) \in (0, \delta_2 \wedge \delta_3)$ such that for all $a \in (0, M]$ and $y \in V := \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_4, |\widetilde{z}| < r_1/4\right\}$

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}) u(y) \le c_2 + c_5 + c_5 \left(\delta_D(y)^{(\alpha/2-\beta)\wedge 0} + |\log \delta_D(y)|\right) - c_4 \delta_D(y)^{p-\alpha} \le 0. \tag{2.31}$$

Let $\eta$ be a non-negative smooth radial function with compact support in $\mathbb{R}^d$ such that $\eta(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}^d} \eta(x)dx = 1$. For $k \ge 1$, define $\eta_k(x) = 2^{kd}\eta(2^k x)$. Set $u^{(k)}(z) := (\eta_k * u)(z)$. As $(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}) u^{(k)} = \eta_k * (\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}) u$, we have by (2.31) that

$$\left(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}\right) u^{(k)} \le 0$$

on $V_k := \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 + 2^{-k} < |z| < r_1 + \delta_4 - 2^{-k} \text{ and } |\widetilde{z}| < r_1/4 - 2^{-k}\right\}$. Since $u^{(k)}$ is a bounded smooth function on $\mathbb{R}^d$ with bounded first and second partial derivatives, by Ito's formula and the Lévy system (2.25),

$$M_t^k := u^{(k)}(\widehat{X}_t^a) - u^{(k)}(\widehat{X}_0^a) - \int_0^t \left(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}\right) u^{(k)}(\widehat{X}_s^a) ds \tag{2.32}$$

is a martingale. Thus it follows from (2.32) that $t \mapsto u^{(k)}\big(\widehat{X}^a_{t \wedge \widehat{\tau}^a_{V_k}}\big)$ is a bounded supermartingale. Since $V_k$ increases to $V$ and $u$ is bounded and continuous on $\overline{V}$, we conclude that

$$t \mapsto u\left(\widehat{X}^a_{t \wedge \widehat{\tau}^a_V}\right) \text{ is a bounded supermartingale.} \tag{2.33}$$

We observe that, since $\phi(x)=0$,

$$u(x) \le \delta_U(x)^{\alpha/2}. \tag{2.34}$$

We also observe that, since $\phi \ge 2g_p$ outside of $\{z \in U : z_d > 0, |\widetilde{z}| < r_1/2\}$ and

$$u(y) \ge \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right)^{\alpha/2} - \left( y_d - \sqrt{r_1^2 - |\widetilde{y}|^2} \right)^p > c_6$$

on $\{z_d > 0, r_1 + \delta_4 \le |z| < 3r_1/2, |\widetilde{z}| < r_1/2\}$, we have

$$u(y) \ge c_7 > 0 \qquad \text{for } y \in V^c \setminus \overline{B(0,r_1)}, \tag{2.35}$$

where $c_7$ depends on $\delta_4$, $\alpha$, $\beta$ and $r_1$. Therefore, by (2.33)-(2.35) we get

$$\delta_U(x)^{\alpha/2} \ge u(x) \ge \mathbb{E}_x \left[ u\big(\widehat{X}_{\widehat{\tau}_V^a}\big) \right] \ge c_7 \mathbb{P}_x \left( \widehat{X}^a_{\widehat{\tau}_V^a} \in V^c \setminus \overline{B(0,r_1)} \right) \ge c_7 \mathbb{P}_x \left( |\widehat{X}^a_{\widehat{\tau}_U^a}| \ge 3r_1/2 \right). \tag{2.36}$$

Note that there exist $c_8 = c_8(\alpha, d, r_1) > 0$ and $c_9 = c_9(\beta, d, r_1) > 0$ such that for $z \in U$,

$$\int_{\{|y| \ge 2r_1\}} \frac{dy}{|z-y|^{d+\alpha}} \le c_8 \int_{\{2r_1 \le |y| < 3r_1\}} \frac{dy}{|z-y|^{d+\alpha}}$$

and

$$\int_{\{|y| \ge 2r_1\}} \frac{dy}{|z-y|^{d+\beta}} \le c_9 \int_{\{2r_1 \le |y| < 3r_1\}} \frac{dy}{|z-y|^{d+\beta}}.$$

Thus by (2.25), there exists a positive constant $c_{10} = c_{10}(d, \alpha, \beta, M)$ such that for any $a \in (0, M]$,

$$\begin{aligned}
\mathbb{P}_x \left( |X^a_{\tau_U^a}| \ge 2r_1 \right) &= \mathbb{E}_x \left[ \int_0^{\tau_U^a} \int_{\{|y| \ge 2r_1\}} J^a(X^a_s, y) dy ds \right] \\
&\le c_{10} \mathbb{E}_x \left[ \int_0^{\tau_U^a} \int_{\{2r_1 \le |y| < 3r_1\}} J^a(X^a_s, y) dy ds \right] \\
&= c_{10} \mathbb{P}_x \left( 3r_1 > |X^a_{\tau_U^a}| \ge 2r_1 \right).
\end{aligned} \tag{2.37}$$

Since $r_1 \le 1/4$ and the processes $X$ and $Y$ do not jump simultaneously, we have by (2.36) that there is a positive constant $c_{11} = c_{11}(d, \alpha, \beta, M, r_1)$ such that for all $a \in (0, M]$,

$$\begin{aligned}
\mathbb{P}_x \left( |X^a_{\tau_U^a}| \ge 3r_1/2 \right) &\le (c_{10}+1) \mathbb{P}_x \left( 3r_1 > |X^a_{\tau_U^a}| \ge 3r_1/2 \right) \\
&= (c_{10}+1) \mathbb{P}_x \left( 3r_1 > |\widehat{X}^a_{\widehat{\tau}_U^a}| \ge 3r_1/2 \right) \\
&\le (c_{10}+1) \mathbb{P}_x \left( |\widehat{X}^a_{\widehat{\tau}_U^a}| \ge 3r_1/2 \right) \le c_{11} \delta_U(x)^{\alpha/2}.
\end{aligned}$$

□

**Lemma 2.6** *Assume $M > 0$ and $r_1 \in (0, \frac{1}{4}]$. Let $E = \{x \in \mathbb{R}^d : |x| > r_1\}$. Then for every $T > 0$, there is a constant $C_{16} = C_{16}(r_1, \alpha, \beta, T, M) > 0$ such that for every $a \in [0, M]$,*

$$p_E^a(t, x, y) \le C_{16}\, \delta_E(x)^{\alpha/2} J^a(x, y) \qquad \textit{for } r_1 < |x| < 5r_1/4,\ |y| \ge 2r_1 \textit{ and } t \le T.$$

**Proof.** Define $U := \{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\}$. Since $X^a$ satisfies the hypothesis **H** in [28], by [28, Theorem 1], $X^a_{\tau^a_U} \notin \partial U$ with probability 1. For $r_1 < |x| < 5r_1/4$, $|y| \geq 2r_1$ and $t \in (0, T]$, it follows from the strong Markov property of $X^a$ and (2.25) that

$$
\begin{aligned}
p^a_E(t,x,y) \;&=\; \mathbb{E}_x\left[p^a_E(t-\tau^a_U, X^a_{\tau^a_U}, y) : \tau^a_U < t\right] \\
&= \int_0^t \int_U p_U(s,x,z)\left(\int_{\{w:|w|>3r_1/2\}} J^a(z,w)\; p^a_E(t-s,w,y)dw\right)dzds \\
&= \int_0^t \int_U p_U(s,x,z)\left(\int_{\{w:\,(3r_1/4)+(|y|/2)\geq |w|>3r_1/2\}} J^a(z,w)p^a_E(t-s,w,y)dw\right)dzds \\
&\quad + \int_0^t \int_U p_U(s,x,z)\left(\int_{\{w:\,|w|>(3r_1/4)+(|y|/2)\}} J^a(z,w)p^a_E(t-s,w,y)dw\right)dzds \\
&=: I + II.
\end{aligned}
$$

Note that for $|w| \leq (3r_1/4) + (|y|/2)$,

$$
|w-y| \geq |y| - |w| \geq \frac{1}{2}\left(|y| - \frac{3r_1}{2}\right) \geq \frac{|y|}{8} \geq \frac{|x-y|}{16}. \tag{2.38}
$$

Since $p^a_E(t-s,w,y) \leq p^a(t-s,w,y)$, by (1.4) and (2.38), there exist constants $c_1 = c_1(\alpha,\beta,M) > 0$ and $c_2 = c_2(\alpha,\beta,M) > 0$ such that for $a \in (0, M]$

$$
\begin{aligned}
I &\leq \int_0^t \int_U p^a_U(s,x,z)\left(\int_{\{w:\,(3r_1/4)+(|y|/2)\geq|w|>3r_1/2\}} J^a(z,w)\, c_1 T J^a(w,y)dw\right)dzds \\
&\leq c_2 T J^a(x,y) \int_0^t \int_U p^a_U(s,x,z)\left(\int_{\{w:\,3|x-y|/4\geq|w|>3r_1/2\}} J^a(z,w)dw\right)dzds \\
&= c_2 T J^a(x,y)\, \mathbb{P}_x\left(3r_1/2 < |X^a_{\tau^a_U}| \leq 3|x-y|/4;\, \tau^a_U \leq t\right) \\
&\leq c_2 T J^a(x,y)\, \mathbb{P}_x\left(|X^a_{\tau^a_U}| > 3r_1/2\right).
\end{aligned}
$$

By Lemma 2.5 , we have for $|x| \in (r_1, 5r_1/4)$,

$$
\mathbb{P}_x\left(|X^a_{\tau^a_U}| > 3r_1/2\right) \leq c_3\, \delta_U(x)^{\alpha/2} = c_3 \delta_E(x)^{\alpha/2}
$$

for some positive constant $c_3 = c_3(r_1,\alpha,\beta,M)$. Thus

$$
I \leq c_4\,(T \vee 1)\,\delta_E(x)^{\alpha/2} J^a(x,y) \tag{2.39}
$$

for some positive constant $c_4 = c_4(r_1,\alpha,\beta,M)$. On the other hand, for $z \in U$ and $w \in \mathbb{R}^d$ with $|w| > (3r_1/4) + (|y|/2)$,

$$
|z-w| \geq |w| - |z| \geq \frac{1}{2}\left(|y| - \frac{3r_1}{2}\right) \geq \frac{|y|}{8} \geq \frac{|x-y|}{16}.
$$

Thus by the symmetry of $p^a_E(t-s,w,y)$ in $(w,y)$, we have

$$
\begin{aligned}
II &\le c_5 J^a(x,y) \int_0^t \int_U p^a_U(s,x,z) \left( \int_{\{w:\,|w|>(3r_1/4)+(|y|/2)\}} p^a_E(t-s,y,w)dw \right) dzds \\
&\le c_5 J^a(x,y) \int_0^\infty \int_U p^a_U(s,x,z)dzds \\
&= c_5 J^a(x,y)\mathbb{E}_x\left[\tau^a_U\right] \;\le\; c_6\,\delta_E(x)^{\alpha/2} J^a(x,y)
\end{aligned}
$$

for some positive constants $c_k = c_k(r_1,\alpha,\beta,M)$, $k=5,6$. In the last inequality, we used Lemma 2.5 to deduce that $\mathbb{E}_x\left[\tau^a_U\right] \le c\delta_U(x)^{\alpha/2} = c\delta_E(x)^{\alpha/2}$ for some positive constant $c = c(r_1,\alpha,\beta,M)$. This together with (2.39) proves the lemma. □

**Theorem 2.7** *Assume that $M>0$ and $D$ is an open set that satisfies the uniform exterior ball condition with radius $r_0>0$. Then for every $T>0$, there is a constant $C_{17}=C_{17}(r_0/T,\alpha,\beta,M)>0$ such that for all $a\in(0,M]$, $\lambda\in(0,T]$ and $x,y\in\lambda^{-1}D$,*

$$
p^a_{\lambda^{-1}D}(1,x,y) \;\le\; C_{17}\,\left(1\wedge J^a(x,y)\right)\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}.
$$

**Proof.** Note that for every $\lambda\in(0,T]$, $\lambda^{-1}D$ satisfies the uniform exterior ball condition with radius $r_0/T$. For $x,y\in\lambda^{-1}D$, let $z\in\partial(\lambda^{-1}D)$ be that $|x-z|=\delta_{\lambda^{-1}D}(x)$. Let $B_z\subset(\lambda^{-1}D)^c$ be the ball with radius $r_1:=4^{-1}\wedge(r_0/T)$ so that $\partial B_z\cap\partial(\lambda^{-1}D)=\{z\}$. Since, by (1.4)

$$
p^a_{\lambda^{-1}D}(1,x,y) \;\le\; p^a(1,x,y) \;\le c\left(1\wedge J^a(x,y)\right),
$$

it suffices to prove the theorem for $x\in\lambda^{-1}D$ with $\delta_{\lambda^{-1}D}(x)<r_1/4$. When $\delta_{\lambda^{-1}D}(x)<r_1/4$ and $|x-y|\ge 5r_1$, we have $\delta_{B_z^c}(y)>2r_1$ and so by Lemma 2.6, there is a constant $c_1>0$ that depends only on $(r_0/T,d,\alpha,\beta,M)$ such that for $t\in(0,1]$,

$$
p^a_{\lambda^{-1}D}(t,x,y) \;\le\; p^a_{(\overline{B}_z)^c}(t,x,y) \;\le\; c_1\,\delta_{(\overline{B}_z)^c}(x)^{\alpha/2}J^a(x,y) = c_1\delta_{\lambda^{-1}D}(x)^{\alpha/2}J^a(x,y). \tag{2.40}
$$

So it remains to show that when $\delta_{\lambda^{-1}D}(x)<r_1/4$ and $|x-y|<5r_1$, there exists a positive constant $c_2=c_2(r_0/T,d,\alpha,\beta,M)$ such that

$$
p^a_{\lambda^{-1}D}(1,x,y) \;\le\; c_2\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}. \tag{2.41}
$$

Let $z_x\in\partial(\lambda^{-1}D)$ be such that $|x-z_x|=\delta_{\lambda^{-1}D}(x)$ and $z_0\in\mathbb{R}^d$ so that

$$
B(z_0,r_1)\subset(\lambda^{-1}D)^c \qquad\text{and}\qquad \partial B(z_0,r_1)\cap\partial(\lambda^{-1}D)=\{z_x\}.
$$

Define $U:=\left\{w\in\mathbb{R}^d:\,|w-z_0|\in(r_1,\,8r_1)\right\}$. Note that

$$
x,y\in U\cap\lambda^{-1}D \qquad\text{and}\qquad \delta_U(x)=\delta_{\lambda^{-1}D}(x).
$$

By the strong Markov property and the symmetry of $p^a_{\lambda^{-1}D}(1,x,y)$ in $x$ and $y$, we have

$$
p^a_{\lambda^{-1}D}(1,x,y) = p^a_{U\cap\lambda^{-1}D}(1,x,y) + \mathbb{E}_y\left[p^a_{\lambda^{-1}D}(1-\tau^a_{U\cap\lambda^{-1}D}, X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x);\;\tau^a_{U\cap\lambda^{-1}D}<1\right].
$$

By the semigroup property and (1.4),

$$
\begin{aligned}
p^a_{U\cap\lambda^{-1}D}(1,x,y) &= \int_{U\cap\lambda^{-1}D} p^a_{U\cap\lambda^{-1}D}(1/2,x,z)p^a_{U\cap\lambda^{-1}D}(1/2,z,y)dz \\
&\le \|p^a(1/2,\cdot,\cdot)\|_\infty \, \mathbb{P}_x\left(\tau^a_{U\cap\lambda^{-1}D} > 1/2\right) \\
&\le c_3\, \mathbb{E}_x\left[\tau^a_{U\cap\lambda^{-1}D}\right] \le c_3\, \mathbb{E}_x\left[\tau^a_U\right] \\
&\le c_4\, \delta_U(x)^{\alpha/2} = c_4\, \delta_{\lambda^{-1}D}(x)^{\alpha/2}.
\end{aligned}
$$

In the last inequality, we used Lemma 2.5.

On the other hand, we have $X^a_{\tau^a_{U\cap\lambda^{-1}D}} \in U^c\cap\lambda^{-1}D$ on $\{\tau^a_{U\cap\lambda^{-1}D}<1\}$, and so

$$
|X^a_{\tau^a_{U\cap\lambda^{-1}D}} - x| \ge 7r_1, \qquad \text{on} \quad \left\{\tau^a_{U\cap\lambda^{-1}D} < 1\right\}.
$$

Consequently, by (2.40) for $p^a_{\lambda^{-1}D}\big(1-\tau^a_{U\cap\lambda^{-1}D}, X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x\big)$,

$$
\begin{aligned}
&\mathbb{E}_y\left[p^a_{\lambda^{-1}D}(1-\tau^a_{U\cap\lambda^{-1}D}, X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x);\ \tau^a_{U\cap\lambda^{-1}D}<1\right] \\
&\le \mathbb{E}_y\left[c_1\,\delta_{\lambda^{-1}D}(x)^{\alpha/2} J^a(X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x);\ \tau^a_{U\cap\lambda^{-1}D}<1\right] \\
&\le c_1((7r_1)^{-d-\alpha} + M^\beta(7r_1)^{-d-\beta})\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}\,\mathbb{P}_y\left(\tau^a_{U\cap\lambda^{-1}D}<1\right) \\
&\le c_1((7r_1)^{-d-\alpha} + M^\beta(7r_1)^{-d-\beta})\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}.
\end{aligned}
$$

This completes the proof for (2.41) and hence the theorem. □

**Theorem 2.8** *Assume that $M>0$ and that $D$ is an open set that satisfies the uniform exterior ball condition with radius $r_0>0$. For every $T>0$, there exists a positive constant $C_{18}=C_{18}(T,r_0,\alpha,\beta,M)$ such that for every $a\in[0,M]$, $t\in(0,T]$ and $x,y\in D$,*

$$
p^a_D(t,x,y) \le C_{18}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge tJ^a(x,y)\right). \tag{2.42}
$$

**Proof.** Fix $T,M>0$. By Theorem 2.7, there exists a positive constant $c_1=c_1(T,r_0,\alpha,\beta,M)$ such that for every $t\in(0,T]$,

$$
p^{at^{(\alpha-\beta)/(\alpha\beta)}}_{t^{-1/\alpha}D}(1,x,y) \le c_1\left(1\wedge J^{at^{(\alpha-\beta)/(\alpha\beta)}}(x,y)\right)\delta_{t^{-1/\alpha}D}(x)^{\alpha/2}. \tag{2.43}
$$

Thus by (1.3), (1.4) and (2.43), for every $t\le T$,

$$
\begin{aligned}
p^a_D(t,x,y) &= t^{-d/\alpha} p^{at^{(\alpha-\beta)/(\alpha\beta)}}_{t^{-1/\alpha}D}(1,t^{-1/\alpha}x,t^{-1/\alpha}y) \\
&\le c_1\,t^{-d/\alpha}\left(1\wedge J^{at^{(\alpha-\beta)/(\alpha\beta)}}(t^{-1/\alpha}x,t^{-1/\alpha}y)\right)\delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha/2} \\
&= c_1\left(t^{-d/\alpha}\wedge tJ^a(x,y)\right)\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \\
&\le c_2\,p^a(t,x,y)\,\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}
$$

By symmetry, the above inequality holds with the roles of $x$ and $y$ interchanged. Using the semi-group property for $t \le T$,

$$\begin{aligned} p_D^a(t,x,y) &= \int_D p_D^a(t/2,x,z)p_D^a(t/2,z,y)dz \\ &\le c_3 \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{t} \int_D p^a(t/2,x,z)p^a(t/2,z,y)dz \\ &\le c_3 \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{t} p^a(t,x,y). \end{aligned}$$

This proves the upper bound (2.42) by noting that

$$(1 \wedge a)(1 \wedge b) = \min\{1, a, b, ab\} \qquad \text{for } a, b > 0.$$

□

# 3 Lower bound estimate

**Lemma 3.1** *For any positive constants $\Lambda$, $\kappa$ and $b$, there exists $C_{19} = C_{19}(\Lambda, \kappa, b, \alpha, \beta, M) > 0$ such that for every $z \in \mathbb{R}^d$, $\lambda \in (0, \Lambda]$ and $a \in (0, M]$,*

$$\inf_{\substack{y \in \mathbb{R}^d \\ |y-z| \le \kappa\lambda^{1/\alpha}}} \mathbb{P}_y \left( \tau^a_{B(z, 2\kappa\lambda^{1/\alpha})} > b\lambda \right) \ge C_{19}.$$

**Proof.** By [8, Proposition 4.9], there exists $\varepsilon = \varepsilon(\Lambda, \kappa, \alpha, \beta) > 0$ such that for every $\lambda \in (0, \Lambda]$,

$$\inf_{y \in \mathbb{R}^d} \mathbb{P}_y \left( \tau^1_{B(y, \kappa\lambda^{1/\alpha}/2)} > \varepsilon\lambda \right) \ge \frac{1}{2}.$$

Suppose $b > \varepsilon$ then by the parabolic Harnack principle in [8, Proposition 4.12]

$$c_1 \, p^1_{B(y,\kappa\lambda^{1/\alpha})}(\varepsilon\lambda, y, w) \le p^1_{B(y,\kappa\lambda^{1/\alpha})}(b\lambda, y, w) \qquad \text{for } w \in B(y, \kappa\lambda^{1/\alpha}/2),$$

where the constant $c_1 = c_1(\kappa, b, \alpha, \beta, \Lambda) > 0$ is independent of $y \in \mathbb{R}^d$, $\lambda \in (0, \Lambda]$. Thus

$$\begin{aligned} \mathbb{P}_y \left( \tau^1_{B(y,\kappa\lambda^{1/\alpha})} > b\lambda \right) &= \int_{B(y,\kappa\lambda^{1/\alpha})} p^1_{B(y,\kappa\lambda^{1/\alpha})}(b\lambda, y, w)dw \\ &\ge \int_{B(y,\kappa\lambda^{1/\alpha}/2)} p^1_{B(y,\kappa\lambda^{1/\alpha})}(b\lambda, y, w)dw \\ &\ge c_1 \int_{B(y,\kappa\lambda^{1/\alpha}/2)} p^1_{B(y,\kappa\lambda^{1/\alpha}/2)}(\varepsilon\lambda, y, w)dw \ge c_1/2. \end{aligned} \tag{3.1}$$

For the general case, by (1.3) and (3.1),

$$
\begin{aligned}
&\inf_{\substack{y\in\mathbb{R}^d\\ |y-z|\le\kappa\lambda^{1/\alpha}}} \mathbb{P}_y\left(\tau^a_{B(z,2\kappa\lambda^{1/\alpha})} > b\lambda\right)\\
&\ge \mathbb{P}_0\left(\tau^a_{B(0,\kappa\lambda^{1/\alpha})} > b\lambda\right)\\
&= \int_{B(0,\kappa\lambda^{1/\alpha})} p^a_{B(0,\kappa\lambda^{1/\alpha})}(b\lambda, 0, w)dw\\
&= \int_{B(0,\kappa\lambda^{1/\alpha}a^{-\frac{\beta}{\alpha-\beta}})} p^1_{B(0,\kappa\lambda^{1/\alpha}a^{-\frac{\beta}{\alpha-\beta}})}(a^{\frac{\alpha\beta}{\alpha-\beta}}b\lambda, 0, z)dz\\
&= \mathbb{P}_0\left(\tau^1(B(0,\kappa\lambda^{1/\alpha}a^{-\frac{\beta}{\alpha-\beta}})) > a^{\frac{\alpha\beta}{\alpha-\beta}}b\lambda\right)\\
&\ge \mathbb{P}_0\left(\tau^1(B(0,\kappa\lambda^{1/\alpha}M^{-\frac{\beta}{\alpha-\beta}})) > M^{\frac{\alpha\beta}{\alpha-\beta}}b\lambda\right) \ge c_2(\Lambda,\kappa,b,\alpha,\beta,M) > 0.
\end{aligned}
$$

This proves the lemma. □

Recall that $\psi^a$ is defined in (2.24).

**Proposition 3.2** *Suppose that $M, T > 0$ and $(t,x,y) \in (0,T] \times D \times D$ with $\delta_D(x) \ge t^{1/\alpha} \ge 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$. Then there exists a positive constant $C_{20} = C_{20}(M,\alpha,\beta,T)$ such that for all $a \in (0,M]$*

$$
p^a_D(t,x,y) \ \ge\ C_{20}\, t^{-d/\alpha}. \tag{3.2}
$$

**Proof.** Let $t \in (0,T]$ and $x, y \in D$ with $\delta_D(x) \ge t^{1/\alpha} \ge 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$. By the parabolic Harnack principle in [8, Proposition 4.12] and the scaling property, there exists $c_1 = c_1(M,\alpha,\beta,T) > 0$ such that for all $a \in (0,M]$,

$$
p^a_D(t/2,x,w) \ \le\ c_1\, p^a_D(t,x,y) \qquad \text{for } w \in B(x, 2t^{1/\alpha}/3).
$$

This together with Lemma 3.1 yields that

$$
\begin{aligned}
p^a_D(t,x,y) &\ \ge\ \frac{1}{c_1\,|B(x,t^{1/\alpha}/2)|}\int_{B(x,t^{1/\alpha}/2)} p^a_D(t/2,x,w)dw\\
&\ \ge\ c_2 t^{-d/\alpha}\int_{B(x,t^{1/\alpha}/2)} p^a_{B(x,t^{1/\alpha}/2)}(t/2,x,w)dw\\
&\ =\ c_2 t^{-d/\alpha}\,\mathbb{P}_x\left(\tau^a_{B(x,t^{1/\alpha}/2)} > t/2\right) \ \ge\ c_3\, t^{-d/\alpha},
\end{aligned}
$$

where $c_i = c_i(T,\alpha,\beta,M) > 0$ for $i = 2,3$. □

**Lemma 3.3** *Suppose that $M, T > 0$, $D$ is an open subset of $\mathbb{R}^d$ and $(t,x,y) \in (0,T] \times D \times D$ with $\min\{\delta_D(x), \delta_D(y)\} \ge t^{1/\alpha}$ and $|x-y|^\alpha \ge 2^{-\alpha}t\psi^a(|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{21} = C_{21}(\alpha,\beta,T,M) > 0$ such that for $a \in (0,M]$*

$$
\mathbb{P}_x\left(X^{a,D}_t \in B\big(y,\, 2^{-1}t^{1/\alpha}\big)\right) \ \ge\ C_{21}\, t^{d/\alpha+1} J^a(x,y).
$$

**Proof.** For $t \in (0,T]$, it follows from Lemma 3.1 that, starting at $z \in B(y,\, 4^{-1}t^{1/\alpha})$, with probability at least $c_1 = c_1(\alpha,\beta,T,M) > 0$, for any $a \in (0,M]$, the process $X^a$ does not move more than $6^{-1}t^{1/\alpha}$ by time $t$. Thus, it suffices to show that there exists a constant $c_2 = c_2(\alpha,\beta,T,M) > 0$ such that

$$\mathbb{P}_x\left(X^{a,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \,\geq\, c_2\, t^{d/\alpha+1} J^a(x,y) \tag{3.3}$$

for all $a \in (0,M]$, $t \in (0,T]$ and $|x-y|^\alpha \geq 2^{-\alpha} t \psi^a(|x-y|)^{\alpha/(d+\alpha)}$.

Let $B_x := B(x,\, 6^{-1}t^{1/\alpha})$, $B_y := B(y,\, 6^{-1}t^{1/\alpha})$ and $\tau^a_x := \tau^a_{B_x}$. It follows from Lemma 3.1 that there exists $c_3 = c_3(\alpha,\beta,T,M) > 0$ such that for $a \in (0,M]$ and $t \in (0,T]$,

$$\mathbb{E}_x\left[t \wedge \tau^a_x\right] \,\geq\, t\mathbb{P}_x\left(\tau^a_x \geq t\right) \,\geq\, c_3\, t. \tag{3.4}$$

By the Lévy system in (2.25),

$$\begin{aligned} & \mathbb{P}_x\left(X^{a,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \\ \geq\;& \mathbb{P}_x(X^a_{t\wedge\tau^a_x} \in B(y,\, 4^{-1}t^{1/\alpha}) \text{ and } t\wedge\tau^a_x \text{ is a jumping time }) \\ \geq\;& \mathbb{E}_x\left[\int_0^{t\wedge\tau^a_x} \int_{B_y} J^a(X^a_s,u)\,du\,ds\right]. \end{aligned} \tag{3.5}$$

Note that

$$|x-y| \geq 2^{-1}t^{1/\alpha}\psi^a(|x-y|)^{1/(d+\alpha)} \geq 2^{-1}t^{1/\alpha}.$$

Moreover, if $s < \tau^a_x$ and $u \in B_y$,

$$|X^a_s - u| \leq |x-y| + |x - X^a_s| + |y-u| \leq 2|x-y|.$$

Thus from (3.5) we get that for any $a \in (0,M]$ and $t \in (0,T]$,

$$\begin{aligned} & \mathbb{P}_x\left(X^{a,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \\ \geq\;& \mathbb{E}_x\left[t\wedge\tau^a_x\right] \int_{B_y} j^a(2|x-y|)\,du \\ \geq\;& c_4\, t\,|B_y|\, j^a(2|x-y|) \,\geq\, c_5\, t^{d/\alpha+1} j^a(2|x-y|) \,\geq\, c_5\, 2^{-d-\alpha} t^{d/\alpha+1} j^a(|x-y|) \end{aligned}$$

for some positive constants $c_i = c_i(\alpha,\beta,T,M)$, $i = 4,5$. Here in the second inequality, (3.4) is used. □

**Proposition 3.4** *Suppose that $T > 0$, $M > 0$, $D$ is an open subset of $\mathbb{R}^d$ and $(t,x,y) \in (0,T] \times D \times D$ with $\min\{\delta_D(x),\, \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x-y|^\alpha \geq 2^{-\alpha-1} t \psi^a(|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{22} = C_{22}(\alpha,\beta,T,M) > 0$ such that for any $a \in (0,M]$,*

$$p^a_D(t,x,y) \,\geq\, C_{22}\, t J^a(x,y). \tag{3.6}$$

**Proof.** By the semigroup property, Proposition 3.2 and Lemma 3.3, there exist positive constants $c_1 = c_1(\alpha, \beta, T, M)$ and $c_2 = c_2(\alpha, \beta, T, M)$ such that for any $t \in (0, T]$ and $a \in (0, M]$

$$\begin{aligned} p^a_D(t, x, y) &= \int_D p^a_D(t/2, x, z) p^a_D(t/2, z, y) dz \\ &\geq \int_{B(y, 2^{-1}(t/2)^{1/\alpha})} p^a_D(t/2, x, z) p^a_D(t/2, z, y) dz \\ &\geq c_1 t^{-d/\alpha} \mathbb{P}_x \left( X^{a,D}_{t/2} \in B(y, 2^{-1}(t/2)^{1/\alpha}) \right) \\ &\geq c_2\, t J^a(x, y). \end{aligned}$$

□

In the rest of this section, we assume that $D$ is an open set in $\mathbb{R}^d$ satisfying the uniform interior ball condition with radius $r_0 > 0$ in the following sense: For every $x \in D$ with $\delta_D(x) < r_0$, there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and $B(x_0, r_0) \subset D$ for $x_0 := z_x + r_0(x - z_x)/|x - z_x|$. Clearly, a (uniform) $C^{1,1}$ open set satisfies the uniform interior ball condition.

The goal of this section is to prove the following lower bound for the heat kernel $p^a_D(t, x, y)$.

**Theorem 3.5** *For any $M > 0$ and $T > 0$, there exists positive constant $C_{23} = C_{23}(\alpha, \beta, T, M, r_0)$ such that for all $a \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$p^a_D(t, x, y) \geq C_{23} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge t J^a(x, y)\right).$$

To prove this result, we will first prove a lower bound estimates on the Green function of $X^{a,U}$

$$G^a_U(x, y) := \int_0^\infty p^a_U(t, x, y) dt$$

when $U$ is a bounded $C^{1,1}$ open set. The tool we use to establish the Green function lower bound is a subordinate killed $\alpha$-stable process in $U$. We first introduce this subordinate killed process first.

Assume that $U$ is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and $R_1$ the radius in the uniform interior and exterior ball conditions. Then it follows from [3, Theorem 1.1] that the killed $\alpha$-stable process $X^U$ on $U$ has a density $p_U(t, x, y)$ satisfying the following condition: for any $T > 0$ there exist positive constants $c_2 > c_1$ depending only on $\alpha, T, R_1$ and $d$ such that for any $(t, x, y) \in (0, T] \times U \times U$,

$$p_U(t, x, y) \geq c_1 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right), \tag{3.7}$$

$$p_U(t, x, y) \leq c_2 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right). \tag{3.8}$$

Let $\{T^a_t : t \geq 0\}$ be a subordinator, independent of $X^a$, with Laplace exponent

$$\phi^a(\lambda) = \lambda + a^\beta \lambda^{\beta/\alpha}.$$

Then the process $\{Z^{a,U}_t : t \geq 0\}$ defined by $Z^{a,U}_t = X^U_{T^a_t}$ is called a subordinate killed stable process in $U$. Since $\phi^a$ is a complete Bernstein function, the subordinate $T^a$ has a decreasing potential density $u^a(x)$. In fact $u^a(x)$ is completely monotone. (See [21, 25] for the details.) Then it follows from [25] that the Green function $R^a_U(x,y)$ of $Z^{a,U}$ is given by

$$R^a_U(x,y) = \int_0^\infty p_U(t,x,y)u^a(t)dt. \tag{3.9}$$

It follows from [26] that the Green function $G^a_U$ of $X^{a,U}$ and the Green function $R^a_U$ of $Z^{a,U}$ satisfy the following relation:

$$R^a_U(x,y) \le G^a_U(x,y) \qquad (x,y)\in U\times U. \tag{3.10}$$

So we can get a lower bound on $G^a_U(x,y)$ be establishing a lower bound on $R^a_U(x,y)$. The following result gives sharp two-sided estimates on $R^a_U(x,y)$ and the idea of the proof is similar to that of [24].

**Theorem 3.6** *Suppose that $M>0$ and $U$ is a bounded $C^{1,1}$ open set in $U$. There exist positive constant $C_{25} > C_{24}$ depending only on $(\alpha,\beta,d,R_1,M,diam(U))$ such that for all $a\in(0,M]$,*

$$R^a_U(x,y) \ge C_{24}\begin{cases} \left(1\wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d} & \text{when } d>\alpha,\\ \log\left(1+\frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d=1=\alpha,\\ \left(\delta_U(x)\delta_U(y)\right)^{(\alpha-1)/2}\wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|} & \text{when } d=1<\alpha,\end{cases}$$

*and*

$$R^a_U(x,y) \le C_{25}\begin{cases} \left(1\wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d} & \text{when } d>\alpha,\\ \log\left(1+\frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d=1=\alpha,\\ \left(\delta_U(x)\delta_U(y)\right)^{(\alpha-1)/2}\wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|} & \text{when } d=1<\alpha.\end{cases}$$

**Proof.** Since the drift coefficient of $T^a$ is 1, we know that $u^a(t)\le 1$ for all $t>0$. Now the upper bound on $R^a_U$ follows immediately from (3.9) and [3, Corollary 1.2]. Thus we only need to prove the lower bound.

By using a scaling argument, one can easily check that

$$u^a(t) = u^1(a^{\frac{\alpha}{\alpha-\beta}}t) \qquad t>0. \tag{3.11}$$

Let $T=\text{diam}(U)$. Since $u^1(t)$ is a completely monotone function with $u^1(0+)=1$, by (3.11),

$$u^a(t) \ge u^1(M^{\frac{\alpha}{\alpha-\beta}}T) \qquad \text{for every } t\in(0,T] \text{ and } a\in(0,M]. \tag{3.12}$$

Using (3.12), (3.9) and [3, (4.2)] we get that

$$\begin{aligned}&\int_0^T \left(1\wedge\frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)u^a(t)dt\\ \ge\ & \frac{u^1(M^{\frac{\alpha}{\alpha-\beta}}T)}{|x-y|^{d-\alpha}}\int_{\frac{|x-y|^\alpha}{T}}^\infty \left(u^{\frac{d}{\alpha}-2}\wedge u^{-3}\right)\left(1\wedge\frac{\sqrt{u}\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\sqrt{u}\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)du.\end{aligned} \tag{3.13}$$

Now we can follow the proof of [3, Corollary 1.2] to get the desired lower bound. In fact, when $d>\alpha$, the desired lower bound follows from (3.13) and [3, (4.3) and (4.7)]. Let

$$u_0 := \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}.$$

When $d=\alpha=1$, by (3.13) and [3, (4.3) and (4.9)],

$$\begin{aligned}
R^a_U(x,y) &\ge u^1(M^{\frac{\alpha}{\alpha-\beta}}T)\int_0^T p_U(t,x,y)dt \\
&\ge c_1\left(1\wedge\frac{\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)+c_1\log(u_0\vee 1)+c_1u_0\left((1/u_0)\wedge 1-\frac{|x-y|^\alpha}{T}\right) \\
&\ge c_2(1\wedge u_0)+c_2\log(u_0\vee 1)+c_2u_0\left((1/u_0)\wedge 1-\frac{|x-y|^\alpha}{T}\right) \\
&\ge c_3(1\wedge u_0)+c_3\log(u_0\vee 1)\ \ge\ c_4\log\left(1+\frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right).
\end{aligned}$$

Lastly, in the case $d=1<\alpha<2$. By (3.13), [3, (4.3) and (4.7)] and the first display in part (iii) of the proof of [3, Corollary1.2], we have

$$\begin{aligned}
R^a_U(x,y) &\ge u^1(M^{\frac{\alpha}{\alpha-\beta}}T)\int_T^\infty p_U(t,x,y)dt \\
&\ge c_5\frac{1}{|x-y|^{1-\alpha}}\,(1\wedge u_0) \\
&\quad +c_5\frac{1}{|x-y|^{1-\alpha}}\left(\left((u_0\vee 1)^{1-(1/\alpha)}-1\right)+c_5u_0\left((u_0\vee 1)^{-1/\alpha}-\left(\frac{|x-y|^\alpha}{T}\right)^{1/\alpha}\right)\right) \\
&\ge c_6\frac{1}{|x-y|^{1-\alpha}}\left(u_0\wedge u_0^{1-(1/\alpha)}\right) \\
&= c_6\left((\delta_U(x)\delta_U(y))^{(\alpha-1)/2}\ \wedge\ \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|}\right).
\end{aligned}$$

□

By integrating the lower bound in Theorem 3.6 with respect to $y$ and applying (3.10), we obtain the following lower bound on $\mathbb{E}_x[\tau^a_U]$

**Corollary 3.7** *Suppose that $M>0$ and $U$ is a bounded $C^{1,1}$ open set in $U$. Then there exists a constant $C_{26}=C_{26}(\alpha,\beta,d,M,R_1,diam(U))>0$ such that for every $a\in(0,M]$ and $x\in U$,*

$$\mathbb{E}_x[\tau^a_U]\ge C_{26}\delta_U(x)^{\alpha/2}.$$

We will first establish Theorem 3.5 for small $T$, that is, we will first assume that

$$t\le T_0:=\left(\frac{r_0}{16}\right)^\alpha. \tag{3.14}$$

By integrating (1.3) with respect to $t$ and $y$, we have that for every open set $U$, $\lambda>0$ and $x\in U$,

$$\mathbb{E}_x[\tau^a_U]=\int_U G^a_U(x,z)dz=\lambda^\alpha\int_{\lambda^{-1}U}G^{a\lambda^{(\alpha-\beta)/\beta}}_{\lambda^{-1}U}(\lambda^{-1}x,y)dy=\lambda^\alpha\mathbb{E}_{\lambda^{-1}x}\left[\tau^{a\lambda^{(\alpha-\beta)/\beta}}(\lambda^{-1}U)\right]. \tag{3.15}$$

**Lemma 3.8** *Suppose that $M > 0$, $\kappa \in (0,1)$ and that $(t,x) \in (0,T_0] \times D$ with $\delta_D(x) \le 3t^{1/\alpha} < r_0/4$. Let $z_x \in \partial D$ be such that $|z_x - x| = \delta_D(x)$ and define $\mathbf{n}(z_x) := (x - z_x)/|x - z_x|$. Put $x_1 = z_x + 3t^{1/\alpha}\mathbf{n}(z_x)$ and $B = B(x_1, 3t^{1/\alpha})$. Suppose that $x_0$ is a point on the line segment connecting $z_x$ and $z_x + 6t^{1/\alpha}\mathbf{n}(z_x)$ such that $B(x_0, 2\kappa t^{1/\alpha}) \subset B \setminus \{x\}$. Then for any $b > 0$, there exists a constant $C_{27} = C_{27}(\kappa, \alpha, \beta, r_0, b, M) > 0$ such that for all $a \in (0, M]$*

$$\mathbb{P}_x\left(X^{a,D}_{bt} \in B(x_0, \kappa t^{1/\alpha})\right) \ge C_{27}\, t^{-1/2}\delta_D(x)^{\alpha/2}. \tag{3.16}$$

**Proof.** Let $0 < \kappa_1 \le \kappa$ and assume first that $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \le 3t^{1/\alpha}$. Repeating the proof of Lemma 3.3, we get that, in this case, there exists a constant $c_1 = c_1(\alpha, \beta, \kappa_1, M, r_0, b) > 0$ such that for all $a \in (0, M]$

$$\mathbb{P}_x\left(X^{a,D}_{bt} \in B(x_0, \kappa_1 t^{1/\alpha})\right) \ge c_1 t^{d/\alpha+1} J^a(x, x_0) \ge c_1 \mathcal{A}(d, -\alpha) t^{d/\alpha+1}|x - x_0|^{-d-\alpha}$$

for all $t \le T_0$. Using the fact that $|x - x_0| \in [2\kappa t^{1/\alpha}, 6t^{1/\alpha}]$ we get that for all $a \in (0, M]$,

$$\mathbb{P}_x\left(X^{a,D}_{bt} \in B(x_0, \kappa_1 t^{1/\alpha})\right) \ge c_2 > 0 \tag{3.17}$$

for some constant $c_2 = c_2(\alpha, \beta, \kappa_1, M, r_0, b)$. By taking $\kappa_1 = \kappa$, this shows that (3.16) holds for all $b > 0$ in the case when $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \le 3t^{1/\alpha}$.

So it suffices to consider the case that $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. We now show that there is some $b_0 > 1$ so that (3.16) holds for every $b \ge b_0$ and $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. For simplicity, we assume without loss of generality that $x_0 = 0$ and let $\widehat{B} := B(0, \kappa t^{1/\alpha})$. Let $x_2 = z_x + 4^{-1}\kappa\mathbf{n}(z_x)t^{1/\alpha}$ and $B_2 := B(x_2, 4^{-1}\kappa t^{1/\alpha})$. Observe that since $B(0, 2\kappa t^{1/\alpha}) \subset B \setminus \{x\}$,

$$\kappa/2t^{1/\alpha} \le |y - z| \le 6t^{1/\alpha} \qquad \text{for } y \in B_2 \text{ and } z \in B(0, \kappa t^{1/\alpha}). \tag{3.18}$$

By the strong Markov property of $X^a$ at the first exit time $\tau^a_{B_2}$ from $B_2$ and Lemma 3.1,

$$\begin{aligned} &\mathbb{P}_x\left(X^a_{bt} \in B(0, \kappa t^{1/\alpha})\right) \\ \ge\ &\mathbb{P}_x\left(\tau^a_{B_2} < bt, X^a_{\tau^a_{B_2}} \in B(0, 2^{-1}\kappa t^{1/\alpha}) \text{ and } |X^a_s - X^a_{\tau_{B_2}}| < \kappa/2 \text{ for } s \in [\tau^a_{B_2}, \tau^a_{B_2} + bt^{1/\alpha}]\right) \\ \ge\ &c_3\, \mathbb{P}_x\left(\tau^a_{B_2} < bt \text{ and } X^a_{\tau^a_{B_2}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\right). \end{aligned} \tag{3.19}$$

It follows from (3.15) and Corollary 3.7 that

$$\begin{aligned} \mathbb{P}_x\left(X^a_{\tau^a_{B_2}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\right) &= \int_{B(0,2^{-1}\kappa t^{1/\alpha})}\int_{B_2} G^a_{B_2}(x,y)J^a(y,z)dydz \\ &\ge \mathcal{A}(d,-\alpha)\int_{B(0,2^{-1}\kappa t^{1/\alpha})}\int_{B_2} G^a_{B_2}(x,y)\frac{dydz}{|y-z|^{d+\alpha}} \\ &\ge \frac{c_4}{t}\mathbb{E}_x\left[\tau^a_{B_2}\right] \\ &= c_4 \mathbb{E}_{x/t^{1/\alpha}}\left[\tau^{at^{(\alpha-\beta)/\alpha\beta}}(B(x_2/t^{1/\alpha}, 4^{-1}\kappa))\right] \\ &\ge c_5\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \end{aligned} \tag{3.20}$$

for some positive constants $c_4, c_5$ depending only on $\alpha, \beta, r_0, \kappa$ and $M$. Note that, by (1.3)

$$\int_{B(x_2, 4^{-1}\kappa t^{1/\alpha})} p^a_{B(x_2, 4^{-1}\kappa t^{1/\alpha})}(bt, x, z)dz = \int_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)} p^{at^{(\alpha-\beta)/\alpha\beta}}_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)}(b, t^{-1/\alpha}x, w)dw.$$

Since $at^{(\alpha-\beta)/\alpha\beta} \le MT_0^{(\alpha-\beta)/\alpha\beta}$, by applying Theorem 2.8 to the right hand side of the above display, we get

$$\begin{aligned}\mathbb{P}_x(\tau^a_{B_2} \ge bt) &\le b^{-d/\alpha} \int_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)} \frac{\delta_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)}(t^{-1/\alpha}x)^{\alpha/2}}{\sqrt{b}} dw \\ &\le c_6\, b^{-d/\alpha - 1/2}\, \delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha/2} = c_6\, b^{-d/\alpha-1/2} \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}, \qquad (3.21)\end{aligned}$$

for some positive constant $c_6$ depending only on $\alpha, \beta, r_0, \kappa$ and $M$. Define

$$b_0 := \left(\frac{2c_6}{c_5}\right)^{\frac{2\alpha}{2d+\alpha}}.$$

We have by (3.19)–(3.21) that for $b \ge b_0$,

$$\begin{aligned}\mathbb{P}_x(X^a_{bt} \in \widehat{B}) &\ge c_3 \left(\mathbb{P}_x\big(X^a_{\tau^a_{B_2}} \in B(0, 2^{-1}\kappa t^{1/\alpha})\big) - \mathbb{P}_x\left(\tau^a_{B_2} \ge bt\right)\right) \\ &\ge c_3\,(c_5/2) \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}. \qquad (3.22)\end{aligned}$$

(3.17) and (3.22) show that (3.16) holds for every $b \ge b_0$ and for every $x \in D$ with $\delta_D(x) \le 3t^{1/\alpha}$.

Now we deal with the case $0 < b < b_0$ and $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. If $\delta_D(x) \le 3(bt/b_0)^{1/\alpha}$, we have from (3.16) for the case of $b = b_0$ that

$$\begin{aligned}\mathbb{P}_x\left(X^a_{bt} \in B(x_0, \kappa t^{1/\alpha})\right) &\ge \mathbb{P}_x\left(X^a_{b_0(bt/b_0)} \in B(x_0, \kappa(bt/b_0)^{1/\alpha})\right) \\ &\ge c_7 \left(\frac{\delta_D(x)}{(bt/b_0)^{1/\alpha}}\right)^{\alpha/2} = c_8 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.\end{aligned}$$

If $3(bt/b_0)^{1/\alpha} < \delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$ (in this case $\kappa > 3\cdot 2^4(b/b_0)^{1/\alpha}$), we get (3.16) from (3.17) by taking $\kappa_1 = (b/b_0)^{1/\alpha}$. The proof of the lemma is now complete. □

**Proposition 3.9** *Suppose that $M > 0$ and $(t, x, y) \in (0, T_0] \times D \times D$ with $|x-y| \le t^{1/\alpha}\psi^a(|x-y|)^{1/(d+\alpha)}$, $\delta_D(x) \le 2t^{1/\alpha}$ and $\delta_D(y) \le r_0/5$. Then there exists a constant $C_{28} = C_{28}(\alpha, \beta, M, r_0) > 0$ such that for all $a \in (0, M]$,*

$$p^a_D(t, x, y) \ge C_{28}\, t^{-d/\alpha - 1}\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}. \qquad (3.23)$$

**Proof.** Under the assumptions of the proposition, there are points $z_x,\ z_y \in \partial D$ and $x_0,\ y_0 \in D$ such that $\delta_D(x) = |x - z_x|$, $\delta_D(y) = |y - z_y|$, $\partial B(x_0, 4t^{1/\alpha}) \cap \partial D = \{z_x\}$ and $\partial B(y_0, 4t^{1/\alpha}) \cap \partial D = \{z_y\}$. Observe that

$$\delta_D(x_0) = \delta_D(y_0) = 4t^{1/\alpha} \quad \text{and} \quad |x - x_0|,\ |y - y_0| \in [t^{1/\alpha}, 4t^{1/\alpha}).$$

By the semigroup property, with $B := B(x_0, 4^{-1}t^{1/\alpha})$ and $\widetilde{B} := B(y_0, 4^{-1}t^{1/\alpha})$

$$\begin{aligned} p_D^a(t,x,y) &= \int_D p_D^a(t/3,x,z) \int_D p_D^a(t/3,z,w) p_D^a(t/3,w,y) dw dz \\ &\geq \int_B p_D^a(t/3,x,z) \int_{\widetilde{B}} p_D^a(t/3,z,w) p_D^a(t/3,w,y) dw dz \\ &\geq \inf_{(z,w)\in B\times\widetilde{B}} p_D^a(t/3,z,w) \int_B p_D^a(t/3,x,z) dz \int_{\widetilde{B}} p_D^a(t/3,w,y) dw. \end{aligned}$$

Since for $z \in B$ and $w \in \widetilde{B}$,

$$\delta_D(z) \geq \delta_D(x_0) - |x_0 - z| \geq t^{1/\alpha}, \quad \delta_D(w) \geq \delta_D(y_0) - |y_0 - w| \geq t^{1/\alpha},$$

$$|z-w| \leq |z-x_0| + |x_0-x| + |x-y| + |y-y_0| + |y_0-w| < 10t^{1/\alpha}\psi^a(|x-y|)^{1/(d+\alpha)},$$

by combining Proposition 3.2 and Proposition 3.4, we have that there exists $c_1 = c_1(\alpha,\beta,r_0,M) > 0$ such that for all $a \in (0, M]$,

$$\inf_{(z,w)\in B\times\widetilde{B}} p_D^a(t/3,z,w) \;\geq\; c_1 t^{-d/\alpha}.$$

Since $\delta_D(x) \leq 2t^{1/\alpha} < r_0/8$ and $\delta_D(y) \leq 3t^{1/\alpha}$, we have by Lemma 3.8

$$p_D^a(t,x,y) \;\geq\; c_2\, t^{-d/\alpha-1}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2}$$

for some positive constant $c_2 = c_2(\alpha,\beta,M,r_0)$. □

**Proposition 3.10** *Suppose that $M > 0$ and $(t,x,y) \in (0,T_0] \times D \times D$ with $\delta_D(x) \leq (t/2)^{1/\alpha} \leq \delta_D(y)$ and $|x-y|^\alpha \geq t\psi^a(|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{29} = C_{29}(\alpha,\beta,M,r_0) > 0$ such that for all $a \in (0,M]$,*

$$p_D^a(t,x,y) \;\geq\; C_{29}\, t^{1/2}\, \delta_D(x)^{\alpha/2} J^a(x,y). \tag{3.24}$$

**Proof.** Since $\delta_D(x) \leq (t/2)^{1/\alpha} \leq r_0/16$, there are $z_x \in \partial D$ and $z_0 \in D$ such that $\delta_D(x) = |x - z_x|$ and $\partial B(z_0, 2t^{1/\alpha}) \cap \partial D = \{z_x\}$. Choose $x_0$ in $B(z_0, 2t^{1/\alpha})$ and $\kappa = \kappa(\alpha) \in (0,1)$ such that

$$B\left(x_0, 2\kappa t^{1/\alpha}\right) \subset B\left(z_0, (2-2^{-2/\alpha})t^{1/\alpha}\right) \cap B\left(x, (1-2^{-1-2/\alpha})t^{1/\alpha}\right).$$

Such a ball $B(x_0, 2\kappa t^{1/\alpha})$ always exists because

$$2 \;<\; (2-2^{-1}) + (1-2^{-2}) \;<\; (2-2^{-2/\alpha}) + (1-2^{-1-2/\alpha}).$$

Since $|x-y| \geq t^{1/\alpha}\psi^a(|x-y|)^{1/(d+\alpha)}$, we get that for every $z \in B(x_0, \kappa t^{1/\alpha})$, $\delta_D(z) \geq (t/4)^{1/\alpha}$ and

$$|y-z| \geq |y-x| - |z-x| \geq 2^{-1}(t/4)^{1/\alpha}\psi^a(|x-y|)^{1/(d+\alpha)}.$$

On the other hand, for every $z \in B(x_0, \kappa t^{1/\alpha})$,

$$|z-y| \;\leq\; |z-x| + |x-y| \;\leq\; (1-2^{-1-2/\alpha})t^{1/\alpha} + |x-y| \;<\; 2|x-y|.$$

Thus by the semigroup property and Propositions 3.2 and 3.4, there exist positive constants $c_1, c_2$ and $c_3$ depending only on $(\alpha, \beta, r_0, M)$ such that for all $a \in (0, M]$,

$$
\begin{aligned}
p_D^a(t,x,y) &= \int_D p_D^a(t/2,x,z)p_D^a(t/2,z,y)dz \\
&\geq \int_{B(x_0,\kappa t^{1/\alpha})} p_D^a(t/2,x,z)p_D^a(t/2,z,y)dz \\
&\geq c_1 t \int_{B(x_0,\kappa t^{1/\alpha})} p_D^a(t/2,x,z)J^a(z,y)dz \\
&\geq c_2 t j^a(2|x-y|) \int_{B(x_0,\kappa t^{1/\alpha})} p_D^a(t/2,x,z)dz \\
&\geq c_3 t j^a(|x-y|)\,\mathbb{P}_x\left(X_{t/2}^{a,D} \in B(x_0,\kappa t^{1/\alpha})\right).
\end{aligned}
$$

Applying Lemma 3.8, we arrive at the conclusion of the proposition. □

**Proposition 3.11** *Suppose that $M > 0$ and $(t, x, y) \in (0, T_0] \times D \times D$ with*

$$\max\{\delta_D(x),\, \delta_D(y)\} \leq (t/2)^{1/\alpha} \leq |x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}.$$

*Then there exists a constant $C_{30} = C_{30}(\alpha, \beta, M, r_0) > 0$ such that for all $a \in (0, M]$,*

$$p_D^a(t,x,y) \,\geq\, C_{30}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}J^a(x,y). \tag{3.25}$$

**Proof.** As in the first paragraph of the proof of Proposition 3.9, set $z_x \in \partial D$ and $x_0 \in D$ so that $|x - z_x| = \delta_D(x)$ and $\partial B(x_0, 3t^{1/\alpha}) \cap \partial D = \{z_x\}$. Let $\kappa := 1 - 2^{-1/\alpha}$. Note that for every $z \in B(x_0, \kappa t^{1/\alpha})$, we have

$$4t^{1/\alpha} \geq \delta_D(z) \geq 2(t/2)^{1/\alpha}.$$

If $|y - z| \leq t^{1/\alpha}\psi^a(|y - z|)^{1/(d+\alpha)}$, we can apply Proposition 3.9 and the assumption

$$(t/2)^{1/\alpha} \leq |x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$$

to get that

$$p^a(t/2,z,y) \geq c_1 t^{1/2}\,\delta_D(y)^{\alpha/2}J^a(x,y).$$

If $|y - z| \geq t^{1/\alpha}\psi^a(|y - z|)^{1/(d+\alpha)}$, we can apply Proposition 3.10 to get that

$$p^a(t/2,z,y) \geq c_2 t^{1/2}\,\delta_D(y)^{\alpha/2}J^a(y,z).$$

For $z \in B(x_0, \kappa t^{1/\alpha})$, we have

$$
\begin{aligned}
|z-y| &\leq |x-y| + |x_0-x| + |x_0-z| \,\leq\, |x-y| + 4t^{1/\alpha} \\
&\leq |x-y| + 2^{2+1/\alpha}(t/2)^{1/\alpha}\psi^a(|x-y|)^{1/(d+\alpha)} \\
&\leq (1+2^{2+1/\alpha})|x-y|.
\end{aligned}
$$

Thus if $|y - z| \geq t^{1/\alpha}\psi^a(|y - z|)^{1/(d+\alpha)}$, we have

$$p^a(t/2,z,y) \geq c_3 t^{1/2}\,\delta_D(y)^{\alpha/2}J^a(x,y).$$

Consequently we have for all $z \in B(x_0, \kappa t^{1/\alpha})$

$$p^a(t/2, z, y) \geq c_4 t^{1/2}\, \delta_D(y)^{\alpha/2} J^a(x, y).$$

Hence by the semigroup property we get

$$\begin{aligned}
p_D^a(t, x, y) &= \int_D p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\
&\geq \int_{B(x_0, \kappa t^{1/\alpha})} p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\
&\geq c_4 \int_{B(x_0, \kappa t^{1/\alpha})} p_D^a(t/2, x, z) t^{1/2}\, \delta_D(y)^{\alpha/2} J^a(x, y) dz \\
&= c_4\, t^{1/2}\, \delta_D(y)^{\alpha/2} J^a(x, y) \int_{B(x_0, \kappa t^{1/\alpha})} p_D^a(t/2, x, z) dz \\
&= c_4\, t^{1/2}\, \delta_D(y)^{\alpha/2} J^a(x, y)\, \mathbb{P}_x \left( X_{t/2}^{a,D} \in B(x_0, \kappa t^{1/\alpha}) \right) \\
&= c_5\, \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} J^a(x, y).
\end{aligned}$$

We arrive at the conclusion of the proposition. □

**Proof of Theorem 3.5.** In this proof, for two non-negative functions $f$ and $g$, the notation $f \asymp g$ means that there are positive constants $c_1$ and $c_2$ depending only on $M$, $d$, $\alpha$ and $\beta$ so that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for $f$ and $g$.

We first assume that $t \leq T_0$.

1. We first consider the case $|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)} \leq t^{1/\alpha}$. We claim that in this case

$$p_D(t, x, y) \geq c t^{-d/\alpha} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right). \tag{3.26}$$

This will be proved by considering the following two possibilities.

(a) $\max\{\delta_D(x), \delta_D(y), |x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}\} \leq t^{1/\alpha}$: Proposition 3.9 and symmetry yield (3.26)

(b) $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq |x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$:
If $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$, (3.26) follows from Proposition 3.2.
If $\min\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x-y| \leq t^{1/\alpha} < 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$,

$$\frac{t\psi^a(|x-y|)}{|x-y|^{d+\alpha}} \asymp t^{-d/\alpha} \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right).$$

If $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$, $\min\{\delta_D(x), \delta_D(y)\} < t^{1/\alpha}$ and $|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)} \leq t^{1/\alpha} < 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$,

$$\left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \asymp \left( 1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left( 1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right)$$

Thus by combining Proposition 3.4 and Proposition 3.9, we get (3.26) for the case of $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x-y| \leq t^{1/\alpha} < 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$.

2. Now we consider the case $|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$ and claim that

$$p_D(t,x,y) \geq c\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) tj^a(|x-y|). \tag{3.27}$$

(a) $\min\{\delta_D(x), \delta_D(y)\} \leq (t/2)^{1/\alpha}$ and $|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$: By symmetry we can assume $\delta_D(x) \leq (t/2)^{1/\alpha}$. Thus combining Propositions 3.10 and 3.11, we have (3.27) for this case.

(b) $\min\{\delta_D(x), \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$. In this case, clearly

$$\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \asymp \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right).$$

Thus Proposition 3.4 yields (3.27).

We have arrived at the conclusion of Theorem 3.5 for $t \leq T_0$.

Assume $T = 2T_0$. Recall that $T_0 = (r_0/16)^\alpha$. For $(t,x,y) \in (T_0, 2T_0] \times D \times D$, let $x_0, y_0 \in D$ be such that $\max\{|x-x_0|, |y-y_0|\} < r_0$ and $\min\{\delta_D(x_0), \delta_D(y_0)\} \geq r_0/2$. Note that, since

$$j^a(r) \leq c_1 j^a(2r), \qquad \text{for all } r > 0, \tag{3.28}$$

if $|x-y| \geq 4r_0$, then $\frac{1}{2}|x-y| \leq |x-y| - 2r_0 \leq |x_0-y_0| \leq |x-y| + 2r_0 \leq \frac{3}{2}|x-y|$, and so $c_2^{-1}J^a(x_0,y_0) \leq J^a(x,y) \leq c_2 J^a(x_0,y_0)$ for some constant $c_2 = c_2(M) > 1$. Thus by considering the cases $|x-y| \geq 4r_0$ and $|x-y| < 4r_0$, we have

$$(t/2)^{-d/\alpha} \wedge \frac{tJ^a(x_0,y_0)}{2} \geq c_3\left(t^{-d/\alpha} \wedge (tJ^a(x,y))\right). \tag{3.29}$$

Similarly, there is a positive constant $c_2$ such that

$$\begin{aligned}
(t/3)^{-d/\alpha} \wedge \frac{tJ^a(x,z)}{3} &\geq c_4\left((t/(12))^{-d/\alpha} \wedge \frac{tJ^a(x_0,z)}{12}\right), \quad z \in D, \\
(t/3)^{-d/\alpha} \wedge \frac{tJ^a(w,y)}{3} &\geq c_4\left((t/(12))^{-d/\alpha} \wedge \frac{tJ^a(w,y_0)}{12}\right), \quad w \in D.
\end{aligned} \tag{3.30}$$

By (3.30) and the lower bound estimate in Theorem 3.5 for $p_D^a$ on $(0,T_0] \times D \times D$, we have

$$\begin{aligned}
p_D^a(t,x,y) &= \int_{D\times D} p_D^a(t/3,x,z)p_D^a(t/3,z,w)p_D^a(t/3,w,y)dzdw \\
&\geq c_5\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/3}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/3}}\right)\int_{D\times D}\left((t/3)^{-d/\alpha} \wedge \frac{tJ^a(x,z)}{3}\right)\left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\
&\qquad \cdot p_D^a(t/3,z,w)\left((t/3)^{-d/\alpha} \wedge \frac{tJ^a(w,y)}{3}\right)\left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dzdw \\
&\geq c_6\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\int_{D\times D}\left(\left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(x_0,z)}{12}\right)\left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\
&\qquad \cdot p_D^a(t/3,z,w)\left(\left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(w,y_0)}{12}\right)\left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dzdw
\end{aligned}$$

for some positive constants $c_i, i = 3, 4$. Let $D_1 := \{z \in D : \delta_D(z) > r_0/4\}$. Clearly, $x_0, y_0 \in D_1$ and

$$\min\{\delta_{D_1}(x_0), \delta_{D_1}(y_0)\} \geq r_0/4 = 4(T_0)^{1/\alpha} \geq 4(t/2)^{1/\alpha}. \tag{3.31}$$

By (1.4) and (3.29), we have

$$\begin{aligned}
&\int_{D\times D} \left( \left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(x_0,z)}{12} \right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\
&\qquad \cdot\, p^a_D(t/3,z,w) \left( \left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(w,y_0)}{12} \right) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dzdw \\
&\geq c_7 \int_{D_1\times D_1} \left( \left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(x_0,z)}{12} \right) p^a_D(t/3,z,w) \left( \left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(w,y_0)}{12} \right) dzdw \\
&\geq c_8 \int_{D_1\times D_1} p^a(t/(12),x_0,z) p^a_{D_1}(t/3,z,w) p^a(t/(12),w,y_0) dzdw \\
&\geq c_8 \int_{D_1\times D_1} p^a_{D_1}(t/(12),x_0,z) p^a_{D_1}(t/3,z,w) p^a_{D_1}(t/(12),w,y_0) dzdw \\
&= c_8\, p^a_{D_1}(t/2,x_0,y_0) \geq c_9 \left( (t/2)^{-d/\alpha} \wedge \frac{tJ^a(x_0,y_0)}{2} \right) \geq c_{10} \left( t^{-d/\alpha} \wedge (tJ^a(x,y)) \right)
\end{aligned}$$

for some positive constants $c_i, i = 7, \cdots, 10$. Here both Propositions 3.2 and 3.4 are used in the third inequality in view of (3.31). By repeating the argument above, we have proved Theorem 3.5. □

**Proof of Theorem 1.1.** Theorems 2.8 and 3.5 give Theorem 1.1(i). By [15, 18], for any bounded open set $D$ in $\mathbb{R}^d$, $X^{a,D}$ is intrinsically ultracontractive. Since the function $\psi^a(|x-y|)$ is bounded above and below by a positive constant if $D$ is bounded, using the intrinsic ultracontractivity of $X^a$ on bounded open set and the continuity of eigenvalues proved in [10], the proof of Theorem 1.1 (ii) is almost identical to the one of [5, Theorem 1.1(ii)]. We omit the details. □

**Proof of Corollary 1.2.** The lower bound estimate in (1.5) follows from (3.10) and Theorem 3.6.

Since the function $\psi^a(|x-y|)$ is bounded above and below by a positive constant if $D$ is bounded, by integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to $t$, the proof of the upper bound estimate in (1.5) is identical to the one of [3, Corollary 1.2] so we omit its details here. □

**Theorem 3.12 (Uniform boundary Harnack principle)** *Suppose $M, R \in (0,\infty)$ and that $D$ is an open set in $\mathbb{R}^d$, $z \in \partial D$, $r \in (0,R)$ and that $B(A,\kappa r) \subset D \cap B(z,r)$. There exists $C_{31} = C_{31}(d,\alpha,\beta,\kappa,M,R) > 1$ such that for every $a \in (0,M]$, and any functions $u, v \geq 0$ on $\mathbb{R}^d$, positive regular harmonic for $X^a$ in $D \cap B(z,2r)$ and vanishing on $D^c \cap B(z,2r)$, we have*

$$C_{31}^{-1} \frac{u(A)}{v(A)} \leq \frac{u(x)}{v(x)} \leq C_{31} \frac{u(A)}{v(A)}, \qquad x \in D \cap B(z,r).$$

**Proof.** Note that by the approximate scaling property in (1.3), we have for every $r > 0$.

$$G^a_{B(0,r)}(x,y) = r^{\alpha-d} G^{ar^{(\alpha-\beta)/\beta}}_{B(0,1)}(x/r, y/r). \tag{3.32}$$

Thus applying [3, Corollary 1.2] and our Corollary 1.2 to (3.32), we have that for every $R, M > 0$, there exists $c = c(\alpha, \beta, R, M) > 0$ such that, for every $a \in (0, M]$ and $0 < r \le R$

$$c^{-1} G_{B(x_0,r)}(x,y) \le G^a_{B(x_0,r)}(x,y) \le c G_{B(x_0,r)}(x,y), \quad \forall x, y \in B(x_0, r). \tag{3.33}$$

Using (3.33), we can get uniform estimates on the Poisson kernel

$$K^a_{B(x_0,r)}(x,z) := \int_{B(x_0,r)} G^a_{B(x_0,r)}(x,y) J^a(y,z) dy$$

of $B(x_0, r)$ with respect to $X^a$ for $r \in (0, R]$. In particular, for $r < |z - x_0| < 2R$, $K^a_{B(x_0,r)}(x,z)$ is comparable to $K_{B(x_0,r)}(x,z)$, the Poisson kernel of $B(x_0, r)$ with respect to $X$ for $r \in (0, R]$. Then using the uniform estimates on $K^a_{B(x_0,r)}(x,z)$ and (3.33) we can easily see that [27, Lemma 3.3] can be proved in the same way. Using the uniform estimates on the Poisson kernel of $B(x_0, r)$, (3.28) and (3.33) we can adapt the argument in [1, 19, 27] to get our uniform boundary Harnack principle. We omit the details. □

**Proof of Theorem 1.3** First we observe that Harnack inequality holds for the process $X := X^1$ by [21]. That is, there exists a constant $c_1 = c_1(\alpha, \beta, M) > 0$ such that for any $r \in (0, M^{\beta/(\alpha-\beta)}]$, $x_0 \in \mathbb{R}^d$ and any function $v \ge 0$ harmonic in $B(x_0, r)$ with respect to $X$, we have

$$v(x) \le c_1 v(y) \qquad \text{for all } x, y \in B(x_0, r/2). \tag{3.34}$$

Note that for any $a \in (0, M]$, $X^a$ has the same distribution as $\{\lambda X_{\lambda^{-\alpha} t}, t \ge 0\}$, where $\lambda = a^{\beta/(\beta-\alpha)} \ge M^{\beta/(\beta-\alpha)}$. Consequently, if $u$ is harmonic in $B(x_0, r)$ with respect to $X^a$, where $r \in (0, 1]$, then $v(x) := u(\lambda x)$ is harmonic in $B(\lambda^{-1} x_0, \lambda^{-1} r)$ with respect to $X$ and $\lambda^{-1} r \le M^{\beta/(\beta-\alpha)}$. So by (3.34)

$$u(\lambda x) = v(x) \le c_1 v(y) = c_1 u(\lambda y) \qquad \text{for all } x, y \in B(\lambda^{-1} x_0, \lambda^{-1} r/2).$$

That is,

$$u(x) \le c_1 u(y) \qquad \text{for all } x, y \in B(x_0, r/2). \tag{3.35}$$

In other words, uniform Harnack inequality holds (for every $r \le 1$) for the family of processes $\{X^a, a \in (0, M]\}$.

Since $D$ is $C^{1,1}$ open set, there exists $r_0 \le R_0$ such that the following holds: for every $Q \in \partial D$ and $r \le r_0$ there is a ball $B = B(z^r_Q, r)$ of radius $r$ such that $B \subset \mathbb{R}^d \setminus \overline{D}$ and $\partial B \cap \partial D = \{Q\}$. In addition, it follows [23, Lemma 2.2] that, for each $Q \in \partial D$, we can choose a constant $c_2 = c_2(d, \Lambda) \in (0, 1/8]$ and a bounded $C^{1,1}$ open set $U_Q$ with uniform characteristics $(R_*, \Lambda_*)$ depending on $(R_0, \Lambda)$ such that $B(Q, c_2 r_0) \cap D \subset U_Q \subset B(Q, r_0) \cap D$ and

$$\delta_D(y) = \delta_{U_Q}(y) \quad \text{for every } y \in B(Q, c_2 r_0) \cap D. \tag{3.36}$$

Assume $a \in [0, M]$, $r \in (0, c_2 r_0]$, $Q \in \partial D$ and $u$ is nonnegative function in $\mathbb{R}^d$ harmonic in $D \cap B(Q, r)$ with respect to $X^a$ and vanishes continuously on $D^c \cap B(Q, r)$. Let $z_Q := z^{c_2 r_0}_Q$. By the boundary Harnack principle (Theorem 3.12), there exists a constant $c_3 = c_3(\alpha, \beta, a, R_0, \Lambda, M)$ such that

$$\frac{u(x)}{u(y)} \le c_3 \frac{G^a_{U_Q}(x, z_Q)}{G^a_{U_Q}(x, z_Q)} \qquad \text{for every } x, y \in B(Q, r/8) \cap D.$$

Now applying Corollary 1.2 to $G^a_{U_Q}(x, z_Q)$ and $G^a_{U_Q}(x, z_Q)$, then using (3.36), we conclude that

$$\frac{u(x)}{u(y)} \le c_4 \frac{\delta^{\alpha/2}_{U_Q}(x)}{\delta^{\alpha/2}_{U_Q}(y)} = c_4 \frac{\delta^{\alpha/2}_{D}(x)}{\delta^{\alpha/2}_{D}(y)} \qquad \text{for every } x, y \in B(Q, c_2 r) \cap D \tag{3.37}$$

for some $c_4 = c_4(\alpha, \beta, a, R_0, \Lambda, M) > 0$.

Now Theorem 1.3 follows from the uniform Harnack principle in (3.35), (3.37) and a standard chain argument. □

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