Dirichlet Heat Kernel Estimates for $\Delta^{\alpha/2} + \Delta^{\beta/2}$

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Abstract

For $d \geq 1$ and $0 < \beta < \alpha < 2$, consider a family of pseudo differential operators $\{\Delta^\alpha + a^\beta \Delta^{\beta/2};\ a \in [0,1]\}$ on $\mathbb{R}^d$ that evolves continuously from $\Delta^{\alpha/2}$ to $\Delta^{\alpha/2} + \Delta^{\beta/2}$. It gives arise to a family of Lévy processes $\{X^a, a \in [0,1]\}$ on $\mathbb{R}^d$, where each X^a is the independent sum of a symmetric α-stable process and a symmetric β-stable process with weight a. For any $C^{1,1}$ open set $D \subset \mathbb{R}^d$, we establish explicit sharp two-sided estimates, which are uniform in $a \in (0,1]$, for the transition density function of the subprocess $X^{a,D}$ of X^a killed upon leaving the open set D. The infinitesimal generator of $X^{a,D}$ is the non-local operator $\Delta^\alpha + a^\beta \Delta^{\beta/2}$ with zero exterior condition on D^c. As consequences of these sharp heat kernel estimates, we obtain uniform sharp Green function estimates for $X^{a,D}$ and uniform boundary Harnack principle for X^a in D with explicit decay rate.

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1 Introduction

It is well-known that, for a second order elliptic differential operator $\mathcal{L}$ on $\mathbb{R}^d$ satisfying some natural conditions, there is a diffusion process X on $\mathbb{R}^d$ with $\mathcal{L}$ as its infinitesimal generator. The fundamental solution $p(t,x,y)$ of $\partial_t u = \mathcal{L}u$ (also called the heat kernel of $\mathcal{L}$) is the transition density function of X. Thus obtaining sharp two-sided estimates for $p(t,x,y)$ is a fundamental problem in both analysis and probability theory. Such relationship is also true for a large class of Markov processes with discontinuous sample paths, which constitute an important family of stochastic processes in probability theory. They have been widely used in various applications.

One of the most important and most widely used family of Markov processes is the family of (rotationally) symmetric α-stable processes on $\mathbb{R}^d$, $0 < \alpha \leq 2$. A symmetric α-stable process $X = \{X_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$ is a Lévy process such that

$$\mathbb{E}_x \left[e^{i\xi\cdot(X_t - X_0)} \right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

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When $\alpha = 2$, X is a Brownian motion on $\mathbb{R}^d$ whose infinitesimal generator is the Laplacian Δ. When $0 < \alpha < 2$, the infinitesimal generator of a symmetric α-stable process X on $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2}u(x) = \mathcal{A}(d,-\alpha) \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : \, |y-x|>\varepsilon\}} (u(y)-u(x)) \frac{dy}{|x-y|^{d+\alpha}} \tag{1.1}$$

for some constant $\mathcal{A}(d,-\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Here and in the sequel, we use $:=$ as a way of definition. Here Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$.

Two-sided heat kernel estimates for diffusions on $\mathbb{R}^d$ have a long history and many beautiful results have been established. See [16, 18] and the references therein. But, due to the complication near the boundary, two-sided estimates for the transition density functions of killed diffusions in a domain D (equivalently, the Dirichlet heat kernels) have been established only recently. See [17, 18, 19] for upper bound estimates and [31] for the lower bound estimates of the Dirichlet heat kernels in bounded $C^{1,1}$ domains. In a recent paper [6], we succeeded in establishing sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ with zero exterior condition on D^c (or equivalently, the transition density function of the killed α-stable process) in any $C^{1,1}$ open set.

The approach developed in [6] provides a road map for establishing sharp two-sided heat kernel estimates of some other jump processes in open subsets of $\mathbb{R}^d$. In [7], the ideas of [6] were adapted to establish two-sided heat kernel estimates of censored stable-like processes in $C^{1,1}$ open subsets of $\mathbb{R}^d$. One of the main tools used in [7] is the boundary Harnack principle established in [2] and [21]. Very recently in [4, 5], the heat kernel of the fractional Laplacian in non-smooth open set was discussed.

In [8], the ideas of [6] were adapted to establish two-sided heat kernel estimates of relativistic stable processes in $C^{1,1}$ open subsets of $\mathbb{R}^d$. One of the main facts used in [8] is that relativistic stable processes can be regarded as perturbations of symmetric stable processes in bounded open sets and therefore the Green functions of killed relativistic stable processes in bounded open sets are comparable to the Green functions of killed stable processes in the same open sets.

The goal of this paper is to establish sharp two-sided heat kernel estimates for a Lévy process Z that is the independent sum of an α-stable process X and a β-stable process Y, $0 < \beta < \alpha < 2$, in $C^{1,1}$ open subsets of $\mathbb{R}^d$. The infinitesimal generator of the Lévy process Z is $\Delta^{\alpha/2} + \Delta^{\beta/2}$. Let $p^1_D(t,x,y)$ and $G^1_D(x,y)$ to denote the transition density function and the Green function of the subprocess Z^D of Z killed upon exiting a $C^{1,1}$ open set $D \subset \mathbb{R}^d$. Let $p_D(t,x,y)$ and $G_D(x,y)$ denote the transition density function and Green function of the subprocess X^D of X killed upon exiting D. Intuitively, one expects the following Duhamel's formulas (or Trotter-Kato formula) hold:

$$p^1_D(t,x,y) = p_D(t,x,y) + \int_0^t \int_D p^1_D(s,x,z)\Delta_z^{\beta/2} p_D(t-s,z,y)dz, \tag{1.2}$$

$$G^1_D(x,y) = G_D(x,y) + \int_D G^1_D(x,z)\Delta_z^{\beta/2} G_D(z,y)dz. \tag{1.3}$$

Although the sharp two-sided estimates on $p_D(t,x,y)$ have been derived recently in [6] while the estimates on $G_D(x,y)$ were obtained sometime ago in [14, 23], no sharp estimates on $\Delta_z^{\beta/2} p_D(s,z,y)$

and $\Delta_z^{\beta/2} G_D(z,y)$ are known and sharp estimates seem to be quite challenging to get, at least for $\Delta_z^{\beta/2} p_D(s,z,y)$. Hence at this stage, Duhamel's formula (1.2) does not seem to be useful in deriving sharp two-sided estimates on $p^1_D(t,x,y)$.

The Lévy process Z runs on two different scales: on the small spatial scale, the α component dominates, while on the large spatial scale the β component takes over. Both components play essential roles, and so in general this process can not be regarded as a perturbation of the α-stable process or of the β-stable process. Note that this process can not be obtained from symmetric stable processes through a combination of Girsanov transform and Feynman-Kac transform. So the method of [8] can not be used to establish the comparability of the Green functions of this process and the Green functions of symmetric stable processes in bounded open sets. Since the differences of the Lévy measures of this process and those of symmetric stable processes have infinite total mass, the methods of [20, 25] also could not be used to establish the comparability of the Green functions of these processes and the Green functions of symmetric stable processes in bounded open sets. The approach of this paper will be described in the second paragraph below after the statement of Corollary 1.2.

Let us first recall some basic facts about the independent sum of stable processes and state our main result.

Throughout the remainder of this paper, we assume that $d \geq 1$ and $0 < \beta < \alpha < 2$. The Euclidean distance between x and y will be denoted as $|x-y|$. We will use $B(x,r)$ to denote the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$

Suppose X is a symmetric α-stable process and Y is a symmetric β-stable process on $\mathbb{R}^d$ and that X and Y are independent. For any $a \geq 0$, we define X^a by $X^a_t := X_t + aY_t$. We will call the process X^a the independent sum of the symmetric α-stable process X and the symmetric β-stable process Y with weight a. The infinitesimal generator of X^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Let $p^a(t,x,y)$ denote the transition density of X^a (or equivalently the heat kernel of $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$) with respect to the Lebesgue measure on $\mathbb{R}^d$. We will use $p(t,x,y) = p^0(t,x,y)$ to denote the transition density of $X = X^0$. Recently it is proven in [13] that

$$p^1(t,x,y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta}\right) \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{t}{|x-y|^{d+\beta}}\right) \quad \text{on } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{1.4}$$

Here and in the sequel, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$; for any two positive functions f and g, $f \asymp g$ means that there is a positive constant $c \geq 1$ so that $c^{-1} g \leq f \leq c\, g$ on their common domain of definition.

For every open subset $D \subset \mathbb{R}^d$, we denote by $X^{a,D}$ the subprocess of X^a killed upon leaving D. The infinitesimal generator of $X^{a,D}$ is $(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2})|_D$, the sum of two fractional Laplacians in D with zero exterior condition. It is known (see [13]) that $X^{a,D}$ has a Hölder continuous transition density $p^a_D(t,x,y)$ with respect to the Lebesgue measure.

Unlike the case of the symmetric α-stable process $X := X^0$, X^a does not have the stable scaling for $a > 0$. Instead, the following approximate scaling property is true and will be used several times in the rest of this paper: If $\{X^{a,D}_t, t \geq 0\}$ is the subprocess of X^a killed upon leaving D, then $\{\lambda^{-1} X^{a,D}_{\lambda^\alpha t}, t \geq 0\}$ is the subprocess of $\{X^{a\lambda^{(\alpha-\beta)/\beta}}_t, t \geq 0\}$ killed upon leaving $\lambda^{-1} D := \{\lambda^{-1} y : y \in D\}$. Consequently, for any $\lambda > 0$, we have

$$p^{a\lambda^{(\alpha-\beta)/\beta}}_{\lambda^{-1}D}(t,x,y) = \lambda^d p^a_D(\lambda^\alpha t, \lambda x, \lambda y) \qquad \text{for } t > 0 \text{ and } x, y \in \lambda^{-1} D. \tag{1.5}$$

In particular, letting $a=1$, $\lambda=a^{\beta/(\alpha-\beta)}$ and $D=\mathbb{R}^d$, we get

$$p^a(t,x,y)=a^{\frac{\beta d}{\alpha-\beta}}p^1(a^{\frac{\alpha\beta}{\alpha-\beta}}t,a^{\frac{\beta}{\alpha-\beta}}x,a^{\frac{\beta}{\alpha-\beta}}y) \qquad \text{for } t>0 \text{ and } x,y\in\mathbb{R}^d.$$

So we deduce from (1.4) that there exists a constants $C>1$ depending only on d, α and β such that for every $a>0$ and $(t,x,y)\in(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$

$$C^{-1}f^a(t,x,y)\le p^a(t,x,y)\le Cf^a(t,x,y), \tag{1.6}$$

where

$$f^a(t,x,y):=\Big((a^\beta t)^{-d/\beta}\wedge t^{-d/\alpha}\Big)\wedge\left(\frac{t}{|x-y|^{d+\alpha}}+\frac{a^\beta t}{|x-y|^{d+\beta}}\right).$$

The purpose of this paper is to establish the following two-sided sharp estimates on $p^a_D(t,x,y)$ in Theorem 1.1 for every $t>0$. To state this theorem, we first recall that an open set D in $\mathbb{R}^d$ (when $d\ge 2$) is said to be a (uniform) $C^{1,1}$ open set if there exist a localization radius $R_0>0$ and a constant $\Lambda_0>0$ such that for every $z\in\partial D$, there exist a $C^{1,1}$-function $\phi=\phi_z:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\phi(0)=0$, $\nabla\phi(0)=(0,\dots,0)$, $\|\nabla\phi\|_\infty\le\Lambda_0$, $|\nabla\phi(x)-\nabla\phi(z)|\le\Lambda_0|x-z|$, and an orthonormal coordinate system CS_z with its origin at z such that

$$B(z,R_0)\cap D=\{y=(\widetilde{y},\,y_d) \text{ in } CS_z:|y|<R_0, y_d>\phi(\widetilde{y})\}.$$

The pair (R_0,Λ_0) is called the characteristics of the $C^{1,1}$ open set D. Note that a $C^{1,1}$ open set D with characteristics (R_0,Λ_0) can be unbounded and disconnected; the distance between two distinct components of D is at least R_0. Let $\delta_{\partial D}(x)$ be the Euclidean distance between x and ∂D. It is well known that any $C^{1,1}$ open set D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0<R_0$ such that for every $x\in D$ with $\delta_{\partial D}(x)<r_0$ and $y\in\mathbb{R}^d\setminus\overline{D}$ with $\delta_{\partial D}(y)<r_0$, there are $z_x,z_y\in\partial D$ so that $|x-z_x|=\delta_{\partial D}(x)$, $|y-z_y|=\delta_{\partial D}(y)$ and that $B(x_0,r_0)\subset D$ and $B(y_0,r_0)\subset\mathbb{R}^d\setminus\overline{D}$ for $x_0=z_x+r_0(x-z_x)/|x-z_x|$ and $y_0=z_y+r_0(y-z_y)/|y-z_y|$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive.

Theorem 1.1 *Suppose $M>0$. Let D be a $C^{1,1}$ open subset of $\mathbb{R}^d$ with characteristics (R_0,Λ_0) and $\delta_D(x)$ the Euclidean distance between x and D^c.*

(i) *For every $T>0$, there is a constant $C_1=C_1(R_0,\Lambda_0,M,\alpha,\beta,T,d)\ge 1$ such that for every $a\in(0,M]$,*

$$C_1^{-1}\,f^a_D(t,x,y)\le p^a_D(t,x,y)\le C_1\,f^a_D(t,x,y),$$

where

$$f^a_D(t,x,y):=\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge\left(\frac{t}{|x-y|^{d+\alpha}}+\frac{a^\beta t}{|x-y|^{d+\beta}}\right)\right).$$

(ii) *Suppose in addition that D is bounded. For every $T>0$, there is a constant $C_2\ge 1$ depending only on diam(D), $R_0,\Lambda_0,M,\alpha,\beta,d$ and T so that for every $a\in(0,M]$ and $(t,x,y)\in[T,\infty)\times D\times D$,*

$$C_2^{-1}\,e^{-\lambda_1 t}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2}\;\le\;p^a_D(t,x,y)\;\le\;C_2\,e^{-\lambda_1 t}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2},$$

where $\lambda_1>0$ is the smallest eigenvalue of $-(\Delta^{\alpha/2}+a^\beta\Delta^{\beta/2})|_D$.

In the above theorem we assumed that the weight a is contained in the compact interval $[0, M]$ for some $M > 0$. Without this assumption, the estimates in the above theorem are not valid. As one can see from (1.6), as $a \uparrow \infty$, the β component will play the dominating role.

The above heat kernel estimates are uniform in $a \in (0, M]$. Letting $a \to 0$, Theorem 1.1 recovers the heat kernel estimates for symmetric α-stable processes obtained in [6]. By integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to t, we obtain the following estimates on the Green function $G^a_D(x,y) := \int_0^\infty p^a_D(t,x,y)dt$.

Corollary 1.2 *Suppose $M > 0$. For any bounded $C^{1,1}$ open set D with characteristics (R_0, Λ_0) in $\mathbb{R}^d$, there is a constant $C_3 = C_3(diam(D), R_0, \Lambda_0, M, \alpha, \beta) \geq 1$ so that for every $a \in (0, M]$,*

$$C_3^{-1}\, g_D(x,y) \leq G^a_D(x,y) \leq C_3\, g_D(x,y) \qquad \text{for } x, y \in D,$$

where

$$g_D(x,y) := \begin{cases} \left(1 \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) |x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log\left(1 + \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d = 1 = \alpha, \\ \big(\delta_D(x)\delta_D(y)\big)^{(\alpha-1)/2} \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha. \end{cases} \tag{1.7}$$

To the best of our knowledge, the above Green function estimates are new, which says that for any bounded $C^{1,1}$ open set D, G^a_D is comparable to the Green function G^0_D of the symmetric α stable process in D. The two-sided estimates for G^0_D were first established independently in [14] and [23], when $d \geq 2$, and in [3] and [6] for $d = 1$.

Theorem 1.1(i) will be established through Theorems 2.8 and 3.5, which give the upper bound and lower bound estimates respectively. Unlike [6, 8], Theorem 1.1(ii) is not a consequence of the intrinsic ultracontractivity of X^a in a bounded open set since constant C_3 depends on D only through its diameter and $C^{1,1}$ characteristics. We will prove Theorem 1.1(ii) using Theorem 1.1(i) and some elementary facts from the spectral theory of compact self-adjoint operators and the estimates of eigenvalues established in [15].

In fact, the upper bound estimates in both Theorem 1.1 and Corollary 1.2 hold for any open set D with a weak version of the *uniform exterior ball condition* in place of the $C^{1,1}$ condition, while the lower bound estimates in both Theorem 1.1 and Corollary 1.2 hold for any open set D with the *uniform interior ball condition* in place of the $C^{1,1}$ condition (see Theorems 2.8 and 3.5, and the proofs for Theorem 1.1(ii) and Corollary 1.2).

Although we follow the general strategy we developed in [6], there are several new difficulties to overcome in obtaining two-sided Dirichlet heat kernel estimates for X^a. First, X^a is not self-similar; it is the mixture of two stable processes with two different parameters. Secondly, even though the boundary Harnack principle has been extended in [22] to a large class of pure jump Lévy processes including X^a, the explicit decay rate of harmonic functions of X^a near the boundary of D was unknown. Instead, following the approach in [11], we establish necessary estimates using suitably chosen subharmonic and superharmonic functions of the process X^a to get the desired boundary decay rate for X^a. As in [11], we need to use the finite range (or truncated) symmetric β-stable process $\widehat{Y}^\lambda$ obtained from Y by suppressing all its jumps of size larger than λ. The infinitesimal generator of $\widehat{Y}^\lambda$ is

$$\widehat{\Delta}^{\beta/2}_\lambda u(x) := \mathcal{A}(d,-\beta) \lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^d:\, \varepsilon<|y-x|\le\lambda\}} (u(y)-u(x)) \frac{dy}{|x-y|^{d+\beta}}. \tag{1.8}$$

When $\lambda = 1$, we will simply denote $\widehat{\Delta}^{\beta/2}_\lambda$ by $\widehat{\Delta}^{\beta/2}$. We first establish the desired estimates for the Lévy process $\widehat{X}^a := X + a\widehat{Y}^{1/a}$. The infinitesimal generator of $\widehat{X}^a$ is $\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}$. The desired estimates for $X^a = X + aY$ can then be obtained by adding back those jumps of Y of size larger than $1/a$. To obtain the lower bound of $p^a_D(t,x,y)$, we use the Dirichlet heat kernel estimates for the fractional Laplacian in [6] and a comparison of the killed subordinate stable process with the subordinate killed stable process where we will use some of the results obtained in [29]. Also some ideas in [5, 10] to obtain the lower bound are adapted in this paper.

We like to point out that, unlike [6], the boundary Harnack principle for X^a is not used in this paper, which indicates that it might be possible to obtain sharp heat kernel estimates for processes for which the boundary Harnack principle fails.

As a consequence of Corollary 1.2, we have the following uniform boundary Harnack principle with explicit decay rate.

Theorem 1.3 *Suppose that $M > 0$. For any $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R_0, Λ_0), there exists a positive constant $C_4 = C_4(\alpha, \beta, d, \Lambda_0, R_0, M) \ge 1$ such that for $a \in [0, M]$, $r \in (0, R_0]$, $Q \in \partial D$ and any nonnegative function u in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$\frac{u(x)}{u(y)} \le C_4 \frac{\delta_D(x)^{\alpha/2}}{\delta_D(y)^{\alpha/2}} \qquad \textit{for every } x, y \in D \cap B(Q, r/2). \tag{1.9}$$

Throughout this paper, we will use capital letters $C_1, C_2, \dots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \dots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the constants on dimension d may not be mentioned explicitly. For every function f, let $f^+ := f \vee 0$. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

2 Upper bound estimate

Throughout this section we assume that D is an open set satisfying the uniform exterior ball condition with radius $r_0 > 0$ in the following sense: for every $z \in \partial D$ and $r \in (0, r_0)$, there is a ball B^z of radius r such that $B^z \subset \mathbb{R}^d \setminus \overline{D}$ and $\partial B^z \cap \partial D = \{z\}$. The goal of this section is to establish the upper bound for the transition density (heat kernel) $p^a_D(t,x,y)$. One of the main difficulties of getting the upper bound for $p^a_D(t,x,y)$ is to obtaining the correct boundary decay rate.

Recall that $\Delta^{\alpha/2}$ and $\widehat{\Delta}^{\beta/2}_\lambda$ are defined by (1.1) and (1.8). The next two lemmas can be proved by direct computation, whose proofs can be found in [21] and [11], respectively.

For $p > 0$, let $w_p(x) := (x_1^+)^p$.

Lemma 2.1 *For any* $x \in (0,\infty)\times\mathbb{R}^{d-1}$*, we have*

$$\Delta^{\alpha/2} w_{\alpha/2}(x) \,=\, 0.$$

Moreover, for every $p \in (\alpha/2, \alpha)$*, there is a positive constant* $C_5 = C_5(d,\alpha,p)$ *such that for every* $x \in (0,\infty)\times\mathbb{R}^{d-1}$

$$\Delta^{\alpha/2} w_p(x) = C_5\, x_1^{p-\alpha}.$$

Lemma 2.2 *There are constants* $R_* \in (0,1)$*,* $C_6 > C_7 > 0$ *depending on* p*,* d *and* α *only such that for every* $x \in (0, R_*]\times\mathbb{R}^{d-1}$

$$C_7 x_1^{p-\alpha} \le \widehat{\Delta}^{\alpha/2} w_p(x) \le C_6 x_1^{p-\alpha} \qquad \text{for } \alpha/2 < p < \alpha,$$

$$|\widehat{\Delta}^{\alpha/2} w_p(x)| \le C_6\,|\log x_1| \qquad \text{for } p = \alpha$$

and

$$|\widehat{\Delta}^{\alpha/2} w_p(x)| \le C_6 \qquad \text{for } p > \alpha.$$

In the remainder of this paper, R_* will always stand for the constant in Lemma 2.2. The following result and its proof are similar to [11, Lemma 3.2] and the proof there. For reader's convenience, we spell out the details of the proof here.

Lemma 2.3 *Assume that* $r_1 \in (0, 1/2]$ *and* $p \ge \frac{\alpha}{2}$*. Let* $\delta_1 := R_* \wedge (r_1/4)$*,* $U := \{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\}$ *and*

$$h_p(y) := \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^p \mathbf{1}_{U\cap\{z_d>0, |\widetilde{z}|<r_1/2\}}(y).$$

Then there exist $C_i = C_i(\alpha, p, r_1) > 0$*,* $i = 8, \cdots, 12$*, such that*

(i) *when* $p \in (\alpha/2,\, \alpha)$*, we have for all* $y \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\}$*,*

$$C_8\left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{p-\alpha} \le \widehat{\Delta}^{\alpha/2} h_p(y) \le C_9\left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{p-\alpha} \tag{2.1}$$

and

$$C_8\left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{p-\alpha} \le \Delta^{\alpha/2} h_p(y) \le C_9\left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{p-\alpha}; \tag{2.2}$$

(ii) *when* $p > \alpha$*, we have*

$$|\widehat{\Delta}^{\alpha/2} h_p(y)| \le C_{10} \quad \text{for all } y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\right\}; \tag{2.3}$$

(iii) *when* $p = \alpha/2$*, we have*

$$|\Delta^{\alpha/2} h_{\alpha/2}(y)| \le C_{11} \quad \text{for all } y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\right\}; \tag{2.4}$$

(iv) *when* $p = \alpha$*, we have for every* $y \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\}$*,*

$$|\widehat{\Delta}^{\alpha/2} h_{\alpha/2}(y)| \le C_{12}\left|\log\left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)\right|. \tag{2.5}$$

Proof. Let

$$\Gamma(\tilde{y}) := \sqrt{r_1^2 - |\tilde{y}|^2} \quad \text{and} \quad \underline{h}(y) := y_d - \Gamma(\tilde{y}), \quad y \in U.$$

Fix $x \in \{z \in \mathbb{R}^d : z_d > 0,\ r_1 < |z| < r_1 + R_* \wedge (r_1/8), |\tilde{z}| < r_1/4\}$ and choose a point $x_0 \in \partial B_+(0, r_1) := \{z_d > 0, |z| = r_1\}$ satisfying $\tilde{x} = \widetilde{x_0}$. Denote by $\overrightarrow{n}(x_0)$ the inward unit normal vector at x_0 for the exterior ball $B(0, r_1)^c$ and set $\Phi(y) = \langle y - x_0, \overrightarrow{n}(x_0)\rangle$ for $y \in \mathbb{R}^d$. $\Pi = \{y : \Phi(y) = 0\}$ is the hyperplane tangent to $\partial B_+(0, r_1)$ at the point x_0. Let $\Gamma^* : \tilde{y} \in \mathbb{R}^{d-1} \to \mathbb{R}$ be the function describing the hyperplane Π, that is, $\langle (\tilde{y}, \Gamma^*(\tilde{y})) - x_0, \overrightarrow{n}(x_0)\rangle = 0$. We also let

$$\begin{aligned} E &:= \{y = (\tilde{y}, y_d) : y \in U,\ \ |y - x| < r_1/4\}, \\ A &:= \{y : \Gamma^*(\tilde{y}) > y_d > \Gamma(\tilde{y}),\ |y - x| < r_1/4\}. \end{aligned}$$

and $\overline{h}(y) := (y_d - \Gamma^*(\tilde{y}))\mathbf{1}_{\{y_d > \Gamma^*(\tilde{y})\}}(y)$ for $y \in \mathbb{R}^d$. Since $\nabla\Gamma(\tilde{x}) - \nabla\Gamma^*(\tilde{x}) = 0$, by the mean value theorem,

$$|\overline{h}(y) - \underline{h}(y)| \le |\Gamma(\tilde{y}) - \Gamma^*(\tilde{y})| \le \Lambda|\tilde{y} - \tilde{x}|^2, \qquad y \in E. \tag{2.6}$$

Let $\delta_\Pi(y) = \mathrm{dist}(y, \Pi)$ for $y \in \mathbb{R}^d$ and $U_{\Gamma^*} = \{y \in \mathbb{R}^d : y_d > \Gamma^*(\tilde{y})\}$. Let $b_x := \sqrt{1 + |\nabla\Gamma(\tilde{x})|^2}$ and

$$h_{x,p}(y) := (\overline{h}(y))^p.$$

Note that $h_{x,p}(x) = h_p(x)$ and $B(x, r_1/4) \cap U \subset E$. Since $\overline{h}(y) = b_x\delta_\Pi(y)$ on D_{Γ^*}, by Lemma 2.1,

$$\Delta^{\alpha/2} h_{x,\alpha/2}(x) \;=\; 0 \tag{2.7}$$

and, if $\alpha/2 < p < \alpha$,

$$\Delta^{\alpha/2} h_{x,p}(x) \;=\; c_1\, b_x^p \delta_\Pi^{p-\alpha}(x) \;=\; c_1\, b_x^\alpha (\underline{h}(x))^{p-\alpha} \tag{2.8}$$

for some $c_1 > 0$. By Lemma 2.2, there are constants $c_i > 0$, $i = 2 \ldots 6$, such that for $y \in D_{\Gamma^*}$ and $\delta_\Pi(y) < R_*$, when $\alpha/2 < p < \alpha$,

$$c_2(\underline{h}(x))^{p-\alpha} \le c_3\, b_x^p \delta_\Pi^{p-\alpha}(x) \le \widehat{\Delta}^{\alpha/2} h_{x,p}(x) \;=\; b_x^p \widehat{\Delta}^{\alpha/2}(\delta_\Pi(x))^p \;\le\; c_4\, b_x^p \delta_\Pi^{p-\alpha}(x) \le c_5(\underline{h}(x))^{p-\alpha}, \tag{2.9}$$

when $p > \alpha$,

$$|\widehat{\Delta}^{\alpha/2} h_{x,p}(x)| \;=\; b_x^p |\widehat{\Delta}^{\alpha/2}(\delta_\Pi(x))^p| \;\le\; c_6 \tag{2.10}$$

and when $p = \alpha$,

$$|\widehat{\Delta}^{\alpha/2} h_{x,p}(x)| \le c_6\, |\log(\underline{h}(x))|. \tag{2.11}$$

Note that

$$\begin{aligned} |\widehat{\Delta}^{\alpha/2}(h_p - h_{x,p})(x)| &= \mathcal{A}(d, -\alpha)\left| \lim_{\varepsilon \downarrow 0} \int_{\{1 \ge |y-x| > \varepsilon\}} \frac{(h_p(y) - h_{p,x}(y))}{|x - y|^{d+\alpha}}\, dy \right| \\ &\le \mathcal{A}(d, -\alpha)\left| \int_{\{1 \ge |y-x| > r_1/4\}} \frac{(h_p(y) - h_{p,x}(y))}{|x - y|^{d+\alpha}}\, dy \right| \\ &\quad + \mathcal{A}(d, -\alpha) \lim_{\varepsilon \downarrow 0} \int_{\{r_1/4 \ge |y-x| > \varepsilon\}} \frac{|h_p(y) - h_{p,x}(y)|}{|x - y|^{d+\alpha}}\, dy \end{aligned}$$

$$\begin{aligned}&\le c_7+\mathcal{A}(d,-\alpha)\int_A \frac{h_p(y)+h_{p,x}(y)}{|x-y|^{d+\alpha}}\,dy+\mathcal{A}(d,-\alpha)\int_E \frac{|h_p(y)-h_{p,x}(y)|}{|x-y|^{d+\alpha}}\\&=:\ c_7+I_1+I_2 \qquad (2.12)\end{aligned}$$

and, similarly,

$$\begin{aligned}&|\Delta^{\alpha/2}(h_p-h_{x,p})(x)|\\&\le\mathcal{A}(d,-\alpha)\Big|\int_{\{|y-x|>r_1/4\}}\frac{(h_p(y)-h_{p,x}(y))}{|x-y|^{d+\alpha}}\,dy\Big|\\&+\mathcal{A}(d,-\alpha)\int_A \frac{h_p(y)+h_{p,x}(y)}{|x-y|^{d+\alpha}}\,dy+\mathcal{A}(d,-\alpha)\int_E \frac{|h_p(y)-h_{p,x}(y)|}{|x-y|^{d+\alpha}}=:I_3+I_1+I_2. \qquad (2.13)\end{aligned}$$

Since for $y\in B(x,r_1/4)^c$,

$$|h_{x,p}(y)-h_{x,p}(x)|\le c_8|y-x|^p \quad \text{and} \quad |h_p(y)|\le c_8$$

and $h_p(y)=0$ for $|\widetilde{y}|>r_1/2$, for $\alpha/2\le p<\alpha$ we get

$$\begin{aligned}I_3\le&\mathcal{A}(d,-\alpha)\int_{B(x,r_1/4)^c}\frac{|h_{x,p}(y)-h_{x,p}(x)|}{|x-y|^{d+\alpha}}\,dy+\mathcal{A}(d,-\alpha)\int_{B(x,r_1/4)^c\cap\{|\widetilde{y}|\le r_1/2\}}\frac{|h_p(y)-h_p(x)|}{|x-y|^{d+\alpha}}dy\\&+\mathcal{A}(d,-\alpha)\Big|\int_{B(x,r_1/4)^c\cap\{|\widetilde{y}|>r_1/2\}}\frac{h_p(x)}{|x-y|^{d+\alpha}}\,dy\Big|\\\le&c_9\int_{B(x,r_1/4)^c}\frac{1}{|x-y|^{d+\alpha-p}}dy+c_9\int_{B(x,r_1/4)^c}\frac{1}{|x-y|^{d+\alpha}}dy\le c_{10}<\infty. \qquad (2.14)\end{aligned}$$

We claim that, if $p\ge\alpha/2$,

$$I_1\ +\ I_2\ \le\ c_{11}\ <\ \infty. \qquad (2.15)$$

Note that for $y\in A$

$$\begin{aligned}|h_{x,p}(y)|+|h_p(y)|&\le|y_d-\Gamma^*(\widetilde{y})|^p+|y_d-\Gamma(\widetilde{y})|^p\le2|\Gamma(\widetilde{y})-\Gamma^*(\widetilde{y})|^p\\&\le2|\Gamma(\widetilde{y})-\Gamma(\widetilde{x})-\nabla\Gamma(\widetilde{x})\cdot(\widetilde{y}-\widetilde{x})|^p\le2c_{12}^p|\widetilde{y}-\widetilde{x}|^{2p}. \qquad (2.16)\end{aligned}$$

Furthermore, since $|\Gamma(\widetilde{y})-\Gamma^*(\widetilde{y})|\le c_{13}|\widetilde{y}-\widetilde{x}|^2\le c_{12}r^2$ on $|y-x|=r$, this together with (2.16) yields that

$$\begin{aligned}I_1\ &\le\ c_{14}\int_0^{r_1/4}r^{2p-\alpha-d}\int_{|y-x|=r}\mathbf{1}_A(y)\ m_{d-1}(dy)dr\\&=\ c_{14}\int_0^{r_1/4}r^{2p-\alpha-d}m_{d-1}\big(\{y:\ |y-x|=r,\ \Gamma^*(\widetilde{y})>y_d>\Gamma(\widetilde{y})\}\big)dr\\&\le\ c_{15}\int_0^{r_1/4}r^{2p-\alpha}dr\ <\ \infty.\end{aligned}$$

Note that, if $y\in E$, $y_d\ge x_d-|x-y|\ge\sqrt{|x|^2-|\widetilde{x}|^2}-r_1/4\ge\sqrt{15}r_1/4-r_1/4>0$ and $|\widetilde{y}|\le|\widetilde{x}|+|x-y|<r_1/2$. Thus $E\subset U\cap\{z_d>0,|\widetilde{z}|<r_1/2\}$, and so we have that for $y\in E$

$$|h_p(y)-h_{x,p}(y)|\le c_{16}|(\overline{h}(y))^p-(\underline{h}(y))^p|\le c_{17}(\overline{h}(y))^{(p-1)-}|\overline{h}(y)-\underline{h}(y)|, \qquad (2.17)$$

where $(p-1)_- := (p-1)\wedge 0$. In the last inequality above, we have used the inequalities

$$|b^p - a^p| \le b^{p-1}|b-a| \qquad \text{for } a,b>0,\ 0<p\le 1$$

and

$$|b^p - a^p| \le (p+1)|b-a| \qquad \text{for } a,b\in(0,1),\ p>1.$$

For $y=(\widetilde{y},y_d)\in\mathbb{R}^d$, we use an affine coordinate system $z=(\widetilde{z},z_d)$ to represent it so that $z_d=y_d-\Gamma^*(\widetilde{y})$ and $\widetilde{z}$ is the coordinates in an orthogonal coordinate system centered at x_0 for the $(d-1)$-dimensional hyperplane Π for the point $(\widetilde{y},\Gamma^*(\widetilde{y}))$. Denote such an affine transformation $y\mapsto z$ by $z=\Psi(y)$. It is clear that there is a constant $c_{18}>1$ so that for every $y\in\mathbb{R}^d$,

$$c_{18}^{-1}|\widetilde{y}-\widetilde{x}|\le|\widetilde{z}|\le c_{18}|\widetilde{y}-\widetilde{x}|, \qquad c_{18}^{-1}|y-x|\le|\Psi(y)-\Psi(x)|\le c_{18}|y-x|$$

and that

$$\Psi(E)\subset\{z=(\widetilde{z},z_d)\in\mathbb{R}^d:\ |\widetilde{z}|<c_{18}r_1 \text{ and } 0<z_d\le c_{18}r_1\}.$$

Denote $x_d-\Gamma^*(\widetilde{x})$ by w; that is, $\Psi(x)=(\widetilde{0},w)$. Hence by (2.6) and (2.17) and applying the transform Ψ, we have by using polar coordinates for $\widetilde{z}$ on the hyperplane Π,

$$\begin{aligned}
I_2 &\le c_{19}\int_E \frac{\overline{h}(y)^{(p-1)_-}\,|\widetilde{y}-\widetilde{x}|^2}{|y-x|^{d+\alpha}}\,dy \le c_{19}\int_{\Psi(E)}\frac{z_d^{(p-1)_-}\,|\widetilde{z}|^2}{|z-(\widetilde{0},w)|^{d+\alpha}}\,dz\\
&\le c_{20}\int_0^{c_{18}r_1} z_d^{(p-1)_-}\left(\int_0^{c_{18}r_1}\frac{r^{d-2}}{(r+|z_d-w|)^{d+\alpha-2}}dr\right)dz_d\\
&\le c_{20}\int_0^{c_{18}r_1} z_d^{(p-1)_-}\left(\int_0^{c_{18}r_1}\frac{1}{(r+|z_d-w|)^{\alpha}}dr\right)dz_d\\
&\le c_{21}\int_0^{c_{18}r_1} z_d^{(p-1)_-}\left(\frac{1}{|z_d-w|^{\alpha-1}}-\frac{1}{(c_{18}r_1+|z_d-w|)^{\alpha-1}}\right)dz_d\\
&< c_{22}\int_0^{c_{18}r_1}\frac{1}{z_d^{(1-p)^+}\ |z_d-w|^{\alpha-1}}dz_d\le c_{23}<\infty,
\end{aligned}$$

where all the constants depend only on α, d, p and r_1. The last inequality is due to the fact that since $p>0$, $0<\alpha<2$ and $(1-p)^+ +\alpha-1=\max\{\alpha-p,\alpha-1\}<1$, by the dominated convergence theorem, $\phi(w):=\int_0^{c_{18}r_1}\frac{1}{z_d^{(1-p)^+}\ |z_d-w|^{\alpha-1}}dz_d$ is a strictly positive continuous function in $x_d\in[0,c_{18}r_1]$ and hence is bounded.

Thus we have proved the claim (2.15). The desired estimates (2.1)-(2.5) now follow from (2.7)-(2.15). □

It is well-known that X^1 has Lévy intensity

$$J^1(x,y)=j^1(|x-y|)=\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}}+\frac{\mathcal{A}(d,-\beta)}{|x-y|^{d+\beta}}.$$

A scaling argument yields that

$$J^a(x,y)=j^a(|x-y|)=\frac{\mathcal{A}(d,-\alpha)}{|x-y|^{d+\alpha}}+a^\beta\frac{\mathcal{A}(d,-\beta)}{|x-y|^{d+\beta}}.$$

Put

$$\psi^a(r) = \left(r^{-\alpha} + a^\beta \frac{\mathcal{A}(d,-\beta)}{\mathcal{A}(d,-\alpha)} r^{-\beta} \right)^{-1}. \tag{2.18}$$

Clearly

$$J^a(x,y) = j^a(|x-y|) = \frac{\mathcal{A}(d,-\alpha)}{|x-y|^d \psi^a(|x-y|)}.$$

The function ψ^a plays an important role in the study of mixed stable-like processes including X^a in [13], which serves as the 'scale function' for the heat kernel estimates and global parabolic Harnack inequality.

The Lévy intensity gives rise to a Lévy system for X^a, which describes the jumps of the process X^a: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s,y,y) = 0$ for all $y \in \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time T (with respect to the filtration of X^a),

$$\mathbb{E}_x \Big[\sum_{s \le T} f(s, X^a_{s-}, X^a_s) \Big] = \mathbb{E}_x \Big[\int_0^T \Big(\int_{\mathbb{R}^d} f(s, X^a_s, y) J^a(X^a_s, y) dy \Big) ds \Big]. \tag{2.19}$$

(See, for example, [12, Proof of Lemma 4.7] and [13, Appendix A].)

For any open set $D \subset \mathbb{R}^d$, let $\tau^a_D = \tau^a(D) := \inf\{t > 0 : X^a_t \notin D\}$ denote the first exit time from D by X^a.

Note that by the approximate scaling property in (1.5), we have for every $r > 0$.

$$G^a_{B(0,r)}(x,y) = r^{\alpha - d} G^{ar^{(\alpha-\beta)/\beta}}_{B(0,1)}(x/r, y/r). \tag{2.20}$$

The next lemma follows immediately from a special case of [22, Proposition 2.10 and Lemma 3.6] and (2.20).

Lemma 2.4 *For any $b, M \in (0,\infty)$, there exists $C_{13} = C_{13}(M, b, \alpha, \beta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $a \in [0, M]$ and $r \in (0, b]$,*

$$\mathbb{E}_x \left[\tau^a_{B(x_0,r)} \right] \le C_{13}\, r^{\alpha/2} (r - |x - x_0|)^{\alpha/2} \qquad \text{for } x \in B(x_0, r). \tag{2.21}$$

For $\lambda > 0$, $\widehat{Y}^\lambda = (\widehat{Y}^\lambda_t, \mathbb{P}_x)$ is a Lévy process on $\mathbb{R}^d$ such that

$$\mathbb{E}_x \left[e^{i\xi \cdot (\widehat{Y}^\lambda_t - \widehat{Y}^\lambda_0)} \right] = e^{-t\psi(\xi)} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

with

$$\psi(\xi) = \mathcal{A}(d,-\beta) \int_{\{|y| \le \lambda\}} \frac{1 - \cos(\xi \cdot y)}{|y|^{d+\beta}} dy.$$

In other words, $\widehat{Y}^\lambda$ is a pure jump symmetric Lévy process on $\mathbb{R}^d$ with a Lévy density given by $\mathcal{A}(d,-\beta)|x|^{-d-\beta} 1_{\{|x| \le \lambda\}}$. For $a > 0$, suppose $\widehat{Y}^{1/a}$ is independent of the symmetric α-stable process X on $\mathbb{R}^d$. Define

$$\widehat{X}^a_t := X_t + a \widehat{Y}^{1/a}_t, \qquad t \ge 0.$$

We will call the process $\widehat{X}^a$ the independent sum of the symmetric α-stable process X and the truncated symmetric β-stable process $\widehat{Y}^{1/a}$ with weight $a > 0$. The infinitesimal generator of $\widehat{X}^a$ is $\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}$.

For any open set $U \subset \mathbb{R}^d$, let $\widehat{\tau}^a_U = \inf\{t > 0 : \widehat{X}^a_t \notin U\}$ be the first exit time from U by $\widehat{X}^a$. The truncated process $\widehat{X}^a$ will be used in the proof of next lemma.

Lemma 2.5 *Assume* $r_1 \in (0, \frac{1}{4}]$ *and* $M > 0$. *Let* $U := \{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\}$. *There are constants* $C_{14} = C_{14}(r_1, \alpha) > 0$ *and* $C_{15} = C_{15}(r_1, M, \alpha, \beta) > 0$ *such that for every* $a \in [0, M]$

$$\mathbb{E}_x[\tau_U^a] \le C_{14}\mathbb{P}_x\left(\left|X^a_{\tau^a_U}\right| \ge 3r_1/2\right) \le C_{15}\delta_U(x)^{\alpha/2}, \qquad \text{for } r_1 < |x| < 5r_1/4. \tag{2.22}$$

Proof. The first inequality in (2.22) is easy. In fact, by the Lévy system (2.19) with

$$f(s, x, y) = \mathbf{1}_U(x)\mathbf{1}_{\{5r_1 < |y| < 10r_1\}}(y)$$

and $T = \tau_U^a$, we have that for $x \in U$

$$\begin{aligned}
&\mathbb{P}_x\left(\left|X^a_{\tau^a_U}\right| \ge 3r_1/2\right) \ge \mathbb{P}_x\left(10r_1 > \left|X^a_{\tau^a_U}\right| > 5r_1\right) \\
&= \mathbb{E}_x\left[\int_0^{\tau_U^a}\int_{\{5r_1<|y|<10r_1\}} J^a(X^a_s, y)dyds\right] \\
&\ge \mathbb{E}_x\left[\int_0^{\tau_U^a}\int_{\{5r_1<|y|<10r_1\}} \frac{\mathcal{A}(d,-\alpha)}{|X^a_s - y|^{d+\alpha}}dyds\right] \ge c_1\mathbb{E}_x[\tau_U^a],
\end{aligned}$$

where $c_1 = c_1(r_1, \alpha) > 0$.

It is enough to prove the second inequality in (2.22) for $r_1 < |x| < r_1 + \delta$ for some small $\delta > 0$. Without loss of generality, we assume $\widetilde{x} = \widetilde{0}$ and $x_d > 0$. Let $p > 0$ be such that $p \ne \beta$ and

$$\alpha - (\beta/2) < p < \alpha \wedge \big(\alpha - (\beta/2) + (\alpha - \beta)/3\big).$$

Note that $\alpha/2 < p < \alpha \wedge (3\alpha/2 - \beta)$. Define

$$h(y) := \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{\alpha/2}\mathbf{1}_{U\cap\{z_d>0, |\widetilde{z}|<r_1/2\}}(y),$$

$$g_p(y) := \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{p}\mathbf{1}_{U\cap\{z_d>0, |\widetilde{z}|<r_1/2\}}(y),$$

and let ϕ be a smooth function on $\mathbb{R}^d$ with bounded first and second partial derivatives such that $\phi(y) = 2^{4+p}|\widetilde{y}|^2/r_1^2$ for $y \in \{z_d > 0, r_1 < |z| < 4r_1/5, |\widetilde{z}| < r_1/4\}$ and $2^p \le \phi(y) \le 4^p$ if $|\widetilde{y}| \ge r_1/2$ or $|y| \ge 3r_1/2$.

Since $r_1 \le 1/4$, it is easy to see that $\|g_p\|_\infty < 1$. Now we define

$$u(y) := h(y) + \phi(y) - g_p(y).$$

By Taylor's expansion with remainder of order 2, we get that for any $a \in (0, M]$ and $y \in \mathbb{R}^d$,

$$\left|(\Delta^{\alpha/2} + a^\beta\widehat{\Delta}^{\beta/2})\phi(y)\right| \le \left\|\Delta^{\alpha/2}\phi\right\|_\infty + M^\beta\left\|\widehat{\Delta}^{\beta/2}\phi\right\|_\infty \le c_2(\alpha, \beta, M) < \infty. \tag{2.23}$$

Moreover, by (2.1)–(2.3), there exist $c_3 = c_3(\alpha, \beta) > 0$, $c_4 = c_4(\alpha, \beta) > 0$ and $\delta_1 = \delta_1(\alpha, \beta) \in (0, r_1/8)$ such that

$$\Delta^{\alpha/2}g_p(y) \ge c_3\delta_U(y)^{p-\alpha} \quad \text{for } y \in \left\{z \in \mathbb{R}^d : z_d > 0,\ r_1 < |z| < r_1 + \delta_1, |\widetilde{z}| < r_1/4\right\}$$

and

$$\widehat{\Delta}^{\beta/2}g_p(y) \geq -c_4\delta_U(y)^{(p-\beta)\wedge 0} \quad \text{for } y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1+\delta_1, |\widetilde{z}| < r_1/4\right\}.$$

Note that $p-\alpha > p-\beta$ and $p-\alpha < 0$. Thus there exist $c_5 = c_5(\alpha,\beta,M) > 0$ and $\delta_2 = \delta_2(\alpha,\beta,M) \in (0,\delta_1)$ such that for all $a \in (0,M]$ and $y \in \{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1+\delta_2, |\widetilde{z}| < r_1/4\}$,

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})g_p(y) \geq c_3\delta_U(y)^{p-\alpha} - c_4M^\beta\, \delta_U(y)^{(p-\beta)\wedge 0} \geq c_5\delta_U(y)^{p-\alpha}. \tag{2.24}$$

Furthermore by (2.1) and (2.3)–(2.5), there exist $c_6 = c_6(\alpha,\beta,M) > 0$ and $\delta_3 = \delta_3(\alpha,\beta) \in (0,\delta_1)$ such that for all $a \in (0,M]$ and for every $y \in \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1+\delta_3, |\widetilde{z}| < r_1/4\right\}$,

$$\begin{aligned}\left|(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})h(y)\right| &\leq \left|\Delta^{\alpha/2}h(y)\right| + M^\beta\left|\widehat{\Delta}^{\beta/2}h(y)\right| \\ &\leq \begin{cases} c_6 + c_6\delta_U(y)^{(\alpha/2-\beta)\wedge 0} & \text{if } \beta \neq \alpha/2, \\ c_6 + c_6|\log\delta_U(y)| & \text{if } \beta = \alpha/2. \end{cases}\end{aligned} \tag{2.25}$$

Since $p-\alpha < \alpha/2-\beta$, by (2.23)– (2.25), there exists $\delta_4 = \delta_4(\alpha,\beta,M) \in (0,\delta_2\wedge\delta_3)$ such that for all $a \in (0,M]$ and $y \in V := \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 < |z| < r_1+\delta_4, |\widetilde{z}| < r_1/4\right\}$

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u(y) \leq c_2 + c_6 + c_6\left(\delta_U(y)^{(\alpha/2-\beta)\wedge 0} + |\log\delta_U(y)|\right) - c_5\delta_U(y)^{p-\alpha} \leq 0. \tag{2.26}$$

Let η be a non-negative smooth radial function with compact support in $\mathbb{R}^d$ such that $\eta(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}^d}\eta(x)dx = 1$. For $k \geq 1$, define $\eta_k(x) = 2^{kd}\eta(2^kx)$. Set $u^{(k)}(z) := (\eta_k * u)(z)$. As $(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u^{(k)} = \eta_k * (\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u$, we have by (2.26) that

$$\left(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}\right)u^{(k)} \leq 0$$

on $V_k := \left\{z \in \mathbb{R}^d : z_d > 0,\, r_1 + 2^{-k} < |z| < r_1+\delta_4 - 2^{-k} \text{ and } |\widetilde{z}| < r_1/4 - 2^{-k}\right\}$. Since $u^{(k)}$ is a bounded smooth function on $\mathbb{R}^d$ with bounded first and second partial derivatives, by Ito's formula and the Lévy system (2.19),

$$M_t^k := u^{(k)}(\widehat{X}_t^a) - u^{(k)}(\widehat{X}_0^a) - \int_0^t \left(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}\right)u^{(k)}(\widehat{X}_s^a)ds \tag{2.27}$$

is a martingale. Thus it follows from (2.27) that $t \mapsto u^{(k)}\big(\widehat{X}^a_{t\wedge\widehat{\tau}^a_{V_k}}\big)$ is a bounded supermartingale. Since V_k increases to V and u is bounded and continuous on $\overline{V}$, we conclude that

$$t \mapsto u\left(\widehat{X}^a_{t\wedge\widehat{\tau}^a_V}\right) \quad \text{is a bounded supermartingale.} \tag{2.28}$$

We observe that, since $\phi(x) = 0$,

$$u(x) \leq \delta_U(x)^{\alpha/2}. \tag{2.29}$$

We also observe that, since $\phi \geq 2g_p$ outside of $\{z \in U :\, z_d > 0,\, |\widetilde{z}| < r_1/2\}$ and

$$u(y) \geq \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^{\alpha/2} - \left(y_d - \sqrt{r_1^2 - |\widetilde{y}|^2}\right)^p > c_7$$

on $\{z_d > 0,\, r_1 + \delta_4 \le |z| < 3r_1/2,\, |\widetilde{z}| < r_1/2\}$, we have

$$u(y) \ge c_8 > 0 \qquad \text{for } y \in V^c \setminus \overline{B(0, r_1)}, \tag{2.30}$$

where c_8 depends on δ_4, α, β and r_1. Therefore, by (2.28)-(2.30) we get

$$\delta_U(x)^{\alpha/2} \ge u(x) \ge \mathbb{E}_x \left[u\big(\widehat{X}^a_{\widehat{\tau}^a_V}\big) \right] \ge c_8 \mathbb{P}_x \left(\widehat{X}^a_{\widehat{\tau}^a_V} \in V^c \setminus \overline{B(0, r_1)} \right) \ge c_8 \mathbb{P}_x \left(|\widehat{X}^a_{\widehat{\tau}^a_U}| \ge 3r_1/2 \right). \tag{2.31}$$

Note that there exist $c_9 = c_9(\alpha, d, r_1) > 0$ and $c_{10} = c_{10}(\beta, d, r_1) > 0$ such that for $z \in U$,

$$\int_{\{|y| \ge 2r_1\}} \frac{dy}{|z - y|^{d+\alpha}} \le c_9 \int_{\{2r_1 \le |y| < 3r_1\}} \frac{dy}{|z - y|^{d+\alpha}}$$

and

$$\int_{\{|y| \ge 2r_1\}} \frac{dy}{|z - y|^{d+\beta}} \le c_{10} \int_{\{2r_1 \le |y| < 3r_1\}} \frac{dy}{|z - y|^{d+\beta}}.$$

Thus by (2.19), there exists a positive constant $c_{11} = c_{11}(d, \alpha, \beta, M)$ such that for any $a \in (0, M]$,

$$\begin{aligned}
\mathbb{P}_x \left(\left|X^a_{\tau^a_U}\right| \ge 2r_1 \right) &= \mathbb{E}_x \left[\int_0^{\tau^a_U} \int_{\{|y| \ge 2r_1\}} J^a(X^a_s, y) dy ds \right] \\
&\le c_{11} \mathbb{E}_x \left[\int_0^{\tau^a_U} \int_{\{2r_1 \le |y| < 3r_1\}} J^a(X^a_s, y) dy ds \right] \\
&= c_{11} \mathbb{P}_x \left(3r_1 > \left|X^a_{\tau^a_U}\right| \ge 2r_1 \right).
\end{aligned}$$

Since $r_1 \le 1/4$ and the processes X and Y do not jump simultaneously, we have by (2.31) that there is a positive constant $c_{12} = c_{12}(d, \alpha, \beta, M, r_1)$ such that for all $a \in (0, M]$,

$$\begin{aligned}
\mathbb{P}_x \left(\left|X^a_{\tau^a_U}\right| \ge 3r_1/2 \right) &\le (c_{11} + 1) \mathbb{P}_x \left(3r_1 > \left|X^a_{\tau^a_U}\right| \ge 3r_1/2 \right) \\
&= (c_{11} + 1) \mathbb{P}_x \left(3r_1 > \left|\widehat{X}^a_{\widehat{\tau}^a_U}\right| \ge 3r_1/2 \right) \\
&\le (c_{11} + 1) \mathbb{P}_x \left(\left|\widehat{X}^a_{\widehat{\tau}^a_U}\right| \ge 3r_1/2 \right) \le c_{12} \delta_U(x)^{\alpha/2}.
\end{aligned}$$

□

Lemma 2.6 *Assume $M > 0$ and $r_1 \in (0, \frac{1}{4}]$. Let $E = \{x \in \mathbb{R}^d : |x| > r_1\}$. Then for every $T > 0$, there is a constant $C_{16} = C_{16}(r_1, \alpha, \beta, T, M) > 0$ such that for every $a \in [0, M]$,*

$$p^a_E(t, x, y) \le C_{16}\, \delta_E(x)^{\alpha/2} J^a(x, y) \qquad \text{for } r_1 < |x| < 5r_1/4,\ |y| \ge 2r_1 \text{ and } t \le T.$$

Proof. Define $U := \left\{ z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2 \right\}$. For $r_1 < |x| < 5r_1/4$, $|y| \ge 2r_1$ and $t \in (0, T]$, it follows from the strong Markov property of X^a and (2.19) that

$$\begin{aligned}
& p^a_E(t, x, y) \\
= & \mathbb{E}_x \left[p^a_E(t - \tau^a_U, X^a_{\tau^a_U}, y);\ \tau^a_U < t,\ (3r_1/4) + (|y|/2) \ge |X^a_{\tau^a_U}| \ge 3r_1/2 \right]
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbb{E}_x \left[p_E^a(t-\tau_U^a, X^a_{\tau_U^a}, y);\, \tau_U^a < t, |X^a_{\tau_U^a}| > (3r_1/4) + (|y|/2) \right] \\
&\le \left(\sup_{\substack{s:\, s\in(0,t) \\ w:\, (3r_1/4)+(|y|/2)\ge |w| \ge 3r_1/2}} p_E^a(t-s,w,y) \right) \mathbb{P}_x \left(\tau_U^a < t, (3r_1/4)+(|y|/2) \ge |X^a_{\tau_U^a}| \ge 3r_1/2 \right) \\
&+ \int_0^t \int_U p_U(s,x,z) \left(\int_{\{w:\, |w|>(3r_1/4)+(|y|/2)\}} J^a(z,w) p_E^a(t-s,w,y)dw \right) dzds \\
&=: I + II.
\end{aligned}
$$

Note that for $|w| \le (3r_1/4) + (|y|/2)$,

$$
|w-y| \ge |y| - |w| \ge \frac{1}{2}\left(|y| - \frac{3r_1}{2}\right) \ge \frac{|y|}{8} \ge \frac{|x-y|}{16}. \tag{2.32}
$$

Since $p_E^a(t-s,w,y) \le p^a(t-s,w,y)$, by (1.6) and (2.32), there exists a constant $c_1 = c_1(\alpha,\beta,M) > 0$ such that for $a \in (0, M]$

$$
I \le c_1 T J^a(x,y)\, \mathbb{P}_x \left(|X^a_{\tau_U^a}| \ge 3r_1/2 \right).
$$

By Lemma 2.5, we have for $|x| \in (r_1, 5r_1/4)$,

$$
\mathbb{P}_x \left(|X^a_{\tau_U^a}| \ge 3r_1/2 \right) \le c_2\, \delta_U(x)^{\alpha/2} = c_2 \delta_E(x)^{\alpha/2}
$$

for some positive constant $c_2 = c_2(r_1,\alpha,\beta,M)$. Thus

$$
I \le c_3\, T\, \delta_E(x)^{\alpha/2} J^a(x,y) \tag{2.33}
$$

for some positive constant $c_3 = c_3(r_1,\alpha,\beta,M)$.

On the other hand, for $z \in U$ and $w \in \mathbb{R}^d$ with $|w| > (3r_1/4) + (|y|/2)$,

$$
|z-w| \ge |w| - |z| \ge \frac{1}{2}\left(|y| - \frac{3r_1}{2}\right) \ge \frac{|y|}{8} \ge \frac{|x-y|}{16}.
$$

Thus by the symmetry of $p_E^a(t-s,w,y)$ in (w,y), we have

$$
\begin{aligned}
II &\le c_4 J^a(x,y) \int_0^t \int_U p_U^a(s,x,z) \left(\int_{\{w:\, |w|>(3r_1/4)+(|y|/2)\}} p_E^a(t-s,y,w)dw \right) dzds \\
&\le c_4 J^a(x,y) \int_0^\infty \int_U p_U^a(s,x,z)dzds \\
&= c_4 J^a(x,y) \mathbb{E}_x [\tau_U^a] \ \le\ c_5\, \delta_E(x)^{\alpha/2} J^a(x,y)
\end{aligned}
$$

for some positive constants $c_k = c_k(r_1,\alpha,\beta,M)$, $k = 4,5$. In the last inequality, we used Lemma 2.5 to deduce that $\mathbb{E}_x[\tau_U^a] \le c\delta_U(x)^{\alpha/2} = c_6 \delta_E(x)^{\alpha/2}$ for some positive constant $c_6 = c_6(r_1,\alpha,\beta,M)$. This together with (2.33) proves the lemma. □

Theorem 2.7 *Assume that $M > 0$ and D is an open set that satisfies the uniform exterior ball condition with radius $r_0 > 0$. Then for every $T > 0$, there is a constant $C_{17} = C_{17}(r_0/T, \alpha, \beta, M) > 0$ such that for all $a \in (0, M]$, $\lambda \in (0, T]$ and $x, y \in \lambda^{-1}D$,*

$$p^a_{\lambda^{-1}D}(1, x, y) \le C_{17}\,(1 \wedge J^a(x, y))\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}.$$

Proof. Note that for every $\lambda \in (0, T]$, $\lambda^{-1}D$ satisfies the uniform exterior ball condition with radius r_0/T. For $x, y \in \lambda^{-1}D$, let $z \in \partial(\lambda^{-1}D)$ be that $|x - z| = \delta_{\lambda^{-1}D}(x)$. Let $B_z \subset (\lambda^{-1}D)^c$ be the ball with radius $r_1 := 4^{-1} \wedge (r_0/T)$ so that $\partial B_z \cap \partial(\lambda^{-1}D) = \{z\}$. Since, by (1.6)

$$p^a_{\lambda^{-1}D}(1, x, y) \le p^a(1, x, y) \le c\,(1 \wedge J^a(x, y)),$$

it suffices to prove the theorem for $x \in \lambda^{-1}D$ with $\delta_{\lambda^{-1}D}(x) < r_1/4$. When $\delta_{\lambda^{-1}D}(x) < r_1/4$ and $|x - y| \ge 5r_1$, we have $\delta_{B^c_z}(y) > 2r_1$ and so, by Lemma 2.6, there is a constant $c_1 > 0$ that depends only on $(r_0/T, d, \alpha, \beta, M)$ such that for $t \in (0, 1]$,

$$p^a_{\lambda^{-1}D}(t, x, y) \le p^a_{(\overline{B}_z)^c}(t, x, y) \le c_1\,\delta_{(\overline{B}_z)^c}(x)^{\alpha/2} J^a(x, y) = c_1 \delta_{\lambda^{-1}D}(x)^{\alpha/2} J^a(x, y). \tag{2.34}$$

So it remains to show that, when $\delta_{\lambda^{-1}D}(x) < r_1/4$ and $|x-y| < 5r_1$, there exists a positive constant $c_2 = c_2(r_0/T, d, \alpha, \beta, M)$ such that

$$p^a_{\lambda^{-1}D}(1, x, y) \le c_2\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}. \tag{2.35}$$

Let $z_x \in \partial(\lambda^{-1}D)$ be such that $|x - z_x| = \delta_{\lambda^{-1}D}(x)$ and $z_0 \in \mathbb{R}^d$ so that

$$B(z_0, r_1) \subset (\lambda^{-1}D)^c \qquad \text{and} \qquad \partial B(z_0, r_1) \cap \partial(\lambda^{-1}D) = \{z_x\}.$$

Define $U := \{w \in \mathbb{R}^d : |w - z_0| \in (r_1, 8r_1)\}$. Note that $x, y \in U \cap \lambda^{-1}D$ and $\delta_U(x) = \delta_{\lambda^{-1}D}(x)$. By the strong Markov property and the symmetry of $p^a_{\lambda^{-1}D}(1, x, y)$ in x and y, we have

$$p^a_{\lambda^{-1}D}(1, x, y) = p^a_{U\cap\lambda^{-1}D}(1, x, y) + \mathbb{E}_y\left[p^a_{\lambda^{-1}D}(1 - \tau^a_{U\cap\lambda^{-1}D}, X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x);\ \tau^a_{U\cap\lambda^{-1}D} < 1\right].$$

By the semigroup property and (1.6),

$$\begin{aligned}
p^a_{U\cap\lambda^{-1}D}(1, x, y) &= \int_{U\cap\lambda^{-1}D} p^a_{U\cap\lambda^{-1}D}(1/2, x, z) p^a_{U\cap\lambda^{-1}D}(1/2, z, y) dz \\
&\le \|p^a(1/2, \cdot, \cdot)\|_\infty\, \mathbb{P}_x\left(\tau^a_{U\cap\lambda^{-1}D} > 1/2\right) \\
&\le c_3\,\mathbb{E}_x\left[\tau^a_{U\cap\lambda^{-1}D}\right] \le c_3\,\mathbb{E}_x\left[\tau^a_U\right] \\
&\le c_4\,\delta_U(x)^{\alpha/2} = c_4\,\delta_{\lambda^{-1}D}(x)^{\alpha/2}.
\end{aligned}$$

In the last inequality, we used Lemma 2.5.

On the other hand, we have $X^a_{\tau^a_{U\cap\lambda^{-1}D}} \in U^c \cap \lambda^{-1}D$ on $\{\tau^a_{U\cap\lambda^{-1}D} < 1\}$, and so

$$\left|X^a_{\tau^a_{U\cap\lambda^{-1}D}} - x\right| \ge 7r_1, \qquad \text{on} \quad \left\{\tau^a_{U\cap\lambda^{-1}D} < 1\right\}.$$

Consequently, by (2.34) for $p^a_{\lambda^{-1}D}\big(1 - \tau^a_{U\cap\lambda^{-1}D}, X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x\big)$,

$$\mathbb{E}_y\left[p^a_{\lambda^{-1}D}(1 - \tau^a_{U\cap\lambda^{-1}D}, X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x);\ \tau^a_{U\cap\lambda^{-1}D} < 1\right]$$

$$
\begin{aligned}
&\le \mathbb{E}_y \left[c_1\, \delta_{\lambda^{-1}D}(x)^{\alpha/2} J^a(X^a_{\tau^a_{U\cap\lambda^{-1}D}}, x);\ \tau^a_{U\cap\lambda^{-1}D} < 1 \right] \\
&\le c_1((7r_1)^{-d-\alpha} + M^\beta (7r_1)^{-d-\beta})\, \delta_{\lambda^{-1}D}(x)^{\alpha/2}\, \mathbb{P}_y \left(\tau^a_{U\cap\lambda^{-1}D} < 1 \right) \\
&\le c_1((7r_1)^{-d-\alpha} + M^\beta (7r_1)^{-d-\beta})\, \delta_{\lambda^{-1}D}(x)^{\alpha/2}.
\end{aligned}
$$

This completes the proof for (2.35) and hence the theorem. □

Theorem 2.8 *Assume that $M > 0$ and that D is an open set that satisfies the uniform exterior ball condition with radius $r_0 > 0$. For every $T > 0$, there exists a positive constant $C_{18} = C_{18}(T, r_0, \alpha, \beta, M)$ such that for every $a \in [0, M]$, $t \in (0, T]$ and $x, y \in D$,*

$$
p^a_D(t, x, y) \le C_{18} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge t J^a(x, y) \right). \tag{2.36}
$$

Proof. Fix $T, M > 0$. By Theorem 2.7, there exists a positive constant $c_1 = c_1(T, r_0, \alpha, \beta, M)$ such that for every $t \in (0, T]$,

$$
p^{at^{(\alpha-\beta)/(\alpha\beta)}}_{t^{-1/\alpha}D}(1, x, y) \le c_1 \left(1 \wedge J^{at^{(\alpha-\beta)/(\alpha\beta)}}(x, y) \right) \delta_{t^{-1/\alpha}D}(x)^{\alpha/2}. \tag{2.37}
$$

Thus by (1.5), (1.6) and (2.37), for every $t \le T$,

$$
\begin{aligned}
p^a_D(t, x, y) &= t^{-d/\alpha} p^{at^{(\alpha-\beta)/(\alpha\beta)}}_{t^{-1/\alpha}D}(1, t^{-1/\alpha}x, t^{-1/\alpha}y) \\
&\le c_1\, t^{-d/\alpha} \left(1 \wedge J^{at^{(\alpha-\beta)/(\alpha\beta)}}(t^{-1/\alpha}x, t^{-1/\alpha}y) \right) \delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha/2} \\
&= c_1 \left(t^{-d/\alpha} \wedge t J^a(x, y) \right) \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \ \le\ c_2\, p^a(t, x, y)\, \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}
$$

By symmetry, the above inequality holds with the roles of x and y interchanged. Using the semigroup property for $t \le T$,

$$
\begin{aligned}
p^a_D(t, x, y) &= \int_D p^a_D(t/2, x, z) p^a_D(t/2, z, y) dz \\
&\le c_3\, \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{t} \int_D p^a(t/2, x, z) p^a(t/2, z, y) dz \\
&\le c_3\, \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{t} p^a(t, x, y).
\end{aligned}
$$

This proves the upper bound (2.36) by noting that $(1 \wedge a)(1 \wedge b) = \min\{1, a, b, ab\}$ for $a, b > 0$. □

3 Lower bound estimate

In this section, we discuss the uniform lower bound estimate of $p^a_D(t, x, y)$. We will first give the uniform interior lower bound estimate of $p^a_D(t, x, y)$ for arbitrary open set D.

Lemma 3.1 *For any positive constants Λ, κ and b, there exists $C_{19} = C_{19}(\Lambda, \kappa, b, \alpha, \beta, M) > 0$ such that for every $z \in \mathbb{R}^d$, $\lambda \in (0, \Lambda]$ and $a \in (0, M]$,*

$$\inf_{\substack{y\in\mathbb{R}^d \\ |y-z|\le\kappa\lambda^{1/\alpha}}} \mathbb{P}_y\left(\tau^a_{B(z,2\kappa\lambda^{1/\alpha})} > b\lambda\right) \ge C_{19}.$$

Proof. By [13, Proposition 4.9], there exists $\varepsilon = \varepsilon(\Lambda, \kappa, \alpha, \beta) > 0$ such that for every $\lambda \in (0, \Lambda]$,

$$\inf_{y\in\mathbb{R}^d} \mathbb{P}_y\left(\tau^1_{B(y,\kappa\lambda^{1/\alpha}/2)} > \varepsilon\lambda\right) \ge \frac{1}{2}.$$

Suppose $b > \varepsilon$ then by the parabolic Harnack inequality in [13, Proposition 4.12]

$$c_1\, p^1_{B(y,\kappa\lambda^{1/\alpha})}(\varepsilon\lambda, y, w) \le p^1_{B(y,\kappa\lambda^{1/\alpha})}(b\lambda, y, w) \qquad \text{for } w \in B(y, \kappa\lambda^{1/\alpha}/2),$$

where the constant $c_1 = c_1(\Lambda, \kappa, \alpha, \beta, b) > 0$ is independent of $y \in \mathbb{R}^d$, $\lambda \in (0, \Lambda]$. Thus

$$\begin{aligned}
\mathbb{P}_y\left(\tau^1_{B(y,\kappa\lambda^{1/\alpha})} > b\lambda\right) &= \int_{B(y,\kappa\lambda^{1/\alpha})} p^1_{B(y,\kappa\lambda^{1/\alpha})}(b\lambda, y, w)dw \\
&\ge \int_{B(y,\kappa\lambda^{1/\alpha}/2)} p^1_{B(y,\kappa\lambda^{1/\alpha})}(b\lambda, y, w)dw \\
&\ge c_1 \int_{B(y,\kappa\lambda^{1/\alpha}/2)} p^1_{B(y,\kappa\lambda^{1/\alpha}/2)}(\varepsilon\lambda, y, w)dw \ge c_1/2. \qquad (3.1)
\end{aligned}$$

For the general case, since $\lambda a^{\frac{\alpha\beta}{\alpha-\beta}} \in (0, \Lambda M^{\alpha\beta/(\alpha-\beta)}]$, by (1.5) and (3.1),

$$\begin{aligned}
&\inf_{\substack{y\in\mathbb{R}^d: \\ |y-z|\le\kappa\lambda^{1/\alpha}}} \mathbb{P}_y\left(\tau^a_{B(z,2\kappa\lambda^{1/\alpha})} > b\lambda\right) \\
&\ge \mathbb{P}_0\left(\tau^a_{B(0,\kappa\lambda^{1/\alpha})} > b\lambda\right) \\
&= \int_{B(0,\kappa\lambda^{1/\alpha})} p^a_{B(0,\kappa\lambda^{1/\alpha})}(b\lambda, 0, w)dw \\
&= \int_{B(0,\kappa\lambda^{1/\alpha}a^{\beta/(\alpha-\beta)})} p^1_{B(0,\kappa\lambda^{1/\alpha}a^{\beta/(\alpha-\beta)})}(a^{\alpha\beta/(\alpha-\beta)}b\lambda, 0, z)dz \\
&= \mathbb{P}_0\left(\tau^1_{B(0,\kappa(\lambda a^{\alpha\beta/(\alpha-\beta)})^{1/\alpha})} > b\,\lambda a^{\alpha\beta/(\alpha-\beta)}\right) \ge c_2(\Lambda, \kappa, \alpha, \beta, b, M) > 0.
\end{aligned}$$

This proves the lemma. □

Recall that ψ^a is defined in (2.18).

Proposition 3.2 *Suppose that $M, T > 0$ and $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \ge t^{1/\alpha} \ge 2\psi^a(|x-y|)^{1/\alpha}$. Then there exists a positive constant $C_{20} = C_{20}(M, \alpha, \beta, T)$ such that for all $a \in (0, M]$*

$$p^a_D(t, x, y) \ge C_{20}\, t^{-d/\alpha}. \qquad (3.2)$$

Proof. Let $t \in (0, T]$ and $x, y \in D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2\psi^a(|x-y|)^{1/\alpha}$. By the parabolic Harnack inequality in [13, Proposition 4.12] and the scaling property, there exists $c_1 = c_1(M, \alpha, \beta, T) > 0$ such that for all $a \in (0, M]$,

$$p_D^a(t/2, x, w) \leq c_1\, p_D^a(t, x, y) \qquad \text{for } w \in B(x, 2t^{1/\alpha}/3).$$

This together with Lemma 3.1 yields that

$$\begin{aligned} p_D^a(t, x, y) &\geq \frac{1}{c_1\, |B(x, t^{1/\alpha}/2)|} \int_{B(x, t^{1/\alpha}/2)} p_D^a(t/2, x, w)dw \\ &\geq c_2 t^{-d/\alpha} \int_{B(x, t^{1/\alpha}/2)} p_{B(x, t^{1/\alpha}/2)}^a(t/2, x, w)dw \\ &= c_2 t^{-d/\alpha}\, \mathbb{P}_x \left(\tau_{B(x, t^{1/\alpha}/2)}^a > t/2\right) \geq c_3\, t^{-d/\alpha}, \end{aligned}$$

where $c_i = c_i(T, \alpha, \beta, M) > 0$ for $i = 2, 3$. □

Lemma 3.3 *Suppose that $M, T > 0$, D is an open subset of $\mathbb{R}^d$ and $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq t^{1/\alpha}$ and $t < 2^\alpha\, \psi^a(|x-y|)$. Then there exists a constant $C_{21} = C_{21}(\alpha, \beta, T, M) > 0$ such that for $a \in (0, M]$*

$$\mathbb{P}_x \left(X_t^{a,D} \in B\big(y,\, 2^{-1}t^{1/\alpha}\big)\right) \geq C_{21}\, t^{d/\alpha+1} J^a(x, y).$$

Proof. For $t \in (0, T]$, it follows from Lemma 3.1 that, starting at $z \in B(y,\, 4^{-1}t^{1/\alpha})$, with probability at least $c_1 = c_1(\alpha, \beta, T, M) > 0$, for any $a \in (0, M]$, the process X^a does not move more than $6^{-1}t^{1/\alpha}$ by time t. Thus, it suffices to show that there exists a constant $c_2 = c_2(\alpha, \beta, T, M) > 0$ such that

$$\mathbb{P}_x \left(X^{a,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \geq c_2\, t^{d/\alpha+1} J^a(x, y) \tag{3.3}$$

for all $a \in (0, M]$, $t \in (0, T]$ and $t < 2^\alpha\, \psi^a(|x-y|)$.

Let $B_x := B(x,\, 6^{-1}t^{1/\alpha})$, $B_y := B(y,\, 6^{-1}t^{1/\alpha})$ and $\tau_x^a := \tau_{B_x}^a$. It follows from Lemma 3.1 that there exists $c_3 = c_3(\alpha, \beta, T, M) > 0$ such that for $a \in (0, M]$ and $t \in (0, T]$,

$$\mathbb{E}_x \left[t \wedge \tau_x^a\right] \geq t\mathbb{P}_x \left(\tau_x^a \geq t\right) \geq c_3\, t. \tag{3.4}$$

By the Lévy system in (2.19),

$$\begin{aligned} &\mathbb{P}_x \left(X^{a,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \\ \geq\ &\mathbb{P}_x(X_{t\wedge\tau_x^a}^a \in B(y,\, 4^{-1}t^{1/\alpha}) \text{ and } t \wedge \tau_x^a \text{ is a jumping time }) \\ \geq\ &\mathbb{E}_x \left[\int_0^{t\wedge\tau_x^a} \int_{B_y} J^a(X_s^a, u)duds\right]. \end{aligned} \tag{3.5}$$

Note that $t < 2^\alpha\, \psi^a(|x-y|) \leq 2^\alpha\, |x-y|^\alpha$. Hence for $s < \tau_x^a$ and $u \in B_y$,

$$|X_s^a - u| \leq |x-y| + |x - X_s^a| + |y-u| \leq 2|x-y|.$$

Thus from (3.5) we get that for any $a \in (0, M]$ and $t \in (0, T]$,

$$
\begin{aligned}
& \mathbb{P}_x \left(X^{a,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t \right) \\
\geq\ & \mathbb{E}_x \left[t \wedge \tau_x^a \right] \int_{B_y} j^a(2|x-y|)\, du \\
\geq\ & c_4\, t\, |B_y|\, j^a(2|x-y|) \ \geq\ c_5\, t^{d/\alpha+1} j^a(2|x-y|) \ \geq\ c_5\, 2^{-d-\alpha} t^{d/\alpha+1} j^a(|x-y|)
\end{aligned}
$$

for some positive constants $c_i = c_i(\alpha, \beta, T, M)$, $i = 4, 5$. Here in the second inequality, (3.4) is used. □

Now we are ready to give the interior lower bound estimate of $p_D^a(t,x,y)$ for arbitrary open set D.

Theorem 3.4 *Suppose that $T > 0$, $M > 0$, D is an open subset of $\mathbb{R}^d$ and $(t,x,y) \in (0,T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq t^{1/\alpha}$. Then there exists a constant $C_{22} = C_{22}(\alpha, \beta, T, M) > 0$ such that for any $a \in (0, M]$,*

$$
p_D^a(t,x,y) \ \geq\ C_{22} \left(t^{-d/\alpha} \wedge t J^a(x,y) \right). \tag{3.6}
$$

Proof. In view of Proposition 3.2, it remains to show that (3.6) holds for $(t,x,y) \in (0,T] \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq (t/2)^{1/\alpha}$ and $t < 2\psi^a(|x-y|)$. By the semigroup property, Proposition 3.2 and Lemma 3.3, there exist positive constants $c_1 = c_1(\alpha, \beta, T, M)$ and $c_2 = c_2(\alpha, \beta, T, M)$ such that for any $t \in (0, T]$ and $a \in (0, M]$

$$
\begin{aligned}
p_D^a(t,x,y) &= \int_D p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\
&\geq \int_{B(y,\, 2^{-1}(t/2)^{1/\alpha})} p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\
&\geq c_1 t^{-d/\alpha} \mathbb{P}_x \left(X_{t/2}^{a,D} \in B(y, 2^{-1}(t/2)^{1/\alpha}) \right) \ \geq\ c_2\, t J^a(x,y).
\end{aligned} \tag{3.7}
$$

Now, combining (3.7) with Proposition 3.2, we have proved the theorem. □

In the rest of this section, we assume that D is an open set in $\mathbb{R}^d$ satisfying the uniform interior ball condition with radius $r_0 > 0$ in the following sense: For every $x \in D$ with $\delta_D(x) < r_0$, there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and $B(x_0, r_0) \subset D$ for $x_0 := z_x + r_0(x - z_x)/|x - z_x|$. Clearly, a (uniform) $C^{1,1}$ open set satisfies the uniform interior ball condition.

The goal of the remainder of this section is to prove the following lower bound for the heat kernel $p_D^a(t,x,y)$.

Theorem 3.5 *For any $M > 0$ and $T > 0$, there exists positive constant $C_{23} = C_{23}(\alpha, \beta, T, M, r_0)$ such that for all $a \in (0, M]$ and $(t,x,y) \in (0,T] \times D \times D$,*

$$
p_D^a(t,x,y) \geq C_{23} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge t J^a(x,y) \right).
$$

To prove this result, we will first prove a lower bound estimates on the Green function of $X^{a,U}$

$$G_U^a(x,y) := \int_0^\infty p_U^a(t,x,y)dt$$

when U is a bounded $C^{1,1}$ open set. The tool we use to establish the Green function lower bound is a subordinate killed α-stable process in U. We introduce this subordinate killed process first.

Assume that U is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and R_1 the radius in the uniform interior and exterior ball conditions. Then it follows from [6, Theorem 1.1] that the killed α-stable process X^U on U has a density $p_U(t,x,y)$ satisfying the following condition: for any $T>0$ there exist positive constants $c_2 > c_1$ depending only on α, T, R_1 and d such that for any $(t,x,y) \in (0,T] \times U \times U$,

$$p_U(t,x,y) \geq c_1 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right), \tag{3.8}$$

$$p_U(t,x,y) \leq c_2 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{3.9}$$

Let $\{T_t^a : t \geq 0\}$ be a subordinator, independent of X^a, with Laplace exponent $\phi^a(\lambda) = \lambda + a^\beta \lambda^{\beta/\alpha}$. Then the process $\{Z_t^{a,U} : t \geq 0\}$ defined by $Z_t^{a,U} = X^U_{T_t^a}$ is called a subordinate killed stable process in U. Since ϕ^a is a complete Bernstein function, the subordinator T^a has a decreasing potential density $u^a(x)$. In fact $u^a(x)$ is completely monotone. (See [24, 28] for the details.) Then it follows from [28] that the Green function $R_U^a(x,y)$ of $Z^{a,U}$ is given by

$$R_U^a(x,y) = \int_0^\infty p_U(t,x,y) u^a(t) dt. \tag{3.10}$$

It follows from [29] that the Green function G_U^a of $X^{a,U}$ and the Green function R_U^a of $Z^{a,U}$ satisfy the following relation:

$$R_U^a(x,y) \leq G_U^a(x,y) \qquad (x,y) \in U \times U. \tag{3.11}$$

So we can get a lower bound on $G_U^a(x,y)$ be establishing a lower bound on $R_U^a(x,y)$. The following result gives sharp two-sided estimates on $R_U^a(x,y)$ and the idea of the proof is similar to that of [27].

Theorem 3.6 *Suppose that $M>0$ and U is a bounded $C^{1,1}$ open set in U. There exist positive constants $C_{25} > C_{24}$ depending only on $(\alpha, \beta, d, R_1, M, diam(U))$ such that for all $a \in (0, M]$,*

$$R_U^a(x,y) \geq C_{24} \begin{cases} \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log\left(1 + \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d = 1 = \alpha, \\ \left(\delta_U(x)\delta_U(y)\right)^{(\alpha-1)/2} \wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha, \end{cases}$$

and

$$R^a_U(x,y) \le C_{25} \begin{cases} \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d} & \text{when } d>\alpha, \\ \log\left(1+\frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d=1=\alpha, \\ \left(\delta_U(x)\delta_U(y)\right)^{(\alpha-1)/2} \wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|} & \text{when } d=1<\alpha. \end{cases}$$

Proof. Since the drift coefficient of T^a is 1, we know that $u^a(t) \le 1$ for all $t>0$. Now the upper bound on R^a_U follows immediately from (3.10) and [6, Corollary 1.2]. Thus we only need to prove the lower bound.

By using a scaling argument, one can easily check that

$$u^a(t) = u^1(a^{\frac{\alpha}{\alpha-\beta}}t), \qquad t>0. \tag{3.12}$$

Let $T = \mathrm{diam}(U)$. Since $u^1(t)$ is a completely monotone function with $u^1(0+)=1$, by (3.12),

$$u^a(t) \ge u^1(M^{\frac{\alpha}{\alpha-\beta}}T) \qquad \text{for every } t\in(0,T] \text{ and } a\in(0,M]. \tag{3.13}$$

Using (3.13), (3.10) and [6, (4.2)] we get that

$$\begin{aligned} &\int_0^T \left(1\wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right)u^a(t)dt \\ \ge\ & \frac{u^1(M^{\frac{\alpha}{\alpha-\beta}}T)}{|x-y|^{d-\alpha}}\int_{\frac{|x-y|^\alpha}{T}}^\infty \left(u^{\frac{d}{\alpha}-2}\wedge u^{-3}\right)\left(1\wedge\frac{\sqrt{u}\,\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\sqrt{u}\,\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)du. \end{aligned} \tag{3.14}$$

Now we can follow the proof of [6, Corollary 1.2] to get the desired lower bound. In fact, when $d>\alpha$, the desired lower bound follows from (3.14) and [6, (4.3) and (4.7)]. Let

$$u_0 := \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}.$$

When $d=\alpha=1$, by (3.14) and [6, (4.3) and (4.9)],

$$\begin{aligned} R^a_U(x,y) &\ge u^1(M^{\frac{\alpha}{\alpha-\beta}}T)\int_0^T p_U(t,x,y)dt \\ &\ge c_1\left(1\wedge\frac{\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right)\left(1\wedge\frac{\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) + c_1\log(u_0\vee 1) + c_1 u_0\left((1/u_0)\wedge 1 - \frac{|x-y|^\alpha}{T}\right) \\ &\ge c_2(1\wedge u_0) + c_2\log(u_0\vee 1) + c_2u_0\left((1/u_0)\wedge 1-\frac{|x-y|^\alpha}{T}\right) \\ &\ge c_3(1\wedge u_0)+c_3\log(u_0\vee 1) \ \ge\ c_4\log\left(1+\frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right). \end{aligned}$$

Lastly, in the case $d=1<\alpha<2$. By (3.14), [6, (4.3) and (4.7)] and the first display in part (iii) of the proof of [6, Corollary1.2], we have

$$R^a_U(x,y) \ \ge\ u^1(M^{\frac{\alpha}{\alpha-\beta}}T)\int_T^\infty p_U(t,x,y)dt$$

$$\geq \quad c_5 \frac{1}{|x-y|^{1-\alpha}} \left((1 \wedge u_0) + \left((u_0 \vee 1)^{1-(1/\alpha)} - 1 \right) + c_6 u_0 \left((u_0 \vee 1)^{-1/\alpha} - \left(\frac{|x-y|^\alpha}{T} \right)^{1/\alpha} \right) \right)$$

$$\geq \quad c_7 \frac{1}{|x-y|^{1-\alpha}} \left(u_0 \wedge u_0^{1-(1/\alpha)} \right) = c_7 \left((\delta_U(x)\delta_U(y))^{(\alpha-1)/2} \wedge \frac{\delta_U(x)^{\alpha/2}\delta_U(y)^{\alpha/2}}{|x-y|} \right).$$

□

By integrating the lower bound in Theorem 3.6 with respect to y and applying (3.11), we obtain the following lower bound on $\mathbb{E}_x[\tau_U^a]$

Corollary 3.7 *Suppose that $M > 0$ and U is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ Then there exists a constant $C_{26} = C_{26}(\alpha, \beta, d, M, R_1, diam(U)) > 0$ such that for every $a \in (0, M]$ and $x \in U$,*

$$\mathbb{E}_x[\tau_U^a] \geq C_{26}\delta_U(x)^{\alpha/2}.$$

By integrating (1.5) with respect to t and y, we have that for every open set U, $\lambda > 0$ and $x \in U$,

$$\mathbb{E}_x[\tau_U^a] = \int_U G_U^a(x,z)dz = \lambda^\alpha \int_{\lambda^{-1}U} G_{\lambda^{-1}U}^{a\lambda^{(\alpha-\beta)/\beta}}(\lambda^{-1}x, y)dy = \lambda^\alpha \mathbb{E}_{\lambda^{-1}x}\left[\tau^{a\lambda^{(\alpha-\beta)/\beta}}(\lambda^{-1}U)\right]. \quad (3.15)$$

Lemma 3.8 *Suppose that $M > 0$, $\kappa \in (0,1)$ and that $(t,x) \in (0, (r_0/16)^\alpha] \times D$ with $\delta_D(x) \leq 3t^{1/\alpha} < r_0/4$. Let $z_x \in \partial D$ be such that $|z_x - x| = \delta_D(x)$ and define $\mathbf{n}(z_x) := (x - z_x)/|x - z_x|$. Put $x_1 = z_x + 3t^{1/\alpha}\mathbf{n}(z_x)$ and $B = B(x_1, 3t^{1/\alpha})$. Suppose that x_0 is a point on the line segment connecting z_x and $z_x + 6t^{1/\alpha}\mathbf{n}(z_x)$ such that $B(x_0, 1.5\kappa t^{1/\alpha}) \subset B \setminus \{x\}$. Then for any $b > 0$, there exists a constant $C_{27} = C_{27}(\kappa, \alpha, \beta, r_0, b, M) > 0$ such that for all $a \in (0, M]$*

$$\mathbb{P}_x\left(X_{bt}^{a,D} \in B(x_0, \kappa t^{1/\alpha})\right) \geq C_{27}\, t^{-1/2}\delta_D(x)^{\alpha/2}. \quad (3.16)$$

Proof. Let $0 < \kappa_1 \leq \kappa$ and assume first that $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$. Repeating the proof of Lemma 3.3, we get that, in this case, there exists a constant $c_1 = c_1(\alpha, \beta, \kappa_1, M, r_0, b) > 0$ such that for all $a \in (0, M]$

$$\mathbb{P}_x\left(X_{bt}^{a,D} \in B(x_0, \kappa_1 t^{1/\alpha})\right) \geq c_1 t^{d/\alpha+1} J^a(x, x_0) \geq c_1 \mathcal{A}(d, -\alpha) t^{d/\alpha+1}|x - x_0|^{-d-\alpha}$$

for all $t \leq (r_0/16)^\alpha$. Using the fact that $|x - x_0| \in [2\kappa t^{1/\alpha}, 6t^{1/\alpha}]$ we get that for all $a \in (0, M]$,

$$\mathbb{P}_x\left(X_{bt}^{a,D} \in B(x_0, \kappa_1 t^{1/\alpha})\right) \geq c_2 > 0 \quad (3.17)$$

for some constant $c_2 = c_2(\alpha, \beta, \kappa_1, M, r_0, b)$. By taking $\kappa_1 = \kappa$, this shows that (3.16) holds for all $b > 0$ in the case when $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$.

So it suffices to consider the case that $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. We now show that there is some $b_0 > 1$ so that (3.16) holds for every $b \geq b_0$ and $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. For simplicity, we assume without loss of generality that $x_0 = 0$ and let $\widehat{B} := B(0, \kappa t^{1/\alpha})$. Let $x_2 = z_x + (\kappa/4)\,\mathbf{n}(z_x)t^{1/\alpha}$ and $B_2 := B(x_2, 4^{-1}\kappa t^{1/\alpha})$. Observe that since $B(0, 2\kappa t^{1/\alpha}) \subset B \setminus \{x\}$,

$$\kappa/2t^{1/\alpha} \leq |y - z| \leq 6t^{1/\alpha} \qquad \text{for } y \in B_2 \text{ and } z \in B(0, \kappa t^{1/\alpha}). \quad (3.18)$$

By the strong Markov property of X^a at the first exit time $\tau^a_{B_2}$ from B_2 and Lemma 3.1,

$$
\begin{aligned}
&\mathbb{P}_x\left(X^a_{bt}\in B(0,\kappa t^{1/\alpha})\right)\\
\geq\ &\mathbb{P}_x\left(\tau^a_{B_2}<bt, X^a_{\tau^a_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha}) \text{ and } |X^a_s-X^a_{\tau_{B_2}}|<2^{-1}\kappa t^{1/\alpha} \text{ for } s\in[\tau^a_{B_2},\tau^a_{B_2}+bt^{1/\alpha}]\right)\\
\geq\ &c_3\,\mathbb{P}_x\left(\tau^a_{B_2}<bt \text{ and } X^a_{\tau^a_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\right). \qquad (3.19)
\end{aligned}
$$

It follows from the Lévy system of X^a, (3.18), (3.15) and Corollary 3.7 that

$$
\begin{aligned}
\mathbb{P}_x\left(X^a_{\tau^a_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\right) &= \int_{B_2} G^a_{B_2}(x,y)\left(\int_{B(0,2^{-1}\kappa t^{1/\alpha})} J^a(y,z)dz\right)dy\\
&\geq \int_{B_2} G^a_{B_2}(x,y)\left(\int_{B(0,2^{-1}\kappa t^{1/\alpha})}\frac{\mathcal{A}(d,-\alpha)}{|y-z|^{d+\alpha}}dz\right)dy\\
&\geq \frac{c_4}{t}\mathbb{E}_x\left[\tau^a_{B_2}\right]=c_4\mathbb{E}_{x/t^{1/\alpha}}\left[\tau^{at^{(\alpha-\beta)/\alpha\beta}}_{B(x_2/t^{1/\alpha},4^{-1}\kappa)}\right]\\
&\geq c_5\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \qquad (3.20)
\end{aligned}
$$

for some positive constants c_4, c_5 depending only on $\alpha, \beta, r_0, \kappa$ and M. Note that, by (1.5)

$$
\int_{B(x_2,4^{-1}\kappa t^{1/\alpha})} p^a_{B(x_2,4^{-1}\kappa t^{1/\alpha})}(bt,x,z)dz=\int_{B(t^{-1/\alpha}x_2,4^{-1}\kappa)} p^{at^{(\alpha-\beta)/\alpha\beta}}_{B(t^{-1/\alpha}x_2,4^{-1}\kappa)}(b,t^{-1/\alpha}x,w)dw.
$$

Since $at^{(\alpha-\beta)/\alpha\beta}\leq MT_0^{(\alpha-\beta)/\alpha\beta}$, by applying Theorem 2.8 to the right hand side of the above display, we get

$$
\begin{aligned}
\mathbb{P}_x\left(\tau^a_{B_2}\geq bt\right) &= \lim_{s\uparrow t}\mathbb{P}_x\left(\tau^a_{B_2}>bs\right)=\int_{B_2} p^a_{B_2}(bt,x,z)dz\\
&\leq b^{-d/\alpha}\int_{B(t^{-1/\alpha}x_2,4^{-1}\kappa)}\frac{\delta_{B(t^{-1/\alpha}x_2,4^{-1}\kappa)}(t^{-1/\alpha}x)^{\alpha/2}}{\sqrt{b}}dw\\
&\leq c_6\, b^{-d/\alpha-1/2}\,\delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha/2}=c_6\, b^{-d/\alpha-1/2}\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2} \qquad (3.21)
\end{aligned}
$$

for some positive constant c_6 depending only on $\alpha, \beta, r_0, \kappa$ and M. Define

$$
b_0:=\left(\frac{2c_6}{c_5}\right)^{\frac{2\alpha}{2d+\alpha}}.
$$

We have by (3.19)–(3.21) that for $b\geq b_0$,

$$
\begin{aligned}
\mathbb{P}_x(X^a_{bt}\in\widehat{B}) &\geq c_3\left(\mathbb{P}_x\left(X^a_{\tau^a_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\right)-\mathbb{P}_x\left(\tau^a_{B_2}\geq bt\right)\right)\\
&\geq c_3\,(c_5/2)\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}. \qquad (3.22)
\end{aligned}
$$

(3.17) and (3.22) show that (3.16) holds for every $b\geq b_0$ and for every $x\in D$ with $\delta_D(x)\leq 3t^{1/\alpha}$.

Now we deal with the case $0 < b < b_0$ and $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. If $\delta_D(x) \le 3(bt/b_0)^{1/\alpha}$, we have from (3.16) for the case of $b = b_0$ that

$$\begin{aligned}\mathbb{P}_x\left(X^a_{bt} \in B(x_0,\, \kappa t^{1/\alpha})\right) &\ge \mathbb{P}_x\left(X^a_{b_0(bt/b_0)} \in B(x_0,\, \kappa(bt/b_0)^{1/\alpha})\right) \\ &\ge c_7\left(\frac{\delta_D(x)}{(bt/b_0)^{1/\alpha}}\right)^{\alpha/2} = c_8\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.\end{aligned}$$

If $3(bt/b_0)^{1/\alpha} < \delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$ (in this case $\kappa > 3\cdot 2^4(b/b_0)^{1/\alpha}$), we get (3.16) from (3.17) by taking $\kappa_1 = (b/b_0)^{1/\alpha}$. The proof of the lemma is now complete. □

Proof of Theorem 3.5. Let $T_0 := \left(\frac{r_0}{16}\right)^\alpha$ and consider the case $t \le T_0$ first.

Let $z_x, z_y \in \partial D$ be such that $|z_x - x| = \delta_D(x), |z_y - y| = \delta_D(y)$ and define $\mathbf{n}(z_x) := (x - z_x)/|x - z_x|$ and $\mathbf{n}(z_y) := (y - z_y)/|y - z_y|$. Since D satisfies the uniform interior ball condition with radius r_0 and $0 < t \le T_0$, we can choose ξ^t_x as follows; if $\delta_D(x) \le 3t^{1/\alpha}$, let $\xi^t_x = z_x + (9/2)t^{1/\alpha}\mathbf{n}(z_x)$ (so that $B(\xi^t_x, (3/2)t^{1/\alpha}) \subset B(z_x + 3t^{1/\alpha}\mathbf{n}(z_x), 3t^{1/\alpha}) \setminus \{x\}$ and $\delta_D(z) \ge 3t^{1/\alpha}$ for every $z \in B(\xi^t_x, (3/2)t^{1/\alpha})$). If $\delta_D(x) > 3t^{1/\alpha}$, choose $\xi^t_x \in B(x, \delta_D(x))$ such that $B(\xi^t_x, (3/2)t^{1/\alpha}) \subset B(x, \delta_D(x)) \setminus \{x\}$ (so that and $\delta_D(z) \ge t^{1/\alpha}$ for every $z \in B(\xi^t_x, 2^{-1}t^{1/\alpha})$). We also define ξ^t_y the same way.

If $\delta_D(x) \le 3t^{1/\alpha}$, by Lemma 3.8 (with $b = 3^{-1}, \kappa = 2^{-1}$),

$$\mathbb{P}_x\left(X^{a,D}_{t/3} \in B(\xi^t_x, 2^{-1}t^{1/\alpha})\right) \ge c_0\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.$$

If $\delta_D(x) > 3t^{1/\alpha}$, by Theorem 3.4,

$$\mathbb{P}_x\left(X^{a,D}_{t/3} \in B(\xi^t_x, 2^{-1}t^{1/\alpha})\right) = \int_{B(\xi^t_x, 2^{-1}t^{1/\alpha})} p^a_D(t/3, x, u)du \ge c_1 t^{-d/\alpha}|B(\xi^t_x, 2^{-1}t^{1/\alpha})| \ge c_2.$$

Thus

$$\mathbb{P}_x\left(X^{a,D}_{t/3} \in B(\xi^t_x, 2^{-1}t^{1/\alpha})\right) \ge c_3\left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right). \tag{3.23}$$

Similarly

$$\mathbb{P}_y\left(X^{a,D}_{t/3} \in B(\xi^t_y, 2^{-1}t^{1/\alpha})\right) \ge c_3\left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right). \tag{3.24}$$

Note that by the semigroup property, Theorem 3.4 and (3.23)–(3.24),

$$\begin{aligned}&p^a_D(t, x, y) \\ \ge& \int_{B(\xi^t_y, 2^{-1}t^{1/\alpha})}\int_{B(\xi^t_x, 2^{-1}t^{1/\alpha})} p^a_D(t/3, x, u)p^a_D(t/3, u, v)p^a_D(t/3, v, y)dudv \\ \ge& c_4\int_{B(\xi^t_y, 2^{-1}t^{1/\alpha})}\int_{B(\xi^t_x, 2^{-1}t^{1/\alpha})} p^a_D(t/3, x, u)(tJ^a(u, v) \wedge t^{-d/\alpha})p^a_D(1/3, v, y)dudv \\ \ge& c_5\left(\inf_{(u,v)\in B(\xi^t_x, 2^{-1}t^{1/\alpha})\times B(\xi^t_y, 2^{-1}t^{1/\alpha})}(tJ^a(u, v) \wedge t^{-d/\alpha})\right)\left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right)\left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right).\end{aligned} \tag{3.25}$$

Since $|u-v| \le t^{1/\alpha} + |x-y|$, by considering the cases $|x-y| \ge t^{1/\alpha}$ and $|x-y| < t^{1/\alpha}$ separately, we have

$$\inf_{(u,v)\in B(\xi^t_x, 2^{-1}t^{1/\alpha})\times B(\xi^t_y, 2^{-1}t^{1/\alpha})} (tJ^a(u,v) \wedge t^{-d/\alpha}) \ge c_6 (tJ^a(x,y) \wedge t^{-d/\alpha}). \tag{3.26}$$

Thus combining (3.25) and (3.26), we conclude that for $t \in (0, T_0]$,

$$p^a_D(t,x,y) \ge c_7 \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) \left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1\right) (tJ^a(x,y) \wedge t^{-d/\alpha}). \tag{3.27}$$

Now we assume $T = 2T_0$. Recall that $T_0 = (r_0/16)^\alpha$. For $(t,x,y) \in (T_0, 2T_0] \times D \times D$, let $x_0, y_0 \in D$ be such that $\max\{|x-x_0|, |y-y_0|\} < r_0$ and $\min\{\delta_D(x_0), \delta_D(y_0)\} \ge r_0/2$. Note that, since for any $M > 0$, there exists $c_8 = c_8(M) > 0$ such that

$$j^a(r) \le c_8 j^a(2r), \qquad \text{for all } r > 0 \text{ and } a \in (0, M], \tag{3.28}$$

if $|x-y| \ge 4r_0$, then $\frac{1}{2}|x-y| \le |x-y| - 2r_0 \le |x_0 - y_0| \le |x-y| + 2r_0 \le \frac{3}{2}|x-y|$, and so $c_9^{-1} J^a(x_0,y_0) \le J^a(x,y) \le c_9 J^a(x_0,y_0)$ for some constant $c_9 = c_9(M) > 1$. Thus by considering the cases $|x-y| \ge 4r_0$ and $|x-y| < 4r_0$ separately, we have

$$(t/2)^{-d/\alpha} \wedge \frac{tJ^a(x_0,y_0)}{2} \ge c_{10} \left(t^{-d/\alpha} \wedge (tJ^a(x,y))\right). \tag{3.29}$$

Similarly, there is a positive constant c_{11} such that

$$\begin{aligned} (t/3)^{-d/\alpha} \wedge \frac{tJ^a(x,z)}{3} &\ge c_{11} \left((t/(12))^{-d/\alpha} \wedge \frac{tJ^a(x_0,z)}{12}\right), \quad z \in D, \\ (t/3)^{-d/\alpha} \wedge \frac{tJ^a(w,y)}{3} &\ge c_{11} \left((t/(12))^{-d/\alpha} \wedge \frac{tJ^a(w,y_0)}{12}\right), \quad w \in D. \end{aligned} \tag{3.30}$$

By (3.30) and (3.27), we have

$$\begin{aligned} p^a_D(t,x,y) &= \int_{D\times D} p^a_D(t/3,x,z) p^a_D(t/3,z,w) p^a_D(t/3,w,y)\, dz dw \\ &\ge c_{12} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/3}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/3}}\right) \int_{D\times D} \left((t/3)^{-d/\alpha} \wedge \frac{tJ^a(x,z)}{3}\right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\ &\qquad \cdot p^a_D(t/3,z,w) \left((t/3)^{-d/\alpha} \wedge \frac{tJ^a(w,y)}{3}\right) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dz dw \\ &\ge c_{13} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \int_{D\times D} \left(\left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(x_0,z)}{12}\right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\ &\qquad \cdot p^a_D(t/3,z,w) \left(\left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{tJ^a(w,y_0)}{12}\right) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dz dw \end{aligned}$$

for some positive constants $c_i, i = 12, 13$. Let $D_1 := \{z \in D : \delta_D(z) > r_0/4\}$. Clearly, $x_0, y_0 \in D_1$ and

$$\min\{\delta_{D_1}(x_0), \delta_{D_1}(y_0)\} \ge r_0/4 = 4(T_0)^{1/\alpha} \ge 4(t/2)^{1/\alpha}. \tag{3.31}$$

By (1.6) and (3.29), we have

$$
\begin{aligned}
&\int_{D\times D}\left(\left(\frac{t}{12}\right)^{-d/\alpha}\wedge\frac{tJ^a(x_0,z)}{12}\right)\left(1\wedge\frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right)\\
&\qquad\cdot\, p^a_D(t/3,z,w)\left(\left(\frac{t}{12}\right)^{-d/\alpha}\wedge\frac{tJ^a(w,y_0)}{12}\right)\left(1\wedge\frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right)dzdw\\
\ge\,& c_{14}\int_{D_1\times D_1}\left(\left(\frac{t}{12}\right)^{-d/\alpha}\wedge\frac{tJ^a(x_0,z)}{12}\right)p^a_D(t/3,z,w)\left(\left(\frac{t}{12}\right)^{-d/\alpha}\wedge\frac{tJ^a(w,y_0)}{12}\right)dzdw\\
\ge\,& c_{15}\int_{D_1\times D_1}p^a(t/(12),x_0,z)p^a_{D_1}(t/3,z,w)p^a(t/(12),w,y_0)dzdw\\
\ge\,& c_8\int_{D_1\times D_1}p^a_{D_1}(t/(12),x_0,z)p^a_{D_1}(t/3,z,w)p^a_{D_1}(t/(12),w,y_0)dzdw\\
=\,& c_{15}\,p^a_{D_1}(t/2,x_0,y_0)\ge c_{16}\left((t/2)^{-d/\alpha}\wedge\frac{tJ^a(x_0,y_0)}{2}\right)\ge c_{17}\left(t^{-d/\alpha}\wedge(tJ^a(x,y))\right)
\end{aligned}
$$

for some positive constants $c_i, i=14,\cdots,17$. Here the interior estimate Theorem 3.4 is used in the second to the last inequality in view of (3.31). By repeating the argument above, we have proved Theorem 3.5. □

Proof of Theorem 1.1. Theorems 2.8 and 3.5 give Theorem 1.1(i).

For the proof of Theorem 1.1(ii), we use ideas used in [10]. For reader's convenience, we give the full details of the proof and specify the dependency of the constants carefully.

Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (R_0,Λ_0). Clearly there is a ball $B=B(x_0,r_0)\subset D$ where r_0 depends only on R_0 and Λ_0. For each $a\ge 0$, the semigroup of $X^{a,D}$ is Hilbert-Schmidt as, by Theorem 1.1(i)

$$\int_{D\times D}p^a_D(t,x,y)^2dxdy=\int_D p^a_D(2t,x,x)dx\le C_1\,(2t)^{-d/\alpha}|D|<\infty,$$

and hence is compact. For $a\ge 0$, let $\{\lambda^{a,D}_k: k=1,2\dots\}$ be the eigenvalues of $-(\Delta^{\alpha/2}+a^\beta\Delta^{\beta/2})|_D$, arranged in increasing order and repeated according to multiplicity, and $\{\phi^{a,D}_k: k=1,2,\dots\}$ be the corresponding eigenfunctions normalized to have unit L^2-norm on D. Note that $\{\phi^{a,D}_k: k=1,2,\dots\}$ forms an orthonormal basis of $L^2(D;dx)$. It is well known that $\lambda^{a,D}_1$ is strictly positive and simple, and that $\phi^{a,D}_1$ can be chosen to be strictly positive on D.

We also let $\{\lambda^{a,B}_k: k=1,2\dots\}$ be the eigenvalues of $-(\Delta^{\alpha/2}+a^\beta\Delta^{\beta/2})|_B$, arranged in increasing order and repeated according to multiplicity. From the domain monotonicity of the first eigenvalue, it is easy to see that $\lambda^{a,B}_1\ge\lambda^{a,D}_1$. Thus, using [15, Theorem 3.4], we have that

$$\lambda^{a,D}_1\le\lambda^{a,B}_1\le(\lambda^B_1)^{\alpha/2}+M^{\beta/2}(\lambda^B_1)^{\beta/2}=:c_1\qquad\text{for every } a\in(0,M]\tag{3.32}$$

where λ^B_1 the first eigenvalue of $-\Delta|_B$. Moreover, by the Cauchy-Shwarz inequality,

$$\int_D\left(1\wedge\delta_D(x)^{\alpha/2}\right)\phi^{a,D}_1(x)dx\le\left(\int_D\left(1\wedge\delta_D(x)^{\alpha}\right)dx\right)^{1/2}=:c_2.\tag{3.33}$$

Recall that $p^a_D(t,x,y)$ admits the following eigenfunction expansion

$$p^a_D(t,x,y)=\sum_{k=1}^\infty e^{-t\lambda^{a,D}_k}\phi^{a,D}_k(x)\phi^{a,D}_k(y) \qquad \text{for } t>0 \text{ and } x,y\in D.$$

This implies that

$$\int_{D\times D}\left(1\wedge\delta_D(x)^{\alpha/2}\right)p^a_D(t,x,y)\left(1\wedge\delta_D(y)^{\alpha/2}\right)dxdy=\sum_{k=1}^\infty e^{-t\lambda^{a,D}_k}\left(\int_D(1\wedge\delta_D(x)^{\alpha/2})\phi^{a,D}_k(x)dx\right)^2. \tag{3.34}$$

Consequently, using the fact that $\{\phi^{a,D}_k: k=1,2,\dots\}$ forms an orthonormal basis of $L^2(D;dx)$, we have

$$\int_{D\times D}\left(1\wedge\delta_D(x)^{\alpha/2}\right)p^a_D(t,x,y)\left(1\wedge\delta_D(y)^{\alpha/2}\right)dxdy\le e^{-t\lambda^{a,D}_1}\int_D\left(1\wedge\delta_D(x)^{\alpha}\right)dx \tag{3.35}$$

for all $a>0$ and $t>0$. On the other hand, since

$$\phi^{a,D}_1(x)=e^{\lambda^{a,D}_1}\int_D p^a_D(1,x,y)\phi^{a,D}_1(y)dy,$$

by the upper bound estimate in Theorem 1.1(i) and (3.33) that for every $a\in(0,M]$ and $x\in D$,

$$\phi^{a,D}_1(x)\le e^{\lambda^{a,D}_1}C_1\left(1\wedge\delta_D(x)^{\alpha/2}\right)\int_D\left(1\wedge\delta_D(y)^{\alpha/2}\right)\phi^{a,D}_1(y)dy\le e^{\lambda^{a,D}_1}c_2C_1\left(1\wedge\delta_D(x)^{\alpha/2}\right).$$

Hence

$$\int_D\left(1\wedge\delta_D(x)^{\alpha/2}\right)\phi^{a,D}_1(x)dx\ge e^{-\lambda^{a,D}_1}(c_2C_1)^{-1}\int_D\phi^{a,D}_1(x)^2dx=e^{-\lambda^{a,D}_1}(c_2C_1)^{-1}.$$

It now follows from (3.34) that for every that for every $a\in(0,M]$ and $t>0$

$$\begin{aligned}&\int_{D\times D}\left(1\wedge\delta_D(x)^{\alpha/2}\right)p^a_D(t,x,y)\left(1\wedge\delta_D(y)^{\alpha/2}\right)dxdy\\ \ge\;& e^{-t\lambda^{a,D}_1}\left(\int_D\left(1\wedge\delta_D(x)^{\alpha/2}\right)\phi^{a,D}_1(x)dx\right)^2\ge e^{-(t+2)\lambda^{a,D}_1}(c_2C_1)^{-2}.\end{aligned} \tag{3.36}$$

It suffices to prove Theorem 1.1(ii) for $T\ge3$. For $t\ge T$ and $x,y\in D$, observe that

$$p^a_D(t,x,y)=\int_{D\times D}p^a_D(1,x,z)p^a_D(t-2,z,w)p^a_D(1,w,y)dzdw. \tag{3.37}$$

Since D is bounded, we have by the upper bound estimate in Theorem 1.1(i), (3.32) and (3.35) that for every $a\in(0,M]$, $t\ge T$ and $x,y\in D$,

$$\begin{aligned}&p^a_D(t,x,y)\\ \le\;& C_1^2\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)\int_{D\times D}\left(1\wedge\delta_D(z)^{\alpha/2}\right)p^a_D(t-2,z,w)\left(1\wedge\delta_D(w)^{\alpha/2}\right)dzdw\\ \le\;& C_1^2\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)e^{-(t-2)\lambda^{a,D}_1}\int_D 1\wedge\delta_D(x)^{\alpha}\,dx\end{aligned}$$

$$\le \quad c_3\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}e^{t\lambda_1^{a,D}}.$$

We also have by the lower bound estimate in Theorem 1.1(i) and (3.36) that for every $a \in (0, M]$, $t \ge T$ and $x, y \in D$,

$$\begin{aligned} p_D^a(t,x,y) &\ge C_1^{-2}(1 \vee \mathrm{diam}(D))^{-2d-2\alpha}\left(1 \wedge \delta_D(x)^{\alpha/2}\right)\left(1 \wedge \delta_D(y)^{\alpha/2}\right) \times \\ &\quad \times \int_{D\times D}\left(1 \wedge \delta_D(z)^{\alpha/2}\right) p_D^a(t-2,z,w)\left(1 \wedge \delta_D(w)^{\alpha/2}\right) dzdw \\ &\ge c_4\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}e^{-t\lambda_1^{a,D}}. \end{aligned}$$

This completes the proof of the theorem. □

Proof of Corollary 1.2. The lower bound estimate in (1.7) follows from (3.11) and Theorem 3.6.

Since the function $\psi^a(|x-y|)$ is bounded above and below by a positive constant if D is bounded, by integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to t, the proof of the upper bound estimate in (1.7) is identical to that of [6, Corollary 1.2] so we omit the details here. □

Theorem 3.9 (Uniform boundary Harnack principle) *Suppose $M, R \in (0,\infty)$, D is an open set in $\mathbb{R}^d$, $z \in \partial D$, $r \in (0, R)$ and that $B(A, \kappa r) \subset D \cap B(z, r)$. There exists $C_{28} = C_{28}(d, \alpha, \beta, \kappa, M, R) > 1$ such that for every $a \in (0, M]$, and all functions $u, v \ge 0$ on $\mathbb{R}^d$, positive regular harmonic for X^a in $D \cap B(z, 2r)$ and vanishing on $D^c \cap B(z, 2r)$, we have*

$$C_{28}^{-1}\frac{u(A)}{v(A)} \le \frac{u(x)}{v(x)} \le C_{28}\frac{u(A)}{v(A)}, \qquad x \in D \cap B(z, r).$$

Proof. Applying [6, Corollary 1.2] and our Corollary 1.2 to (2.20), we have that for every $R, M > 0$, there exists $c = c(\alpha, \beta, R, M) > 0$ such that, for every $a \in (0, M]$ and $0 < r \le R$

$$c^{-1}G_{B(x_0,r)}(x,y) \le G^a_{B(x_0,r)}(x,y) \le cG_{B(x_0,r)}(x,y), \quad \forall x,y \in B(x_0,r). \tag{3.38}$$

Using (3.38), we can get uniform estimates on the Poisson kernel

$$K^a_{B(x_0,r)}(x,z) := \int_{B(x_0,r)} G^a_{B(x_0,r)}(x,y)J^a(y,z)dy$$

of $B(x_0, r)$ with respect to X^a for $r \in (0, R]$. In particular, for $r < |z - x_0| < 2R$, $K^a_{B(x_0,r)}(x, z)$ is comparable to $K_{B(x_0,r)}(x, z)$, the Poisson kernel of $B(x_0, r)$ with respect to X for $r \in (0, R]$. Then using the uniform estimates on $K^a_{B(x_0,r)}(x, z)$ and (3.38) we can easily see that [30, Lemma 3.3] can be proved in the same way. Using the uniform estimates on the Poisson kernel of $B(x_0, r)$, (3.28) and (3.38) we can adapt the argument in [1, 22, 30] to get our uniform boundary Harnack principle. In [9], such ideas are used and the uniform boundary Harnack principle is established for the relativistic stable processes. Since the details of the proof is almost identical to those in [9], we omit the details. □

Proof of Theorem 1.3. First we observe that the Harnack inequality holds for the process X^1 by [24]. That is, there exists a constant $c_1 = c_1(\alpha, \beta, M) > 0$ such that for any $r \in (0, M^{\beta/(\alpha-\beta)}]$, $x_0 \in \mathbb{R}^d$ and any function $v \geq 0$ harmonic in $B(x_0, r)$ with respect to X, we have

$$v(x) \leq c_1 v(y) \qquad \text{for all } x, y \in B(x_0, r/2). \tag{3.39}$$

Note that for any $a \in (0, M]$, X^a has the same distribution as $\{\lambda X^1_{\lambda^{-\alpha}t}, t \geq 0\}$, where $\lambda = a^{\beta/(\beta-\alpha)} \geq M^{\beta/(\beta-\alpha)}$. Consequently, if u is harmonic in $B(x_0, r)$ with respect to X^a, where $r \in (0, 1]$, then $v(x) := u(\lambda x)$ is harmonic in $B(\lambda^{-1}x_0, \lambda^{-1}r)$ with respect to X and $\lambda^{-1}r \leq M^{\beta/(\beta-\alpha)}$. So by (3.39)

$$u(\lambda x) = v(x) \leq c_1 v(y) = c_1 u(\lambda y) \qquad \text{for all } x, y \in B(\lambda^{-1}x_0, \lambda^{-1}r/2).$$

That is,

$$u(x) \leq c_1 u(y) \qquad \text{for all } x, y \in B(x_0, r/2). \tag{3.40}$$

In other words, the uniform Harnack inequality holds (for every $r \leq 1$) for the family of processes $\{X^a, a \in (0, M]\}$.

Since D is a $C^{1,1}$ open set, there exists $r_0 \leq R_0$ such that the following holds: for every $Q \in \partial D$ and $r \leq r_0$ there is a ball $B = B(z^r_Q, r)$ of radius r such that $B \subset D$ and $\partial B \cap \partial D = \{Q\}$. In addition, it follows [26, Lemma 2.2] that, for each $Q \in \partial D$, we can choose a constant $c_2 = c_2(d, \Lambda) \in (0, 1/8]$ and a bounded $C^{1,1}$ open set U_Q with uniform characteristics (R_*, Λ_*) depending on (R_0, Λ) such that $B(Q, c_2 r_0) \cap D \subset U_Q \subset B(Q, r_0) \cap D$ and

$$\delta_D(y) = \delta_{U_Q}(y) \quad \text{for every } y \in B(Q, c_2 r_0) \cap D. \tag{3.41}$$

Assume $a \in [0, M]$, $r \in (0, c_2 r_0]$, $Q \in \partial D$ and u is nonnegative function in $\mathbb{R}^d$ harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q, r)$. Let $z_Q := z^{c_2 r_0}_Q$. By the boundary Harnack principle (Theorem 3.9), there exists a constant $c_3 = c_3(\alpha, \beta, a, R_0, \Lambda, M)$ such that

$$\frac{u(x)}{u(y)} \leq c_3 \frac{G^a_{U_Q}(x, z_Q)}{G^a_{U_Q}(x, z_Q)} \quad \text{for every } x, y \in B(Q, r/8) \cap D.$$

Now applying Corollary 1.2 to $G^a_{U_Q}(x, z_Q)$ and $G^a_{U_Q}(x, z_Q)$, then using (3.41), we conclude that

$$\frac{u(x)}{u(y)} \leq c_4 \frac{\delta^{\alpha/2}_{U_Q}(x)}{\delta^{\alpha/2}_{U_Q}(y)} = c_4 \frac{\delta^{\alpha/2}_{D}(x)}{\delta^{\alpha/2}_{D}(y)} \quad \text{for every } x, y \in B(Q, c_2 r) \cap D \tag{3.42}$$

for some $c_4 = c_4(\alpha, \beta, a, R_0, \Lambda, M) > 0$.

Now Theorem 1.3 follows from the uniform Harnack inequality in (3.40), (3.42) and a standard chain argument. □

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References

[1] K. Bogdan, The boundary Harnack principle for the fractional Laplacian. *Studia Math.* **123** (1997), 43–80.

[2] K. Bogdan, K. Burdzy and Z.-Q. Chen, Censored stable processes. *Probab. Theory Related Fields*, **127** (2003), 89–152.

[3] K. Bogdan and T. Byczkowski, Potential theory of Schrdinger operator based on fractional Laplacian. *Probab. Math. Statist.*, **20** (2000), 293–C335.

[4] K. Bogdan and T. Grzywny, Heat kernel of fractional Laplacian in cones. *Colloq. Math.* **118(2)** (2010), 365–377

[5] K. Bogdan, T. Grzywny and M. Ryznar, Heat kernel estimates for the fractional Laplacian with Dirichlet conditions. *Ann. Probab.* **38** (2010), 1901–1923.

[6] Z.-Q. Chen, P. Kim, and R. Song, Heat kernel estimates for Dirichlet fractional Laplacian. *J. European Math. Soc.*, **12** (2010), 1307–1329.

[7] Z.-Q. Chen, P. Kim and R. Song, Two-sided heat kernel estimates for censored stable-like processes. *Probab. Theory Relat. Fields*, **146** (2010), 361–399.

[8] Z.-Q. Chen, P. Kim and R. Song, Sharp heat kernel estimates for relativistic stable processes in open sets. *Ann. Probab.* (to appear).

[9] Z.-Q. Chen, P. Kim and R. Song, Green function estimates for relativistic stable processes in half-space-like open sets. Preprint, 2010.

[10] Z.-Q. Chen, P. Kim and R. Song, Heat kernel estimate for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets. *J. London Math. Soc.* (to appear).

[11] Z.-Q. Chen, P. Kim, R. Song and Z. Vondraček, Boundary Harnack pinciple for $\Delta + \Delta^{\alpha/2}$. Preprint, 2009, arXiv:0908.1559 [math.PR].

[12] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d-sets, *Stoch. Proc. Appl.*, **108** (2003), 27–62.

[13] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Relat. Fields*, **140** (2008), 277–317.

[14] Z.-Q. Chen and R. Song, Estimates on Green functions and Poisson kernels of symmetric stable processes, *Math. Ann.*, **312** (1998), 465-601.

[15] Z.-Q. Chen and R. Song, Two-sided eigenvalue estimates for subordinate processes in domains. *J. Funct. Anal.*, **226(1)** (2005), 90–113

[16] E. B. Davies, Explicit constants for Gaussian upper bounds on heat kernels. *Amer. J. Math.* **109** (1987), 319–333.

[17] E. B. Davies, The equivalence of certain heat kernel and Green function bounds. *J. Funct. Anal.* **71** (1987), 88–103.

[18] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[19] E. B. Davies and B. Simon, Ultracontractivity and heat kernels for Schrödinger operator and Dirichlet Laplacians. *J. Funct. Anal.* **59** (1984), 335-395.

[20] T. Grzywny and M. Ryznar, Estimates of Green functions for some perturbations of fractional Laplacian. *Illinois J. Math.* **51** (2007), 1409–1438.

[21] Q.-Y. Guan, Boundary Harnack inequality for regional fractional Laplacian. arXiv:0705.1614 [math.PR]

[22] P. Kim, R. Song and Z. Vondraček, Boundary Harnack principle for subordinate Brownian motion. *Stoch. Proc. Appl.*, **119** (2009), 1601–1631.

[23] T. Kulczycki, Properties of Green function of symmetric stable processes, *Probab. Math. Stat.* **17** (1997), 381–406.

[24] M. Rao, R. Song and Z. Vondraček, Green function estimates and Harnack inequality for subordinate Brownian motions. *Potential Anal.* **25** (2006), 1–27

[25] M. Ryznar, Estimates of Green function for relativistic α-stable process. *Potential Anal.* **17** (2002), 1–23.

[26] R. Song, Estimates on the Dirichlet heat kernel of domains above the graphs of bounded $C^{1,1}$ functions. *Glas. Mat.* **39** (2004), 273–286.

[27] R. Song and Z. Vondracek, Sharp bounds for Green functions and jumping functions of subordinate killed Brownian motions in bounded $C^{1,1}$ domains. *Electron. Comm. Probab.*, **9** (2004), 96–105.

[28] R. Song, Z. Vondracek, Potential theory of special subordinators and subordinate killed stable processes. *J. Theoret. Probab.*, **19** (2006), 817–847.

[29] R. Song, Z. Vondracek, On the relationship between subordinate killed and killed subordinate processes, *Elect. Commun. Probab.*, **13** (325–336) 2008.

[30] R. Song and J. Wu, Boundary Harnack principle for symmetric stable processes, *J. Funct. Anal.* **168(2)** (1999), 403–427.

[31] Q. S. Zhang, The boundary behavior of heat kernels of Dirichlet Laplacians, *J. Differential Equations*, **182** (2002), 416–430.

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